Engineering Mechanics

Third Revised Edition

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Tata McGraw-Hill

Tata McGraw-Hill

Sixth reprint 2006
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ISBN 0-07-049473-8

Published by Tata McGraw-Hill Publishing Company Limited, 7 West Patel Nagar, New Delhi 110 008 and typeset in Times at The Composers, 20/5 Old Market, West Patel Nagar, New Delhi 110 008 and printed at Saurabh Printers Pvt. Ltd., A-16, Sector IV, Noida

Cover Design: B & M Design, Noida
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1 ENGINEERING MECHANICS: WHAT AND WHY?

1.1 ENGINEERING AND ENGINEERING SCIENCES

Engineering is an activity concerned with the *creation of new systems* for the benefit of mankind. The process of creativity proceeds by way of research, design and development; new systems emerge from innovation and systems may be constituted by mechanical, electromechanical, hydraulic, thermal or other elements. Creation of new systems is thus basic to all engineering. The Living Webster Encyclopedic Dictionary aptly defines engineering as *the art of executing a partial application of scientific knowledge.*

It is important to understand the difference between engineering and science. Science is concerned with a systematic understanding and gathering of the facts, laws and principles governing natural phenomena. Engineering, on the other hand, is an art of utilisation of the established facts, laws and principles to create certain desired phenomena as shown in Fig. 1.1. The activities of science and engineering are thus mutually opposite. Both may proceed through similar ways and means of analysis and synthesis but are oppositely directed. The training of scientists and engineers should be correspondingly designed for their respective objectives.

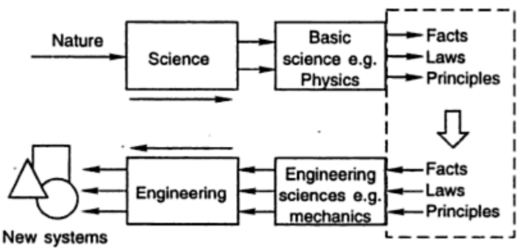


Fig. 1.1 ***Role of Engineering Sciences***

The sets of core courses meant for engineering students are called *engineering sciences.* These are essentially basic sciences compartmentalised and labelled specially for engineering students with regard to their future responsibility. The existing laws and principles are conveyed to the students by the engineering-science courses and emphasis is laid on their application to real-life problems. Some of the engineering-science courses being offered in India and abroad are: Mechanics, Manufacturing Processes, Energy Conversions, Transport Phenomena, Material Science and Design Engineering.

1.2 MECHANICS AND ITS RELEVANCE TO ENGINEERING

Mechanics is the physical science concerned with the dynamical behaviour of material bodies in the presence of mechanical disturbances. Since such behaviour is of interest to mechanical, civil, electrical, chemical, aeronautical, textile, metallurgical and mining engineers, it is appropriate to conclude that the subject of mechanics lies at the core of all engineering analysis.

Engineering mechanics refers to a course in mechanics tailored exclusively for engineers. Essential features of such a course are:

1. The subject matter is not presented as rigorously as a course in analytical or axiomatic mechanics may demand.
2. On the other hand, the contents are not just a series of applications as implied by Applied Mechanics but a thorough grounding of the basic principles together with engineering applications.
3. The course is integrated to provide a sound foundation in engineering-science.
4. The course comprises the foundation for a number of courses that are to be built upon it. Some of them are shown in Fig. 1.2.

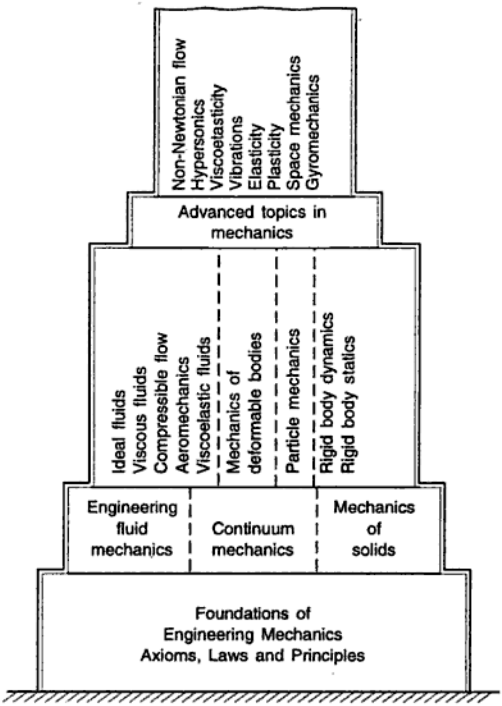


Fig. 1.2 ***Mechanics as the Foundation of a Number of Courses***

1.3 DIFFERENT FORMULATIONS OF MECHANICS

The subject of mechanics has been dealt with by a number of scientists from Archimedes (287-212 B.C.) to Einstein (1878-1955). Apart from the historical development of the subject, the following three broad classifications have come to stay in view of the different axioms and principles employed:

1. Classical mechanics
2. Quantum or wave mechanics
3. Relativistic mechanics

The subject of classical mechanics rests on the classical foundations laid by Galileo, Kepler, Newton and Euler. The laws of linear motion due to Newton and the law of angular motion due to Euler have stood the test of time remarkably well. These are valid for the dynamic behaviour of most of the observable bodies. Alternative foundations to classical mechanics were provided by Lagrange in terms of the Lagrangian equation and by Hamilton in terms of the canonical equations. Later, the 'principle of least action' on the basis of variational concepts was proposed as the single principle governing the behaviour of bodies is most circumstances. The word 'classical' therefore, is justifiable with respect to its dictionary meaning:

Classical ≡ Traditionally accepted, long established; excellent, standard

The classical pattern breaks down for a body approaching the speed of light on the one hand and for particles of size comparable with atoms on the other. It is for these reasons that the structure of an atom remained unexplained until the principles of Quantum Mechanics were framed and the problems of very high-speed bodies remained a mystery until the formulation of the special and general theories of relativity by Albert Einstein in the twentieth century. Relativistic mechanics is based upon novel concepts of space and time, mass and energy, and the frame of reference.

Table 1.1 gives the names of the scientists in relation to their respective regimes of mechanics. The regimes of different formulations in mechanics are represented schematically in Fig. 1.3.

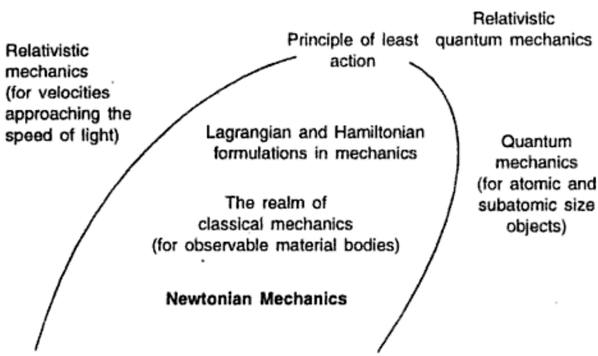


Fig. 1.3 ***Regimes of Mechanics***

Table 1.1 Pioneers of Mechanics

Quantum or Wave Mechanics	*Classical Mechanics*	*Relativistic Mechanics*
Schrödinger	≡ Non-relativistic	Lorentz
(1887-1961)	≡ Newtonian	(1853-1928)
Broglie	Newton (1642-1727)	Einstein
(1892-1965)	Equivalent foundations by	(1878-1955)
	Lagrange (1736-1813)	Bose
	Hamilton (1805-1865)	(1897-1974)
	D'Alembert (1717-1783)	
	Jacobi (1804-1851)	
	Contributions mainly by	
	Kepler (1571-1630)	
	Archimedes (287-212 B.C.)	
	Galileo (1564-1642)	
	Euler (1707-1783)	
	Coulomb (1736-1806)	
	Coriolis (1792-1843)	
	Foucault (1819-1868)	
	Bernoulli (1667-1748)	

Further advances have not diminished the value of the Newtonian or classical 'universal-time and absolute-distance' concepts except for setting the upper bounds. The bulk of the engineering work rests upon the foundation of classical mechanics and it is, therefore, important to lay utmost emphasis on the teaching of classical mechanics to engineers.

1.4 FOUNDATIONAL CONCEPTS

The axiomatic foundations of mechanics, have the following ingredients:

Undefined terms and concepts ⇒ Defined entities ⇒ Axioms and laws ⇒ Theorems and principles

Some terms and concepts cannot be absolutely defined but are developed for axiomatic thinking and mutual understanding. Examples of such classical terms are: point, line and plane; space, time and matter; mass, force and energy.

Space refers to the unlimited general expanse of physical dimensions in which all material objects are located. Measurements and locations in space involve the concepts of *point, direction, length* and *displacement.* A *point,* for example, is just an exact indication of a location in space, requiring no space at all for itself. *Time* refers to the sequence of events. It is related to the concepts of *before, after* and *simultaneous* occurrence of two or more events. Measurement of time is made with the help of a clock. *Matter* refers to the substance of which physical objects are composed, the constituent substances are indeed the *atoms* and *molecules*. The quantity of matter associated with an object is measured as its *mass*. A physical object may consist of matter which is uniformly or non-uniformly distributed. Bodies with the same quantity of matter or the same mass can possess different shapes and sizes depending upon the distribution of matter in them.

Defined entities include momentum, moment of a force, impulse, work, equilibrium, rigid body, etc., Axioms are the relatively universal statements relating undefined concepts and defined concepts. Examples of axioms are Newton's 2nd law, laws of friction and the law of gravitation.

Theorems and principles are derived from the axioms. Theorems and principles can be proved. Examples of principles are Work-Energy principle, Lami's theorem and Impulse-Momentum principle.

1.5 FRAMES OF REFERENCE: INERTIAL AND MOVING

It is necessary to refer the motion of a body under study to some datum or reference space and clock. A reference frame, therefore, consists of a space and a clock to measure time. A reference frame should be such that the relative location of any two arbitrary points in it remains the same. It follows that the distance between any two points in the reference frame should remain invariant. A reference frame is called *fixed frame of reference* or *absolute frame* if each point on the frame is at 'absolute rest'. It is impossible to locate a fixed frame of reference in the universe. Rectangular Cartesian axes can be embedded in a frame of reference. The origin and orientation of axes can be according to convenience.

A reference frame is termed as *moving frame of reference* if each point on the frame is not at rest. A moving frame may be *inertial* or *non-inertial*. An inertial frame is one which moves at a constant velocity, i.e., the velocity of each point identified on the frame is the same and remains constant. Obviously, an inertial frame can move in a straight line at constant speed. In other words, an inertial frame can be defined as a frame which does not have any acceleration. An inertial frame is also known as *Galilean frame.*

The state of *rest* of a body refers to the absence of motion relative to some coordinate system. By *absolute rest* we mean a state of fixedness in space. Such a state could provide an absolute reference for the motion of other objects. However, it is doubtful if any such reference exists in the solar system or in the entire universe. It is, therefore, appropriate to speak of *relative rest* of a body with reference to a moving frame of reference. A reference frame fixed on the earth is both an approximation of the rest-frame and a convenient choice for all earth-bound objects for most engineering applications. A better choice from the point of view of physicists and mathematicians would be the centres of the earth, solar system, galaxy, and so on.

1.6 IDEALISATION OF BODIES

A *body* is a distinct mass, continuously distributed over a volume V enclosed in a surface S. An element of a body is referred to occupy a small volume ΔV and have a small mass Δm.

The words 'body' and 'system' are often used interchangeably. By general consensus, a body implies a single material configuration and a system refers to a combination of bodies. For example, a car is said to be a body if we were to consider the whole car as a single lump of mass but the car is referred to as a system of engine, chassis and wheels if we were to identify these items collectively.

It may be understood that the mathematical modelling of a system should be done for the specific purpose in view: different mathematical models of a system are made for different objectives of analysis. For example, an aeroplane may be regarded as (a) a concentrated mass with negligible dimensions for the purpose of tracing its trajectory when it is flying sky-high, (b) merely a wing with a large span for the analysis of its lifting characteristics, (c) a distributed mass system for the stability analysis under different flight conditions and (d) a deformable shell for the purpose of calculations of the strain when subjected to different pressures inside and outside the cabin.

Different idealisations of bodies bear standard nomenclature and have specific implications. These are:

1. Particle
2. System of particles
3. Continuum
4. Rigid body
5. Deformable body
6. Fluid
7. Solid

Particle

When the dimensions or size of a body are considered to be negligible and are irrelevant to the description of its motion, the body is modelled as a particle. A particle is a point mass or a material point in the abstract sense. A body is, therefore, represented as a particle if its dimensions are small compared to the coordinates describing its motion as shown in Fig. 1.4.

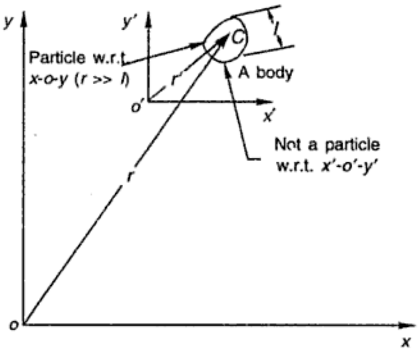


Fig. 1.4 ***Criteria of Idealisation of Bodies***

Examples A cricket ball as viewed by a spectator; a distant aeroplane tracked by a ground observer; a satellite orbiting the earth and seen by an observer on the earth; a planet as seen from another planet.

System of Particles

When two or more bodies are represented by particles and are dealt with together, a system of particles is constituted. A system of particles is an idealisation of a collection of point masses. A body or a set of bodies is, therefore, represented as a system of particles if each part of the body or each body individually qualifies to be represented by a particle. A system of particles may comprise a rigid collection or a deformable collection in accordance with the criteria of rigid or deformable bodies which follow.

Examples Billiard balls observed by a viewer in the gallery; sun-earth-moon system; electron-proton-neutron nature of atom.

Continuum

When the microscopic nature of matter is disregarded and properties of the substance are defined assuming a continuous distribution of mass, the embodiment of matter so modelled is called a continuum. In a continuum, the gross effects of the actions of the molecules and atoms are conveyed by the concepts of density, pressure and temperature which simplify our study considerably.

The mass density ρ at a point P in a continuum is defined as the ratio of the mass element Δm to the volume ΔV enclosing the point, in the limit when ΔV tends to zero

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV} \tag{1.1a}$$

The expression of the mass of a body in terms of its density is, therefore,

$$\boxed{m = \int \rho \, dV} \tag{1.1b}$$

where ρ, the density of the continuum may be constant or may vary continuously with the space coordinates.

A body may, therefore, be represented as a continuum if the approximations about the continuity of mass and the continuous variation of the physical variables are acceptable in terms of analysis and results.

A continuum may be a rigid or a deformable medium in accordance with the definitions which follow.

Rigid Body

When the dimensions, linear and angular, of a body do not change during the course of observation, the body is modelled as a rigid body. A rigid body, in other words, is the one in which the distance between any two arbitrary points is invariant. A body, therefore, qualifies to be represented as a rigid body if the deformation between its parts is negligible in the course of its analysis.

Examples An aeroplane observed in roll, pitch and yaw; a spinning top; a wheel of a cart.

Deformable Body

When the dimensions, linear or angular, of a body change during its analysis, the body is modelled as a deformable body. Deformation may be brought about in a variety of ways; it may be temporary or permanent, instantaneous or continuous. A body is, therefore, represented as a deformable body if the relative deformation between its parts cannot be ignored in the course of its analysis.

Examples A beam deflecting under the application of a load; a liquid flowing in any situation; a shaft twisting under the application of a torque.

Fluid

A substance which deforms continuously under the application of shear stresses, however small, is called a fluid. The process of continuous deformation is called a flow. A fluid must, therefore, flow when subjected to a shear stress. In the absence of shear stresses the fluids behave as static masses or as rigid bodies in motion.

Examples A liquid, e.g., water, oil or molten metal; a gas, e.g., air, oxygen or supercritical steam; a vapour, e.g., dry saturated subcritical steam; blood, slurry, ink, milk and beer.

Solid

A substance characterised by some preferred configuration of its own, i.e., possessing a definite shape and a definite volume, is called a *solid*. Any change of shape or volume of a solid is accompanied by its tendency to regain its original configuration or stay in the new configuration with a change in some of its properties. Solids can be regarded as rigid bodies or as deformable bodies depending on their mathematical-modelling requirements.

Examples A straight metre scale for linear measurements is regarded as a rigid-solid body; a metre rod bent to draw a curve is regarded as a deformable solid.

1.7 INTERNAL AND EXTERNAL FORCES

Internal forces are those which hold together the material content of the body or the parts of the system under consideration. Internal forces resist or tend to resist the external forces. For example, if a car is pulling a trailer by a rope coupling the two as shown in Fig. 1.5, then the force in the rope is an internal force for the 'car-and-trailer' system. The tractive force developed by a vehicle is transmitted through a series of components between the engine and the wheels; these forces are internal as far as the whole vehicle is concerned.

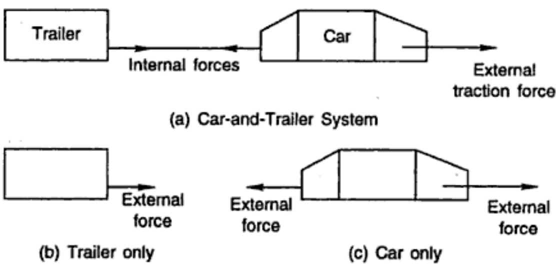


Fig. 1.5 ***Internal and External Forces***

External forces are those which act on a body or a system from outside. It is indeed the forces exerted on a body from outside that govern its state of motion. For example, for the trailer pulled by the car by a rope; the effect of the car is experienced by the force in the rope; hence, for the trailer, the pull by the car, i.e., the tension in the rope is an external force as shown in Fig. 1.5(b). Similarly, the tension in the rope is an external force for the car, as shown in Fig. 1.5(c).

If a component, say a gear, is to be considered for analysis, the forces exerted by the other gears and components on it are external forces.

It should be clear that a force is classified as internal or external depending upon the boundaries of the system. For example, the force between the earth and the moon is external if we were to consider the motion of the moon alone but the same force is internal if we were to consider the motion of the earth-moon system.

The concept of internal and external action is equally valid for moments also. Internal moments are those originating from inside a body or a system, whereas external moments are by virtue of sources outside the body or the system under consideration.

1.8 PRINCIPLE OF TRANSMISSIBILITY OF FORCE

The principle of transmissibility of force states that the condition of motion of a rigid body remains unchanged if a force F of a given magnitude, direction and sense acts anywhere along the same line of action on the rigid body. For example a force F acting at a_2 along the line of action a_1a_2 is equivalent to a force F acting at a_1 along the same line of action a_1a_2 as shown in Fig. 1.6. Another example of a rigid

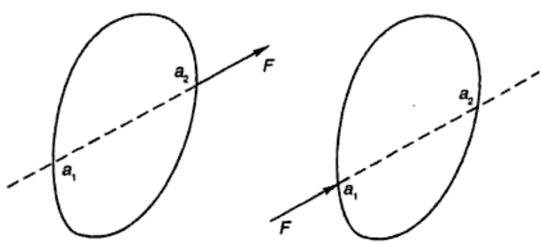


Fig. 1.6 ***Transmissibility of Force***

body motion is provided if a mass m being lifted with an acceleration $\mathbf{a}$ by means of a force applied differently at different places by along the same line of action passing through the centre of gravity C as shown in Fig. 1.7. A string with tension T pulling it up or an upward force F applied from below or a combination of the two such that

$$T = F = T_1 + F_1$$

result in the same motion of the body due to the principle of transmissibility:

$$T - mg = ma \quad \text{or} \quad a = (T - mg)/m$$
$$F - mg = ma \quad \text{or} \quad a = (F - mg)/m$$
$$T_1 + F_1 - mg = ma \quad \text{or} \quad a = (T_1 + F_1 - mg)/m$$

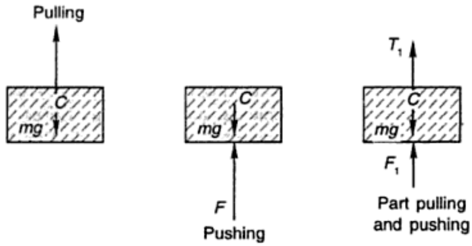
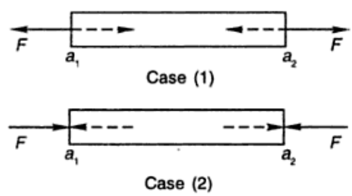
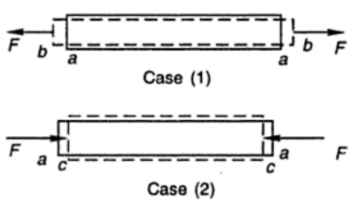


Fig. 1.7 ***Examples of Transmissibility of Force***

The principle of transmissibility applies only to a rigid body and is valid only from the point of view of the net external effect for the state of motion of the rigid body. It applies neither to the rigid body from the point of view of internal resistance or internal forces developed in a body nor to deformable bodies under any circumstances. Consider, for example, a rigid bar under the action of two equal and opposite forces as shown in Fig. 1.8(i). The principle of transmissibility would state that the forces in case 1 and in case 2 are equivalent and in each case the net external force is zero. This statement is true only from the point of view of external behaviour of the body. Let us look at the development of the internal forces to keep the body and its parts in equilibrium. The resistive forces are developed at a_1 and a_2 as shown dotted in the two cases. Clearly, the bar in case 1 is in tension and the bar in case 2 is in compression. These are entirely different effects. If the bar in question was non-rigid or deformable as shown in Fig. 1.8(ii), the bar would also yield in tension or in compression. In case 1, under tension, the bar elongates



(i) Internal Forces in a Rigid Bar



(ii) Deformation in a Deformable Bar

Fig. 1.8 ***Internal Forces and Deformation***

longitudinally whereas in case 2, under compression, the bar contracts longitudinally. The behaviour of the bar is in contravention of the principle of transmissibility which would have stated that the bar is subjected to zero external force and there is no net effect on the body.

In conclusion it may be stated that the principle of transmissibility, which requires the force to be a free vector, is valid only for a rigid body from the point of view of net external effect for the state of motion of the body.

1.9 CONCEPT OF FREE-BODY DIAGRAM

No system, natural or man-made, consists of a single body alone or is complete by itself. A single body or a part of the system can, however, be isolated from the rest by appropriately accounting for its effect. A free-body diagram consists of a diagrammatic representation of a single body or a subsystem of bodies isolated from its surroundings but shown under the action of forces and moments due to external actions.

Consider, for example, a book lying flat on a table. The book exerts its weight on the table and the table exerts its own weight as well as transmits the weight of the book on the ground. A free-body diagram for the book alone would consist of its weight **W** acting through the centre of gravity and the reaction exerted on the book by the table top as shown in Fig. 1.9. The reaction per unit area can be shown as **R**/*A* as in Fig. 1.9(b) or as a single resultant reaction force **R** collinear with the weight **W**.

Consider, as another example, two cylinders placed in a V-groove. Free-body diagrams of the two bodies isolated from the V-groove as well as of each body separately are shown in Figs. 1.9 (d), (e) and (f) respectively.

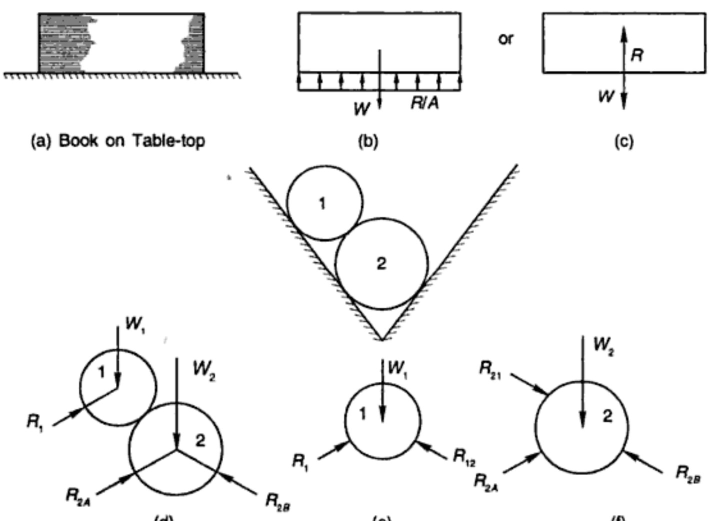


Fig. 1.9 ***(a) Book on Table-top (b), (c) Free-body Diagram of the Book (d), (e), (f) Free-body Diagram of Cylinders or Spheres***

A free-body diagram may be drawn for any single member of a system, any subsystem of the system or the entire system irrespective of whether the system is in equilibrium: at rest, in uniform motion or in a dynamic state of motion.

The example of a book lying flat on a table is that of static equilibrium. In such cases the forces and moments acting on the body must be in conformity with the conditions of equilibrium which are dealt with in detail in Chapter 3.

Free-body diagram of a single member or a subsystem of a dynamic system, on the other hand, would reveal an unbalance of the forces and moments; the unbalanced resultants causing accelerations, linear or angular. Further discussion on the subject of free-body diagram for dynamic systems will be resumed in the chapters on dynamics and when the concept of *inertia forces* is introduced.

1.10 LAWS OF MECHANICS

Instead of stating the laws straight away, let us examine the contribution due to Newton first. The three laws of motion and the law of gravitation due to Newton, a literal translation from the original Latin 'Principia Mathematica Philosophia Naturalis' written in 1667 are collated:

Law 1 Every body perseveres in its state of rest, or of uniform motion in a right line unless it is compelled to change that state by forces impressed thereon.

Law 2 The alteration (acceleration) of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

Law 3 To every action, there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

The laws due to Newton reproduced as above are indeed philosophical and useful but are, by no means, the laws governing the motion of bodies in general. A comprehensive criticism may not be in order but one can appreciate some of the points to prove the assertion:

1. The word 'body' is undefined. It either refers to a particle only or to the centre of mass of a rigid body.
2. The term 'motion in a right line' appears in the first and second law but no attempt has been made to govern the rotational and general motion of the bodies of finite size.
3. Only the 'forces' have been considered; the action of a moment is not included.
4. The second law which relates acceleration to the forces impressed assumes the constancy of mass of the body.
5. If 'force' is recognised as a primitive concept, then the first law can be considered to be contained in the second law.
6. The action-reaction principle put forth by the third law can also be derived from the second law for rigid bodies.
7. The first and third laws are, therefore, not entirely independent of the second law; the message can be conveyed by the second law alone.

The second law applies to a body of mass m under the application of the motive force $\mathbf{F}$. Mathematically, the acceleration

$$\mathbf{a} \propto \mathbf{F}$$

or

$$\mathbf{F} = k\mathbf{a}$$

where k is a constant of proportionality. This constant, determined experimentally, equals the mass of the system.

Hence,

$$\boxed{\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt}} \tag{1.2a}$$

It is obvious that the force $\mathbf{F}$ and acceleration $\mathbf{a}$ must be collinear, vector $\mathbf{F}$ being m times vector $\mathbf{a}$. Further, the units of force are derived from the base units of mass and acceleration.

Mass	kg
Acceleration	m/s^2
Force	kg × m/s^2 = kg m/s^2 ≡ N or newton

Quantitatively, a force of 1 N causes an acceleration of 1 m/s^2 of a body of mass 1 kg.

The second law is not immediately applicable to the systems of variable mass. The law can, however, be reframed to cover the motion of constant-mass and variable-mass bodies by writing

$$\boxed{\mathbf{F} = \frac{d}{dt}(m\mathbf{V})} \tag{1.2b}$$

The bracketed term ($m\mathbf{V}$) is the *momentum* of the body of mass m moving at a velocity $\mathbf{V}$. The second law, in other words, states:

The rate of change of momentum of a body equals the force impressed upon it.

In view of the fact that the first and third laws are contained in this law, only this law will be retained and henceforth referred to as *Newton's law.*

In order to appreciate that the first and third laws of motion due to Newton are substantially contained in the second law, we proceed as follows:

From the second law,

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt}$$

If

$$\mathbf{F} = 0, \mathbf{a} = 0 = \frac{d\mathbf{V}}{dt}$$

whence $\mathbf{V}$ = zero or constant.

It follows that, in the absence of an external force, a body will continue to be in a state of rest or of uniform velocity. This is, in essence, the statement of the first law.

This reduction may also be seen graphically by plotting $\mathbf{F}$ *vs* $\mathbf{a}$ as shown in Fig. 1.10. The resulting straight line with slope m passes through the origin where,

for $\mathbf{F} = 0$, $\quad \mathbf{a} = 0$

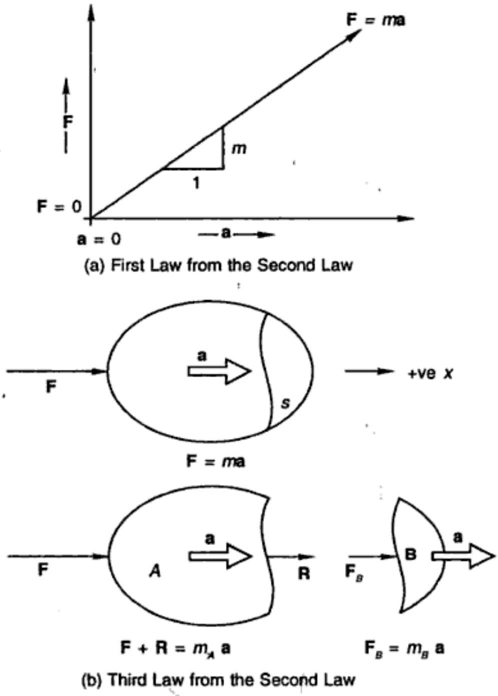


Fig. 1.10 ***Derivations from the Second Law***

Again, consider a rigid body moving at an acceleration **a** under the application of a force **F**. By the second law,

$$\mathbf{F} = m\mathbf{a} \tag{i}$$

Imagine the body to be constituted of two sub-bodies A and B such that the surface of contact is S as shown in Fig.1.10. The sub-bodies have their masses m_A and m_B for A and B respectively such that

$$m_A + m_B = m \tag{ii}$$

Since the whole body was moving at an acceleration **a**, every part of the body must have the same acceleration **a**. The total applied force **F** is, however, distributed over the parts of the body to bring about this state. In particular, let the force acting to accelerate the part B be $\mathbf{F}_B$ such that, by the second law,

$$\mathbf{F}_B = m_B\,\mathbf{a} \tag{iii}$$

In other words, the action of the body A on B is the force $\mathbf{F}_B$. Let the reaction of B on A be **R**. Consequently, the net external force acting on A is given by

$$\mathbf{F} + \mathbf{R} = m_A\,\mathbf{a} \tag{iv}$$

Adding (iii) and (iv),

$$\mathbf{F}_B + \mathbf{F} + \mathbf{R} = m_B\mathbf{a} + m_A\mathbf{a} = (m_A + m_B)\mathbf{a}$$

Employing (ii),

$$\mathbf{F}_B + \mathbf{F} + \mathbf{R} = m\mathbf{a}$$

and comparing with (i),

$$\mathbf{F} = m\mathbf{a}$$

it follows that

$$\mathbf{F}_B + \mathbf{F} + \mathbf{R} = \mathbf{F}$$

whence

$$\mathbf{F}_B + \mathbf{R} = 0$$

and

$$\mathbf{R} = -\mathbf{F}_B \qquad \text{(v)}$$

Relation (v) proves that the reaction force **R** by the body *B* is equal in magnitude and direction to the action $\mathbf{F}_B$ exerted on it but is opposite in sense. In other words, to an action $\mathbf{F}_B$, there is an equal and opposite reaction **R**. This is, in essence, the third law of motion due to Newton.

While claiming to prove that the second law contains the first and third laws one must not underestimate the conditions of validity:

1. The first law would have served to define the terms 'force', 'frame of reference' and 'state of rest'. These terms need to be defined axiomatically if the first law is regarded derivable from the second law.
2. The third law can be derived from the second law under the condition of transmissibility of force which is only valid for rigid bodies.
3. The third law as stated by Newton does not restrict the action and reaction principle to forces only. Since the first two laws relate to forces and their actions only and the concept of moment was not introduced by Newton, the third law as stated by Newton refers to the action and reaction of forces alone.
4. The action and reaction principle, in general, is valid for moments also. The concept of reaction may, therefore, be introduced axiomatically referring both to forces and moments as actions.

Having recognised that Newton's laws are, by themselves, inadequate to govern the general motion of finite-size bodies under the action of forces and moments and also that only the restated second law is carried over, it is but natural to decide and state a *complete set of laws* for the general motion of observable bodies. The complete set of laws should include the laws governing the behaviour of mass, momentum and energy. The basic assumptions in classical mechanics are that the mass must be conserved and the energy must be conserved separately. Rates of changes of linear momentum and angular momentum must be governed by the laws of motion.

In quite the same way as Newton's law governs the motion of a particle (or of the centre of mass of a body), Euler's law governs the motion of a rigid body. Euler's law states that

Table 1.2 Laws of Mechanics

Entity	*Law*	*Statement*	*Mathematical Formulation*
Mass	Law of conservation of mass	Mass can neither by created nor destroyed by any physical or chemical means.	$\frac{d}{dt}(m) = 0$
Linear momentum	Newton's law	The rate of change of momentum of a body equals the force impressed upon it.	$\mathbf{F} = \frac{d}{dt}(m\mathbf{V})$ $= \frac{d}{dt}(\mathbf{p}) = \dot{\mathbf{p}}$
Angular momentum	Euler's law	The rate of change of angular momentum of a body about an origin O equals the moment impressed upon it about the origin.	$\mathbf{M} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{V})$ $= \frac{d}{dt}\mathbf{H} = \dot{\mathbf{H}}$
Energy	Law of conservation of energy	The rate of change of internal energy and kinetic energy of any mechanical system equals the sum of the rates of work done by the external forces and the energy flux across the boundary as well as the energy developed within the system.	

$$\mathbf{M} = \frac{d}{dt}(\mathbf{H}) = \dot{\mathbf{H}}$$

where **H** is the angular momentum of the body about a point and **M** is the moment of the external forces acting on the body about that point.

It is interesting to note the similarity of the Newton's and Euler's laws:

$$\text{Newton's Law} \quad \mathbf{F} = \frac{d}{dt}(\mathbf{p}) = \dot{\mathbf{p}} \tag{1.3}$$

$$\text{Euler's Law} \quad \mathbf{M} = \frac{d}{dt}(\mathbf{H}) = \dot{\mathbf{H}} \tag{1.4}$$

The role of force in the rate of change of linear momentum is similar to the role of moment in the rate of change of angular momentum. In fact, both the laws relate to the rate of change of momentum; Newton's for the linear momentum and Euler's for the angular momentum. The force and the moment refer to the external action; the force for translational motion and the moment for rotational motion or tendencies thereof. A general statement to include both the laws may be made thus:

The rate of change of momentum of a body is proportional to the external action impressed upon it.

It should be clear that the word 'action' implies 'force' or 'moment' and the corresponding 'momentum' is linear or angular.

The law of conservation of energy may at first appear redundant to the problems in mechanics. This is not true because the law explains a number of dissipative phenomena on the one hand and degenerates to a simple form on the other hand for reversible phenomena. A pictorial representation of the laws of mechanics is given in Fig. 1.11.

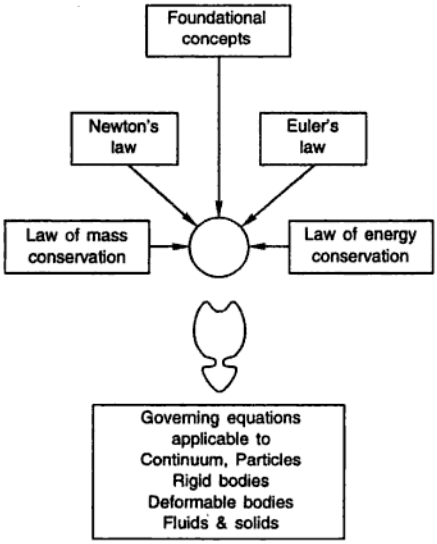


Fig. 1.11 ***Law of Mechanics***

1.11 LAW OF GRAVITATION—WEIGHT OF BODIES

In addition to the fundamental laws of mechanics, there are some more laws concerned with the origin and nature of forces. The law of gravitation due to Newton is perhaps the closest to the foundational laws and is discussed below.

Any two particles will be attracted towards each other along a line connecting their centres with a mutual force whose magnitude is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The law of gravitation requires that the force of attraction between two particles of masses m_1 and m_2 separated by a distance r as shown in Fig. 1.12 is given by

$$\boxed{F = G\frac{m_1 m_2}{r^2}} \tag{1.5}$$

where G is the universal constant of gravitation; its value being 6.67×10^{-11} N m^2/kg^2 or m^2/kg s^2. Quantitatively, an attractive force of 6.67×10^{-11} N is exerted by a body of mass 1 kg on another body of mass 1 kg at 1 m distance from it. Obviously, the

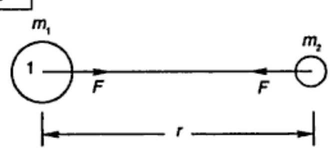


Fig. 1.12 ***Concept of Gravitation***

attractive force of reaction by the other body on it must also be equal to the same value.

The law of gravitation helps in defining the *weight* of a body. The weight of a body is the force exerted on it by the planet. For an earth-bound object of mass m, the weight is approximately given by

$$W = G\frac{M_e m}{r^2} \tag{1.6}$$

where M_e is the mass of the earth = 5.9761×10^{24} kg and r is the radial distance between the centres of the earth and the object.

It is customary to write

$$W = mg \tag{1.7}$$

where

$$g = \frac{GM_e}{R_e^2} = 9.806\,65 \text{ m/s}^2 \tag{1.8}$$

and R_e= mean radius of the earth ≈ 6371 km.

Since g is a constant for a planet and, when multiplied by the mass of a body, it provides the force on the body, it is termed as acceleration due to gravity. It is indeed the acceleration acquired by a body falling freely, i.e., without resistance, in the gravitational field of the planet.

Concept Review Questions

1. Comment on the scope of classical mechanics vis-a-vis other formulations in mechanics.
2. Comment on the need to idealise a body as a particle, a rigid body, a deformable body or a continuum.
3. What is the advantage of drawing a free-body diagram? Is it possible to draw a free-body diagram of a body or a system undergoing acceleration? Give examples.
4. State Newton's second law of motion and show that the first and third laws are contained in it.
5. What is meant by the state of equilibrium of a body? State the dynamical conditions of equilibrium and comment whether the conditions are both necessary and sufficient or not.

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions.

1. In all engineering problems a frame of reference at rest with respect to the earth is taken as an inertial frame. The assumption is valid because
   (a) the centrifugal force on the earth and the force of attraction between the earth and the sun balance each other
   (b) the acceleration and angular velocity of the earth is so small that the error caused is negligible

(c) the error due to the acceleration of the earth is taken care of by the experimental calculation of the value of g
(d) the earth does not have any acceleration

2. Zero work done by a system of forces acting on a body implies that
   (a) the resultant of the system of forces is zero
   (b) the cross product of the resultant of the system of forces and the vector in the direction of motion of the body is zero
   (c) the body does not have any motion
   (d) the motion of the body is in a direction perpendicular to the direction of the simplest resultant of the system of forces
3. An inertial frame of reference is one which necessarily has
   (a) fixed directions of its coordinate axes but the origin can move with constant speed
   (b) fixed directions of its axes but the origin can move with constant velocity
   (c) a fixed origin but directions can change with time
   (d) fixed origin and fixed directions of its axes
4. A free-body diagram of a body shows a body
   (a) isolated from all external effects
   (b) isolated from its surroundings and all external forces acting on it
   (c) isolated from its surroundings and all external actions acting on it
   (d) separately from its surroundings and all external and internal forces acting on it
5. The free-body diagram of a satellite rotating about the earth will show the satellite isolated from its surroundings and
   (a) no force acting on it
   (b) its velocity shown on it
   (c) the force of gravity and centrifugal force acting on it
   (d) only the force of gravity acting on it
6. An implication of Newton's law is that
   (a) the total momentum (linear + angular) of the body is conserved
   (b) the linear and angular momentum of the body are conserved separately
   (c) only the linear momentum of the body is conserved
   (d) a rigid body will tend to rotate if a force is applied at a point other than the centre of mass of the body
7. An implication of Euler's law is that
   (a) a rotating wheel will not change the orientation of its axis of rotation unless acted upon by an external torque
   (b) a rotating body will not change its angular velocity unless a couple is applied to it
   (c) a stationary body cannot be made to rotate by the application of a single force only
   (d) the total momentum of a body is conserved.
8. The momentum of a particle
   (a) does not depend on the frame of reference at all
   (b) does not depend on the frame of reference so long as it is an inertial frame of reference
   (c) is zero if no external force is acting on it
   (d) is conserved under all circumstances
9. If a body moving in a horizontal line with a certain velocity starts ejecting mass downwards at a constant rate, the horizontal velocity of the body will
   (a) remain unchanged

(b) start decreasing at a constant rate
(c) start increasing at a constant rate
(d) start increasing at an increasing rate

10. The force of gravitation between two bodies will be inversely proportional to the square of the distance between their centre of masses if the bodies
(a) are of constant densities
(b) are symmetrical about their centres of mass
(c) are of any arbitrary shape
(d) are of same shape, size and orientation

11. A man falling down from a height h starts rotating mid-way of his fall. The vertical velocity with which he will touch the ground will be
(a) $\sqrt{2gh}$
(b) less than $\sqrt{2gh}$
(c) more than $\sqrt{2gh}$
(d) less or greater but never equal to $\sqrt{2gh}$

Answers to Multiple-Choice Questions

1 (b),	2 (e),	3 (b),	4 (c),	5 (c),
6 (d),	7 (b),	8 (a),	9 (a),	10 (b),
11 (b)				

R
REVIEW SECTION

In this section, an attempt has been made to review the SI units and vector operations which are required throughout the study of mechanics:

- ❑ R1 **REVIEW OF SI UNITS**
- ❑ R2 **REVIEW OF VECTORS**

R1 Review of SI Units

A review of the Systéme International d' Unit'es, abbreviated as SI Units with special reference to mechanics is presented as follows:

R1.1 SI BASE UNITS

Base Units	*Unit Symbol*	*Dimensions*	*Physical Quantity*
kilogram	kg	M	mass
metre	m	L	length
second	s	T	time
kelvin	K	t	temperature
ampere	A	I	current
candela	cd		luminous intensity
mole	mol		amount of substance
Supplementary Units			
radian	rad	—	plane angle
steradian	sr	—	solid angle

R1.2 SI DERIVED UNITS WITH NEW NAMES

Derived Unit	*Unit Symbol*	*Physical Quantity*
newton	$N = kg\ m/s^2$	force
joule	$J = Nm = kg\ m^2/s^2$	energy, work, heat
watt	$W = J/s = N\ m/s = kg\ m^2/s^3$	power
pascal	$Pa = N/m^2 = kg/ms^2$	pressure, stress
hertz	$Hz = s^{-1}$	frequency

Notes:

A. Note that kilogram is written as kg and not as kg_m, kg_f, etc. Similarly second as s, not sec or sec., etc. No full stops, plurals, dots or dashes should be used. For example, torque is in Nm, not N.m, N-m, etc.

B. The unit of force is newton with symbol N and there is no such thing as kilogram force in SI units; just N and its multiple and submultiples. The unit of energy in any form is joule, J = Nm

   No horsepower or metric horsepower; just watt, W = J/s = N m/s and its multiples and submultiples.

C. Always leave a space between the number and the unit symbol, e.g., 23.2 cm and 2500 N.
D. For numbers less than unity, zero must be put on the left of the decimal and for larger numbers exceeding five figures, one space after every three digits counting from the right end must be left blank without any commas, e.g. 0.23 cm and 15 232 756 are the correct ways of writing these numbers.
E. It is permissible and perhaps advisable that one space be left between any two unit symbols and no space be left after a multiple or submultiple symbol, e.g., kg m^2/s, kJ/kg K.

R1.3 UNITS OF SOME COMMON PHYSICAL QUANTITIES

Physical Quantity	*Unit*	*Unit Symbol*
Acceleration	metre/second^2	m/s^2
Angular acceleration	radian/second^2	rad/s^2
Angular displacement	radian	rad
Angular momentum	kilogram metre^2/second	kg m^2/s
Angular velocity	radian/second	rad/s
Area	Square metre	m^2
Couple, moment	newton metre	N m
Density	kilogram/metre^3	kg/m^3
Discharge	metre^3/second	m^3/s
Displacement	metre	m
Energy	joule	J(= N m)
Force	newton	N(= kg m/s^2)
Frequency	per second	Hz(= /s)
Length	metre	m
Mass	kilogram	kg
Moduli of elasticity	newton/metre^2	Pa (= N/m^2)
Moment	newton metre	N m
Momentum	kilogram metre/second	kg m/s (= Ns)
Moment of inertia	kilogram metre^2	kg m^2
Plane angle	radian	rad
Power	watt	W(= N m/s)
Pressure, Stress	newton/metre^2, Pascal	Pa (= N/m^2)
Specific energy	joule/kilogram	J/kg
Specific volume	kilogram/metre^3	Kg/m^3
Speed	metre/second	m/s
Time	second	s
Torque	newton metre	N m
Velocity	metre/second	m/s
Velocity potential	metre^2/second	m^2/s
Viscosity (dynamic)	newton second/metre^2	N s/m^2
	or	(= Pa s)
	kilogram/metre second	kg/m s
Volume	metre^3	m^3
Weight	newton	N (= kg m/s^2)
Work	joule	J (= N m)

R1.4 MULTIPLES AND SUBMULTIPLES

tera	T	10^{12}	milli	m	10^{-3}
giga	G	10^{9}	micro	μ	10^{-6}
mega	M	10^{6}	nano	n	10^{-9}
kilo	k	10^{3}	pico	p	10^{-12}
deci	d	10^{-1}	femto	f	10^{-15}
centi	c	10^{-2}	atto	a	10^{-18}

R1.5 SOME CONVERSION FACTORS

To convert the following	*Into*	*Multiply by*	*Conversely multiply by*
inches	cm	2.5400	0.3937
feet	m	0.3048	3.2808
miles	km	1.6093	0.6214
gallons	m^3	4.546×10^{-3}	220
pints	m^3	0.5683×10^{-3}	1.76×10^{3}
gallons (US)	m^3	3.785×10^{-3}	264.2
degrees	rad	0.017 45	57.2957
pounds (lb)	kg	0.4536	2.2046
tons	kg	1016.0	9.842×10^{-4}
tonne	kg	1000.0	10^{-3}
knots	m/s	0.5144	1.943
r.p.m.	rad/s	0.1047	9.550
pound/feet^3	kg/m^3	16.02	0.0624
cusecs	m^3/s	0.0283	35.33
g.p.m.	m^3/s	0.0758×10^{-9}	13.20×10^{9}
lb_f	N	4.448	0.2248
kg_f	N	9.807	0.1019
ton_f	kN	9.964	0.1003
kgf/cm^2	kPa	98.07	0.0102
p.s.i.	kPa	6.895	1.1450
inches (water gauge)	kPa	0.2491	4.015
inches (Mercury)	kPa	3.386	0.2953
torr	kPa	1.333	0.7502
foot pounds	J	1.356	0.7375
h.p.	kW	0.7457	1.341
poise	N s/m^2	0.1	10
stokes	m^2/s	10^{-4}	10^{4}

R1.6 VALUES OF SOME USEFUL CONSTANTS

Constant Quantity	*Symbol*	*Value*	*SI Units*
Speed of light in vacuum	c	$2.997\,925 \times 10^{8}$	m/s
Planck's constant	h	6.6253×10^{-34}	J s

(Contd.)

Constant Quantity	*Symbol*	*Value*	*SI Units*
Gravitational constant	G	6.670×10^{-11}	N m^2/kg^2
Universal gas constant	R_0	8314.4	J/K mol
Zero Celsius (centigrade)	$0^{\circ}C$	273.15	K
Triple point of water	tr	273.16	K
Characteristic gas constant for air	Ra	287	J/kg K
Mean molecular weight of air	Ma	28.966	
Mean ICAO* air density	ρ_a	1.225	kg/m^3
Mean ICAO* air viscosity	μ_a	18×10^{-6}	Ns/m^2
Mean density of dry air (S.T.P.)	ρ	1.205	kg/m^3
Standard atmosphere (pressure)	atm	101.325	kN/m^2
Standard atmosphere (temperature)	T_s	288.15	K
Lapse rate for standard atmosphere	L	6.5	K/km
Mass of atmosphere		5.27×10^{18}	kg
Voltage gradient, fine weather; average		100	V/m
Solar constant for earth	S_c	1400	J/m^2 s
Sonic speed in air at STP	a	340.3	m/s
Gravitational parameter	GM	3.986×10^{14}	m^3/s^2
Mass of the earth	M	5.976×10^{24}	kg
Standard gravitational acceleration	g_c	9.806 65	m/s^2
Mean radius of the earth	r_e	6371	km
Mean density of the earth	ρ_e	55.17	kg/m^3
Escape velocity at the surface	V_e	11.2	km/s
Rotational velocity at the equator		465	km/s
Mean velocity in orbit		29.78	km/s
Approximate age of the earth		4.5×10^9	years
Area of land surface		148.9×10^{12}	m^2
Area of water surface		362.2×10^{12}	m^2
Height of Mount Everest		8847.7	m
Depth of Marianas Trench		11.033	km

Acceleration $g = 9.80616 - 0.025928 \cos 2\lambda + 0.000069 \cos^2\lambda - 0.000003h$ m/s^2 at a place with latitude λ and at height h metres above the sea level for the earth.

*International Civil Aviation Organisation.

R1.7 PROPERTIES OF WATER, MERCURY AND AIR

Fluid Properties	*Water*	*Mercury*	*Air*
Density kg/m^3 (at 20 °C)	1000	13546	1.20
Viscosity, N s/m^2	1×10^{-3}	1.55×10^{-3}	18×10^{-6}
Surface tension, N/m	0.073	0.472	—
Melting point, K	273	234	—
Boiling point, K	373	630	83
Sonic speed m/s (at 1 bar)	1410	1370	340

Test Your SI Power

1. Write down the seven base units and the two supplementary units in SI. Show, by way of expressing the following physical quantities in terms of these units, that this is a complete set of the base and supplementary units:
   (a) force
   (b) energy
   (c) pressure
   (d) charge
   (e) illuminance

   (**Ans.** kg, m, s, K, A, cd, mol; rad, sr: kg m/s^2, kg m^2/s^2, kg/ms^2, As, cd sr/m^2)
2. Recognise the following units and express them in equivalent symbolic forms: newton, joule, watt, pascal, poise, tesla.

   (**Ans.** N ≡ kg m/s^2, J ≡ N m ≡ kg m^2/s^2
   W ≡ J/s ≡ N m/s ≡ kg m^2/s^3, Pa ≡ N/m^2 ≡ kg/m s^2
   P ≡ g/Ms ≡ 0.1 kg/m s ≡ 0.1 N s/m^2 ≡ 0.1 Pa s
   T ≡ Wb/m^2 ≡ V s/m^2 ≡ Ws A m^2.)
3. Pick up the correct SI abbreviations:
   N m/S, N-m/s, N/m^2.s, Ns/m^2.s, N s/m^2, Pa-s, WS/A^2, N.m/S, kg_f/m^2, kg_fm, kg/s^2.m, kg m/s^2, Cd sr., N m/kg_f, N m/N, C,°K.

   (**Ans.** N s/m^2, kg m/s^2, N m/N)
4. Convert the following quantities into coherent SI units:
   1 kg_f/cm^2, 20 kN/cm^2, 2 $grams/cm^3$, 5 foot pounds, 20 Chu, 10 metric horse power, 20 knots, 3000 r.p.m., 2 quintals, 0.05 cumecs, 2 centipoise, 10 centistokes, 20°C, 1 kcal/kg °C, 10 $lumens/foot^2$, 2 light years.

   (**Ans.** 98.07×10^3 N/m^2; 200×10^6 N/m^2, 2000 kg/m^3, 6.78 J, 38×10^3 J.
   7.355×10^3 W, 10.288 m/s, 314.1 rad/s, 200 kg, 0.05 m^3/s,
   0.002 N s/m^2, 10×10^{-6} m^2/s, 293.15 K, 4.187×10^3 J/kg K,
   107.64 lx, 18.921×10^{15} m)
5. State the value of the universal gas constant in SI units and hence express the characteristic gas constants for the following gases:

Gas	Molecular Weight
(a) air	28.966
(b) carbon dioxide	44.01
(c) oxygen	32.00
(d) hydrogen	2.016

   (**Ans.** 8314.4 J/k mol; 287, 188.9, 259.8, 4124.2 J/kg K)
6. Write down the accurate mean value of *g*, the acceleration due to gravity on the earth. What is the approximation usually made by engineers? State the circumstances under which a quantity should be multiplied by *g* in SI units?

   (**Ans.** 9.806 65 m/s^2, 9.81 m/s^2; only to calculate the weight of a given mass, e.g., 1 kg mass weighs 1×9.81 kg m/s^2 or 9.81 N)

R2 Review of Vectors

R2.1 NUMBERS, SCALARS AND VECTORS

The magnitude of a physical variable in terms of a pre-determined unit of measurement is expressed in *numbers*.

The quantities which are specified completely by the magnitude and units are called *scalars* or *scalar quantities.*

Vector quantities are those which are specified completely by the *magnitude with units, direction and sense*. Vectors must, in addition, obey the laws of vector operations and, in particular, the parallelogram law of addition.

Examples

Numbers	1, 2, 3.14159, 9.80665, 10, 20
Scalars	2 kg mass, 3.14159 m length, 1 s time, 10 m/s speed
Vectors	20 N force vertically downwards, 2 m/s velocity along the forward tangent to the path,
	9.80665 m/s^2 acceleration directed towards the centre of the earth,
	10 N m torque about the positive z-axis.

A vector is represented by a bold-faced letter such as **A** and **B** in print and by overbars in handwriting such as $\overline{A}$ and $\overline{B}$.

Geometrically, a vector is represented by a bold line segment with an arrow at one end such that (a) the length of the line represents the magnitude A with units, the orientation of the line shows the direction of the vector and the arrow mark specifies the sense of the vector, i.e., to or from a point.

A vector **A** is geometrically represented as in Fig. R2.1(a). In a right-handed system or dextral system of coordinates, a vector represented by an arrowed-line segment in a certain direction may also imply its rotational character governed by the right-handed screw-rule. For example, a vector **A** representing angular velocity, angular acceleration or moment would imply the sense by the right-handed screw rule as shown in Fig. R2.1(b).

Vectors are categorised as sliding, free, or bound as follows:

A *sliding vector* or *transmissible vector* may be applied anywhere along its line of action; the line segment can be taken anywhere on the line of action so that the magnitude, direction and sense as well as the line of action remain the same. A force acting on a rigid body and producing acceleration is a transmissible vector. The principle of transmissibility of force is taken up further in Art. 2.6. A *free* *vector* may be moved anywhere in space provided its magnitude, direction and sense remain the same. A *bound vector* must be specified with a point of application; a bound vector has the magnitude, direction and sense as well as the point of application specified. A representation of these concepts is made in Figs. R2.1(c) and (d).

A vector is said to be a *unit vector* if its magnitude equals unity. A unit vector may, therefore, be chosen in any direction and with any sense. In particular, the unit vector along a vector **A** or in the direction of the vector **A** must be

$$\mathbf{e} = \frac{\mathbf{A}}{A} = \frac{1}{A}\mathbf{A} \qquad \text{(R2.1)}$$

which is in the same direction as **A** but with a magnitude of unity as shown in Fig. R2.1 (e).

The unit vectors along the coordinate axes are given a special status.
In the rectangular coordinates,

$\mathbf{i} \equiv$ unit vector along the x-axis

$\mathbf{j} \equiv$ unit vector along the y-axis

$\mathbf{k} \equiv$ unit vector along the z-axis

In the cylindrical coordinates,

$\mathbf{e}_r \equiv$ unit vector radially outwards in the x-y plane

$\mathbf{e}_\theta \equiv$ unit vector in cricumferential direction in the x-y plane.

$\mathbf{e}_z \equiv$ Unit vector along the z-axis

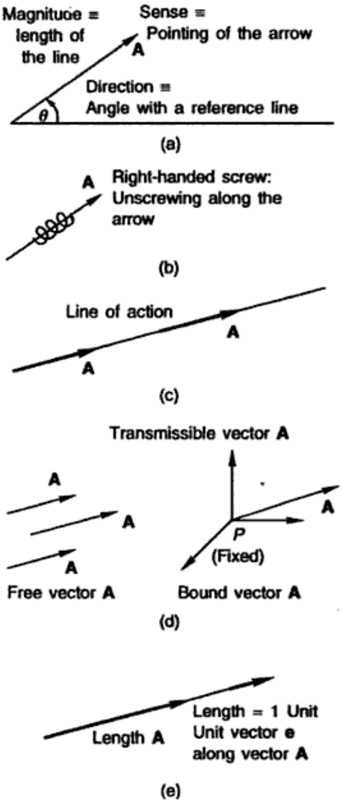


Fig. R2.1 ***Vector Representation***

In the spherical coordinates,

$\mathbf{e}_R \equiv$ unit vector radially outwards in space

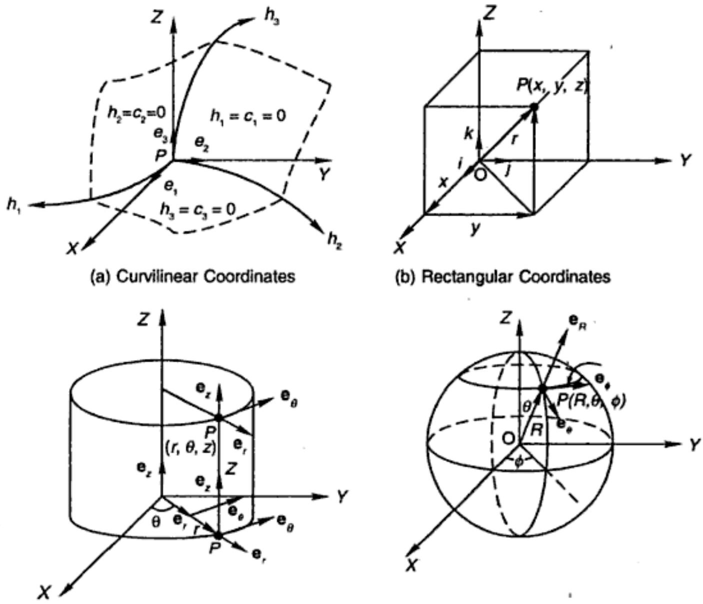
$\mathbf{e}_\theta \equiv$ unit vector in circumferential direction referred to the z-axis

$\mathbf{e}_\phi \equiv$ unit vector in circumferential direction in the x-y plane.

The unit vectors in different coordinate systems are illustrated in Fig. R2.2.

A *null* or *zero vector* is defined as a vector whose magnitude is zero. The role of a zero vector in vector operations is equivalent to the role of zero value in scalar operations. Interestingly, a zero vector may be thought of as parallel to any direction for convenience since a zero vector must be parallel to all directions simultaneously.

Two vectors are said to be *equal vectors* if their magnitudes, directions and sense are the same. Two vectors are said to be *equivalent* or *equipollent vectors* if,

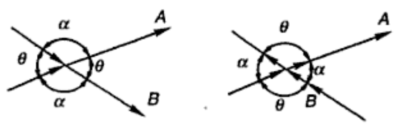


(a) Curvilinear Coordinates (b) Rectangular Coordinates

(c) Cylindrical Coordinates (d) Spherical Coordinates

Fig. R2.2 ***Coordinate Systems***

in a certain sense, they produce the same effect. It may be mentioned that the equality of vectors does not necessarily mean their equivalence of effect. A vector is said to be *negative* of another vector, if they have the same magnitude and direction but are opposite in sense.



θ, Correct Angle and α, Incorrect Angle

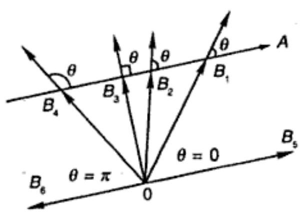


Fig. R2.3 ***Angle between Two Vectors***

It is necessary to understand the concept of the angle between two vectors. The included angle is defined as the angle, restricted to the interval $0 \leq \theta \leq \pi$, formed between two vectors when both are taken from or towards a common initial point. In other words, it is the smaller of the two angles formed between the directions of the vectors when their arrows point out or towards a common point. Clearly, the concept of the angle between two vectors is restricted to a pair of coplanar vectors. Vectors may be mutually orthogonal or parallel if the angle between them is $\pi/2$ or 0 respectively. Two vectors are said to be skew if a common plane cannot be passed through them.

The correct angle between two vectors **A** and **B** has been shown as in different situations in Fig. R2.2. In particular, $\mathbf{B}_3$ is perpendicular to **A**, $\mathbf{B}_5$ is parallel to **A** and $\mathbf{B}_6$ is antiparallel or parallel and opposed to **A**.

R2.2 ADDITION OF VECTORS

The most fundamental law of vector algebra is the parallelogram law of vector addition; so much so that the quantities possessing direction, magnitude and sense may be denied the vectorial status, if they do not obey the parallelogram law. Conformity with the law may as well be incorporated in the definition of the vector quantities.

The parallelogram law of vector addition states that if two vectors comprise the adjacent sides of a parallelogram, pointing towards or away from the point of intersection, then the diagonal of the parallelogram passing through the same point and with the same sense represents the sum of the two vectors. The addition of vectors **A** and **B** requires that

1. **A** and **B** be placed together to point towards or away from a point *O*
2. A parallelogram be made with **A** and **B** as adjacent sides
3. The diagonal of the parallelogram passing through *O* with the arrow pointing towards or away from *O* as the case may be, represents **C** the *sum of two vectors* **A** and **B** as shown in Fig. R2.4(a).

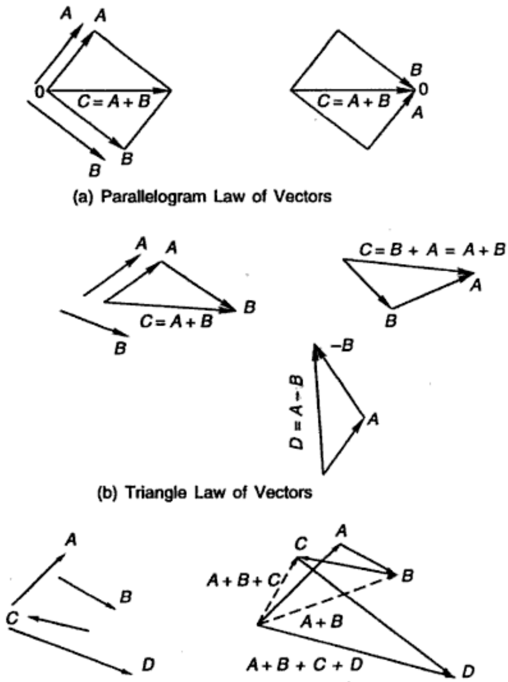
A corollary of the parallelogram law is the law of triangle of vectors illustrated in Fig. R2.4(b). The additive vectors **A** and **B** are placed one after the other in the same sense to constitute two sides of a triangle, the third side of which, drawn from the initial point of **A** to the final point of **B**, represents the sum of the vectors **A** and **B**. It can be seen that **A** added to **B** or **B** added to **A** results in the same vector, i.e.,

$$\boxed{\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}}$$

The difference of two vectors can be obtained by adding the additive vector to the negative of the subtractive vector, i.e.,

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

If it is desired to add more than two vectors, then the parallelogram law can be used to continue adding two at a time or the triangle law can be extended to comprise the polygon law as demonstrated in Fig. R2.4(c).



(c) Polygon Law of Vectors

Fig. R2.4 ***Addition of Vectors***

Two intersecting vectors must lie in a plane. The addition of a number of vectors may, therefore, imply that a number of parallelograms in the planes of the pairs of the vectors must be drawn. Instead, the addition may be performed by a space polygon of the vectors. In engineering it is so often desired to obtain the sum of plane or spatial vectors that the geometrical methods of parallelograms of polygons prove to be inconvenient.

The addition of a vector to itself results in a vector twice its magnitude but the same in direction and sense. In general, a vector is *n* times another vector if its magnitude is *n* times that of the other and the direction and sense are the same. The laws of vector addition and rules of multiplication by scalars are given below.

Addition

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(Commutative law of addition)}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \quad \text{(Associative law of addition)} \qquad \text{(R2.2)}$$

Multiplication by Scalars

$$n\mathbf{A} = \mathbf{A}n \quad \text{(Commutative)}$$

$$m(n\mathbf{A}) = (mn)\mathbf{A} \quad \text{(Associative)}$$

$$(n + m)\mathbf{A} = n\mathbf{A} + m\mathbf{A} \quad \text{(Distributive)} \quad \text{(R2.3)}$$

$$n(\mathbf{A} + \mathbf{B}) = n\mathbf{A} + n\mathbf{B} \quad \text{(Distributive)}$$

where n and m are scalars.

R2.3 RESOLUTION OF VECTORS

The resolution of a vector into its constituent vectors is defined as the reverse action of addition of the component vectors to result in the given vector. Thus if $\mathbf{A} + \mathbf{B} = \mathbf{C}$, i.e., if **A** and **B** can add to give **C**, then **C** can be resolved to give **A** and **B**, i.e., $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

In general, a vector can be resolved into an infinite pair of constituent vectors but the resolution of a vector in any two stipulated directions coplanar with the given vectors is unique. Just as a number of vectors can be added to comprise a resultant, a given vector can be resolved into a number of constituent vectors.

In particular, it is important to understand the resolution of a vector into three mutually orthogonal component vectors. A vector **A** is resolved into three components corresponding to its projections along the three orthogonal coordinate axes, i.e., $\mathbf{A}_x$, $\mathbf{A}_y$ and $\mathbf{A}_z$ along the x, y and z axes respectively as shown in Fig. R2.5. It can be seen that $\mathbf{A}_x$ and $\mathbf{A}_y$ add up to constitute $\mathbf{O}_Q$ which when added to $\mathbf{A}_z$ results in vector **A**. Conversely, the vector **A** is considered resolved into $\mathbf{A}_z$ and $\mathbf{O}_Q$ and $\mathbf{O}_Q$ further resolved into $\mathbf{A}_x$ and $\mathbf{A}_y$ giving rise to $\mathbf{A}_x$, $\mathbf{A}_y$ and $\mathbf{A}_z$ as the three orthogonal components.

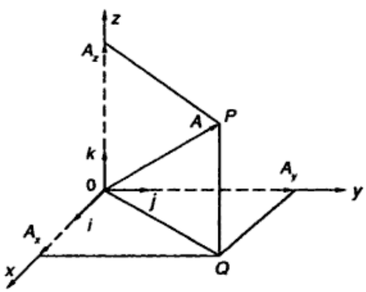


Fig. R2.5 ***Components of a Vector***

In terms of the unit vectors **i**, **j** and **k** in the Cartesian system of coordinates,

$$\mathbf{A}_x = A_x\mathbf{i}$$

$$\mathbf{A}_y = A_y\mathbf{j}$$

$$\mathbf{A}_z = A_z\mathbf{k}$$

We may, therefore, state that the scalar components of a vector **A** are A_x, A_y and A_z along the x,y and z directions respectively. The scalar components are generally referred to as the components of the vector. It also follows by the Pythagoras theorem that

$$\boxed{A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}} \quad \text{(R2.4)}$$

It may be noted here that the process of arriving at the components of a vector by our analysis has been long and requires the use of the parallelogram law of vectors. In fact, it was stated that the parallelogram law is fundamental to the existence of the vectors and that it could be included in the definition of a vector quantity. This is indeed the modern approach where the parallelogram law is not talked of and the vector components are defined straightaway. By definition, then, a vector is a quantity possessing n components, i.e.,

$$\mathbf{A} = \mathbf{A}(r_1, r_2, \ldots, r_i, \ldots, r_n) \tag{R2.5}$$

such that the components commute, associate, etc., according to a set of rules. The components can then be sepecialised for the orthogonal systems as arrived at here.

A vector **A** can be expressed in terms of its scalar components as

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

Similarly

$$\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$$

$$\mathbf{C} = C_x\,\mathbf{i} + C_y\,\mathbf{j} + C_z\,\mathbf{k}$$

and

$$\mathbf{D} = D_x\,\mathbf{i} + D_y\,\mathbf{j} + D_z\,\mathbf{k}$$

(i) If

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

then

$$\mathbf{C} = C_x\,\mathbf{i} + C_y\,\mathbf{j} + C_z\,\mathbf{k}$$
$$= (A_x + B_x)\,\mathbf{i} + (A_y + B_y)\,\mathbf{j} + (A_z + B_z)\,\mathbf{k}$$

because the scalar components can be added numerically.

Hence,

$$C_x = A_x + B_x$$
$$C_y = A_y + B_y$$
$$C_z = A_z + B_z$$

(ii) If

$$\mathbf{D} = \mathbf{A} - \mathbf{B}$$

then

$$\mathbf{D} = D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k}$$
$$= (A_x - B_x)\,\mathbf{i} + (A_y - B_y)\,\mathbf{j} + (A_z - B_z)\,\mathbf{k}$$

and

$$D_x = A_x - B_x$$
$$D_y = A_y - B_y$$
$$D_z = A_z - B_z$$

In general, if $\mathbf{R} = \mathbf{A} + \mathbf{B} - \ldots$

$$R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_z\,\mathbf{k} = (A_x + B_x - \ldots)\,\mathbf{i} + (A_y + B_y - \ldots)\,\mathbf{j} + (A_z + B_z - \ldots)\,\mathbf{k} \tag{R2.6}$$

***Example* R2.1** A vector of magnitude 10 units is directed 30 degrees north of east. Represent it graphically and analytically and determine its components due east and north.

Solution Vector **A** shown in Fig. Ex. R2.1 with its length OP of 10 units to a chosen scale, direction 30° with the east, the x-axis and arrow to indicate its sense O to P, is the required vector. Its components due east and north, found by measuring its projections on the x and y axes respectively to the same scale, are 8.66 units and 5 units as shown. Alternatively, its component due east is

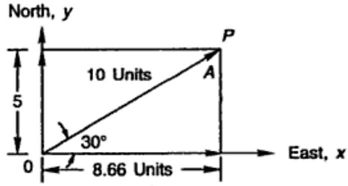


Fig. Ex. R2.1

$$OP \cos 30° = A \cos 30° = 10 \times \cos 30° = 8.66 \text{ units}$$

and its component due north is

$$OP \sin 30° = A \sin 30° = 10 \times \sin 30° = 5.00 \text{ units}$$

The vector may therefore be expressed analytically as

$$\mathbf{A} = 8.66\,\mathbf{i} + 5\,\mathbf{j}$$

where **i** and **j** are the unit vectors along x and y axes respectively. The analytical representation implies that the magnitude of the vector is

$$A = \sqrt{8.66^2 + 5^2} = 10 \text{ units}$$

and that the vector makes an angle θ with the x-axis.

$$= \sin^{-1}\left(\frac{5.00}{10}\right) = \cos^{-1}\left(\frac{8.66}{10}\right) = \tan^{-1}\left(\frac{5}{8.66}\right)$$

$$= 30° = \pi/6 \text{ rad}$$

***Example* R2.2** A vector of magnitude 100 units makes an angle of 30° with the z-axis and its projection on the x-y plane makes an angle of 45° with the x-axis. Determine (a) the components of the vector and (b) the angles of the vector with the axes.

Solution The vector **A** represented in Fig. Ex. R2.2 has the components represented by OX along the x-axis, OY along the y-axis and OZ along the z-axis.

The projection of **A** on the x-y plane is OQ which is composed of OX and OY component.

$$OZ = A \cos 30° = 100 \times 0.866 = 86.6 \text{ units}$$

$$OQ = A \sin 30° = 100 \times 0.500 = 50.0 \text{ units}$$

whence, by further resolution,

$$OX = OQ \cos 45° = 50 \times 0.707 = 35.35 \text{ units}$$

$$OY = OQ \sin 45° = 50 \times 0.707 = 35.35 \text{ units}$$

The components of the vector along the x, y and z axes are 35.35, 35.35 and 86.6 units respectively.

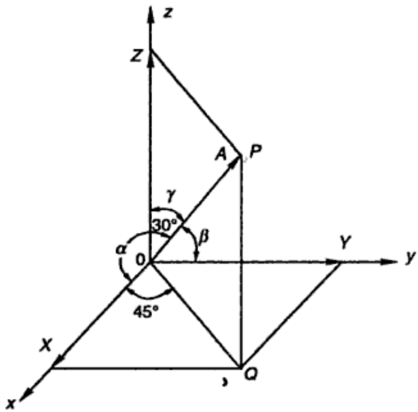


Fig. Ex. R2.2

$$\mathbf{A} = 35.35\,\mathbf{i} + 35.35\,\mathbf{j} + 86.6\,\mathbf{k}$$

It may be checked that

$$\sqrt{(35.35)^2 + (35.35)^2 + (86.6)^2} = 100 \text{ units as expected}$$

The direction cosines are determined as follows:

$$l = \cos\alpha = \frac{OX}{OP} = \frac{35.35}{100} = 0.3535$$

$$m = \cos\beta = \frac{OY}{OP} = \frac{35.35}{100} = 0.3535$$

$$n = \cos\gamma = \frac{OX}{OP} = \frac{86.6}{100} = 0.866$$

It follows that the angles in the respective axes are

$$\alpha = 69.3°, \qquad \beta = 69.3°, \qquad \gamma = 30°$$

It may be checked that

$$\sqrt{0.3535^2 + 0.3535^2 + 0.866^2} = 1$$

in accordance with the relationship

$$l^2 + m^2 + n^2 = 1$$

***Example* R2.3** The coordinates of the initial and terminal points of a vector are (3, 1, –2) and (4, –7, 10) respectively. Determine the components of the vector and its angles with the axes. Specify the vector.

Solution

The components of the vector are:

$$4 - 3 = 1 \qquad \text{along the } x\text{-axis}$$

$$-7 - 1 = -8 \qquad \text{along the } y\text{-axis}$$

$$10 - (-2) = 12 \qquad \text{along the } z\text{-axis}$$

The magnitude of the vector is, therefore,

$$A = \sqrt{1^2 + (-8)^2 + 12^2} = 14.46$$

and its direction cosines are

$$l = \cos\alpha = \frac{1}{14.46} = 0.069$$

$$m = \cos\beta = \frac{-8}{14.46} = -0.553$$

$$n = \cos\gamma = \frac{12}{14.46} = 0.830$$

whence, $\alpha = 86.04°$, $\beta = 123.57°$, $\gamma = 33.9°$

The vector is specified as

$$\mathbf{A} = 1\,\mathbf{i} - 8\,\mathbf{j} + 12\,\mathbf{k}$$

or, alternatively stated as a vector of magnitude 14.46 units making angles of 86.04°, –56.43° and 33.9° with the x, y and z axes respectively with its sense from the initial to the terminal point.

Example **R2.4** Two vectors **A** and **B** are added and subtracted to comprise vectors **C** and **D**. Determine these vectors and the unit vector along them.

$$\mathbf{A} = 2\,\mathbf{i} + 3\,\mathbf{j}$$

$$\mathbf{B} = 3\,\mathbf{i} - 2\,\mathbf{j}$$

Evaluate also the magnitude of vector **E** = 2**C** + 0.75**D**.

Solution

For **A**, $A_x = 2$ $\quad A_y = 3$

For **B**, $B_x = 3$ $\quad B_y = -2$

For **C**, $C_x = A_x + B_x = 5$ $\quad C_y = A_y + B_y = 1$

For **D**, $D_x = A_x - B_x = -1$ $\quad D_y = A_y - B_y = 5$

Hence,

$$\mathbf{C} = 5\,\mathbf{i} + \mathbf{j}$$

$$\mathbf{D} = -\mathbf{i} + 5\,\mathbf{j}$$

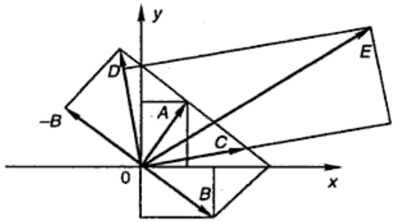


Fig. Ex. R2.4

The magnitude of **C** is given by

$$C = \sqrt{5^2 + 1^2} = 5.1 \text{ units}$$

Hence the unit vector along **C** must be given by

$$\frac{\mathbf{C}}{C} = \frac{5}{5.1}\mathbf{i} + \frac{1}{5.1}\mathbf{j} = 0.981\mathbf{i} + 0.196\,\mathbf{j}$$

Clearly, the magnitude of the unit vector can be checked to be

$$\sqrt{0.981^2 + 0.196^2} = 1 \text{ as expected}$$

Similarly, the unit vector along **D** is given by

$$\frac{-1}{\sqrt{1^2 + 5^2}}\mathbf{i} + \frac{5}{\sqrt{1^2 + 5^2}}\mathbf{j} = -0.196\,\mathbf{i} + 0.981\,\mathbf{j}$$

which is also unity in magnitude.

Graphically, **A** and **B** are added to yield **C** by the parallelogram method of addition. **B** is subtracted from **A** if –**B** is added to **A** to comprise **D** by the same procedure.

$$\begin{aligned}\mathbf{E} &= 2\,\mathbf{C} + 0.75\,\mathbf{D}\\ &= 2(5\,\mathbf{i} + 1\,\mathbf{j}) + 0.75(-1\,\mathbf{i} + 5\,\mathbf{j})\\ &= 9.25\,\mathbf{i} + 5.75\,\mathbf{j}\end{aligned}$$

Magnitude of $E = \sqrt{9.25^2 + 5.75^2} = 10.89$ units

The unit vector along **E** is given by

$$\frac{9.25}{10.89}\mathbf{i} + \frac{5.75}{10.89}\mathbf{j} = 0.849\,\mathbf{i} + 0.528\,\mathbf{j}$$

***Example* R2.5** Find the resultant of four given vectors

$$\mathbf{A} = 3\,\mathbf{i} + 2\,\mathbf{j}$$

$$\mathbf{B} = 2\,\mathbf{i} + 3\,\mathbf{j}$$

$$\mathbf{C} = 3\,\mathbf{i} - 4\,\mathbf{j}$$

$$\mathbf{D} = -5\,\mathbf{i} - 2\,\mathbf{j}$$

Solution Analytically, the components of the resultant vector **R** are given by

$$R_x = A_x + B_x + C_x + D_x$$

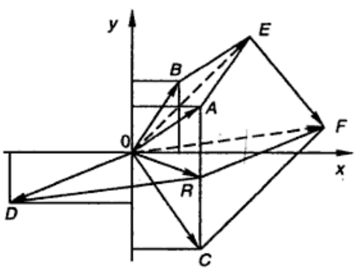
$$= 3 + 2 + 3 - 5 = 3$$

$$R_y = A_y + B_y + C_y + D_y$$

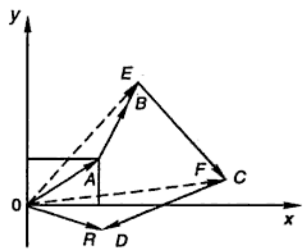
$$= 2 + 3 - 4 - 2 = -1$$

Hence, $$\mathbf{R} = 3\,\mathbf{i} - \mathbf{j}$$

Graphically, **A** and **B** are added to comprise **E** which when added to **C** provides **F** and that added to **D** results in the final vector **R** as shown in Fig. Ex. R2.5(a). Alternatively, the polygon method of vector addition as shown in Fig. Ex. R2.5(b) requires vector **B** to be placed at the tip of vector **A**, **C** at the tip of **B** and **D** at the tip of **C**. The closing line of the polygon directed from the starting point O is the resultant **R**.



(a) By Parallelograms



(b) By Polygon Method

Fig. Ex. R2.5

Example **R2.6** Three vectors **A**, **B** and **C** are given as

$$\mathbf{A} = 2\,\mathbf{i} + 3\,\mathbf{j} - 4\,\mathbf{k}$$

$$\mathbf{B} = 3\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}$$

$$\mathbf{C} = 2\,\mathbf{i} - 3\,\mathbf{k}$$

Determine (a) the resultant vector **R**, (b) the vector **E** to make the sum of **A**, **B**, **C** and **E** zero and (c) the vector **D** = (**A** + **B** – 2**C**).

Solution

(a) Writing **i**, **j** and **k** components columnwise,

$$\mathbf{A} = 2\,\mathbf{i} + 3\,\mathbf{j} - 4\,\mathbf{k}$$

$$\mathbf{B} = 3\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}$$

$$\mathbf{C} = 2\,\mathbf{i} + 0\,\mathbf{j} - 3\,\mathbf{k}$$

$\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} = 7\,\mathbf{i} - 1\,\mathbf{j} - 2\,\mathbf{k}$

(b) Since $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{R}$
E must be equal to –**R** so that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{E} = 0$$

Hence, $\mathbf{E} = -\mathbf{R} = -7\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$

(c) Writing **i**, **j** and **k** components of **A**, **B** and –2**C** columwise,

$$\begin{aligned} \mathbf{A} &= 2\,\mathbf{i} + 3\,\mathbf{j} - 4\,\mathbf{k} \\ \mathbf{B} &= 3\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k} \\ -2\mathbf{C} &= -4\,\mathbf{i} + 0\,\mathbf{j} + 6\,\mathbf{k} \\ \hline \mathbf{A} + \mathbf{B} - 2\mathbf{C} &= 1\,\mathbf{i} - 1\,\mathbf{j} + 7\,\mathbf{k} \end{aligned}$$

R2.4 CONCEPT OF PRODUCTS

Numbers, scalars and vectors can be multiplied to yield meaningful results:

(a) *Product of a scalar by a number* means the magnitude multiplication of the scalar.

Example 10 m length multiplied by 4.2 means 42 m length

(b) *Product of a scalar by a scalar* means the multiplication of the magnitudes as well as that of the units to result in the units of the resulting physical quantity.

Example 10 m/s speed for 5 s time results in a distance of 50 m.

(c) *Product of a vector by a number* means that the magnitude of the vector is multiplied by the number, the direction and sense of the vector remaining the same.
Example 5 times the acceleration of 9.81 m/s^2 means an acceleration of 49.05 m/s^2 in the same direction and sense.

(d) *Product of a vector by a scalar* implies the multiplication of their magnitudes as well as that of the units resulting in a new vector quantity with its magnitude as the product of their magnitudes and the direction and sense the same as before.
Example A velocity of 10 m/s for 5 s time produces a displacement of 50 m directed forward along the velocity vector.

(e) *Product of a vector by a vector* should also provide meaningful results. There are two types of vector products of interest to us, i.e., *scalar or dot product* and *vector or cross product.*

(el) *Scalar or dot product* of two coplanar vectors **A** and **B** denoted by $\mathbf{A} \cdot \mathbf{B}$ and read as '**A** dot **B**' implies a scalar quantity equal to (a) the magnitude of **A** times the magnitude of the projection of **B** on **A** or (b) the magnitude of **B** times the magnitude of the projection of **A** on **B** or (c) the product of the magnitudes of **A** and **B** and the cosine of the smaller angle between them.

$$\boxed{\mathbf{A} \cdot \mathbf{B} = AB\cos\theta} \tag{R2.7}$$

as shown in Fig. R2.5(a).

The dot product of two skew vectors is not defined. The dot product of two

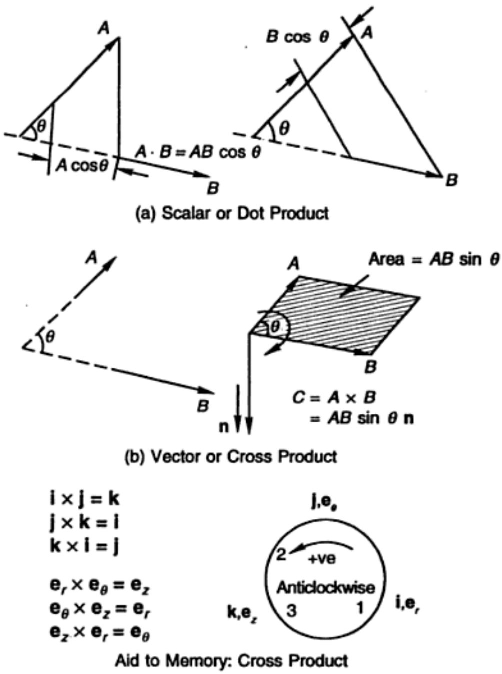


Fig. R2.5 ***Products of Vectors***

collinear or parallel vectors must result in a scalar quantity equal to the multiplication of the magnitudes of the two vectors since $\cos\theta = 1$ for $\theta = 0$.

$$\mathbf{A} \cdot n\,\mathbf{A} = n\,A^2$$

The dot product of two equal vectors results into the square of its magnitudes

$$\mathbf{A} \cdot \mathbf{A} = A^2 \qquad \text{(R2.8)}$$

The dot product of two orthogonal vectors must be zero since $\cos 90° = 0$. In particular, the dot products of the unit vectors are:

$$\begin{aligned}
\mathbf{i} \cdot \mathbf{j} &= 1 \times 1 \cos 0° = 1 \\
\mathbf{j} \cdot \mathbf{j} &= 1 \times 1 \cos 0° = 1 \\
\mathbf{k} \cdot \mathbf{k} &= 1 \times 1 \cos 0° = 1 \\
\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} &= 1 \times 1 \cos 90° = 0 \qquad \text{(R2.9)} \\
\mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} &= 1 \times 1 \cos 90° = 0 \\
\mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} &= 1 \times 1 \cos 90° = 0
\end{aligned}$$

In terms of the rectangular components,

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

$$\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$$

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z = \mathbf{B} \cdot \mathbf{A} \qquad \text{(R2.10)}$$

Since $\quad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$

and

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \ldots = 0$$

$$\mathbf{A} \cdot \mathbf{A} = A^2 = A_x^2 + A_y^2 + A_z^2$$

The scalar or dot product is both commutative and distributive

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \qquad \text{(R2.11)}$$

$$n(\mathbf{A} \cdot \mathbf{B}) = n\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot n\mathbf{B}$$

If $\mathbf{A} \cdot \mathbf{B} = 0$, then either $\mathbf{A} = 0$ or $\mathbf{B} = 0$ or both are zero (trivial case), or **A** and **B** are mutually perpendicular vectors. The angle between two vectors **A** and **B** is given by

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

The magnitude of a vector A is given by

$$A = \sqrt{A^2} = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Example If a force $\mathbf{F} = 10$ N acts upon a particle causing a displacement $S = 3$ m at an angle of 60° to the direction of the force, the work done on the particle equals

$$10 \times 3 \times \cos 60° = 15 \text{ N m}$$

(e2) *Vector* or *cross product* of two coplanar vectors **A** and **B** is a vector **C** denoted by $\mathbf{A} \times \mathbf{B}$ and read as '**A** cross **B**' such that (a) its magnitude equals the product of the magnitudes of the vectors **A** and **B** and the sine of the smaller angle between them, (b) its direction is perpendicular to the plane of **A** and **B**, and (c) its sense is given by the right-handed screw rule.

$$\boxed{\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB \sin\theta_{\mathbf{n}}} \qquad \text{(R2.12)}$$

Geometrically, the magnitude of the cross product vector **C** equals the area of the parallelogram bounded by the vectors **A** and **B** as the adjacent sides as shown in Fig. R2.5(b). The direction of vector **C** is perpendicular to this area with an arrow-head decided by the right-handed screw rule.

The cross product of two skew vectors is not defined.

The cross product of two collinear or parallel vectors must vanish since $\sin 0° = 0$.

The cross product of two orthogonal vectors must be a vector directed along the third orthogonal axis given by the cross-product rule.

In particular, the cross products of the unit vectors are:

$$\begin{array}{|ccc|} \mathbf{i}\times\mathbf{i}=0 & \mathbf{j}\times\mathbf{j}=0 & \mathbf{k}\times\mathbf{k}=0 \\ \mathbf{i}\times\mathbf{j}=\mathbf{k} & & \mathbf{j}\times\mathbf{i}=-\mathbf{k} \\ \mathbf{j}\times\mathbf{k}=\mathbf{i} & & \mathbf{k}\times\mathbf{j}=-\mathbf{i} \\ \mathbf{k}\times\mathbf{i}=\mathbf{j} & & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \end{array} \tag{R2.13}$$

In terms of the rectangular components,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{R2.14}$$

$$\mathbf{A}\times\mathbf{A} = 0$$

The vector or cross product is *not commutative* but obeys the distributive law:

$$\mathbf{A}\times\mathbf{B} \neq \mathbf{B}\times\mathbf{A} \tag{R2.15}$$

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

$$n(\mathbf{A}\times\mathbf{B}) = n\mathbf{A}\times\mathbf{B} = \mathbf{A}\times n\mathbf{B}$$

If $\mathbf{A}\times\mathbf{B} = 0$, then either $\mathbf{A} = 0$ or $\mathbf{B} = 0$ or both are zero (trivial case), or **A** is parallel to **B**, or **A** and **B** are collinear.

The angle between two vectors **A** and **B** is given by

$$\sin\theta = \frac{|\mathbf{A}\times\mathbf{B}|}{AB} \tag{R2.16}$$

(f) *Triple products of vectors* can also be meaningfully defined as follows:

(f1) *Scalar triple product* of three vectors **A**, **B** and **C** is defined as

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{R2.17}$$

where
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$
$$\mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k}$$

The scalar triple product results in a scalar quantity represented by the volume of the parallelopiped having **A**, **B** and **C** as the adjacent edges.

This can be shown with reference to Fig. R2.7

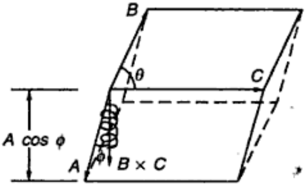


Fig. R2.7 ***Representation of* A · (B × C)**

$$\mathbf{B} \times \mathbf{C} = BC \sin\theta\, \mathbf{n}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = BC \sin\theta\, A \cos\phi$$

= Area of the parallelogram contained by **B** and **C** multiplied by the perpendicular distance between the face *OBC* and the one parallel to it

= Volume of the parallelopiped contained by the three vectors **A**, **B** and **C**

Volume is a scalar quantity. One can arrive at the volume of a parallelopiped by multiplying the area of its faces by the perpendicular distance between that face and the one parallel to it. Hence the order of vectors in a scalar triple product is immaterial. Hence

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \qquad \text{(R2.18)}$$

If the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is zero, then either **A** or **B** or **C** is zero singly or in combination, or they are coplanar vectors or two or all are collinear vectors.

(f2) *Vector triple product* of three vectors **A**, **B** and **C** is defined as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\, \mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \qquad \text{(R2.19)}$$

and $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

If the vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = 0$ then either **A** or **B** or **C** is zero singly or in combination, or **A** is in the plane containing **B** and **C**.

***Example* R2.7** Determine the components of the 500 N force shown along the *aa* and *bb* axes.

Solution Let the *x* axis be along *oa* and the *y* axis perpendicular to it.

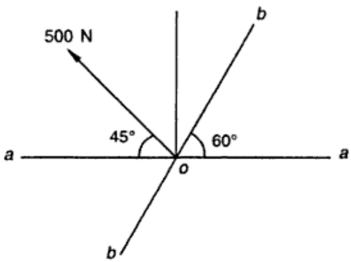


Fig. Ex R2.7

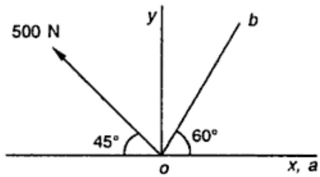


Fig. Ex R2.7 *(Solution)*

Unit vector along *oa* is **i** and the unit vector along *ob* is cos 60 **i** + sin 60 **j** = 0.5 **i** + 0.866 **j**.

The 500 N force is expressed as

$$-500 \cos 45\, \mathbf{i} + 500 \sin 45\, \mathbf{j} \text{ N}$$

or $$-353.5\, \mathbf{i} + 353.5\, \mathbf{j} \text{ N}$$

The component of the force along *oa* is

$$(-353.5\, \mathbf{i} + 353.5\, \mathbf{j}) \cdot \mathbf{i} = -353.5 \text{ N}$$

and the component along *ob* is

$$(-353.5\, \mathbf{i} + 353.5\, \mathbf{j}) \cdot (0.5\, \mathbf{i} + 0.866\, \mathbf{j}) = -176.8 + 306.2 = 129.4 \text{ N}$$

One can as well determine the desired components geometrically by estimating the projections of the 500 N force along *oa* and *ob* respectively. The projection along *oa* is –500 cos 45°, i.e., –353.5 N and the projection along *ob* is 500 cos (90 – 60 + 45), i.e., 500 cos 75° or 129.4 N.

***Example* R2.8** Determine the component of the vector

$$(3\,\mathbf{i} + 2\,\mathbf{j} - 5\,\mathbf{k})$$

along the vector (4 **i** – 3 **j**).

Solution The dot product of a vector with another results in the product of the projection of one vector along the other and the magnitude of the other vector.

$$\mathbf{A} \cdot \mathbf{B} = A \cos \theta \quad B = AB \cos \theta$$

where θ is the angle between **A** and **B**.
If **e** is unit vector along *B* then

$$\mathbf{A} \cdot \mathbf{e} = A \cos \theta$$

the component of **A** along **B**.
The unit vector along (4 **i** – 3 **j**) is

$$\mathbf{e} = \frac{(4\,\mathbf{i} - 3\,\mathbf{j})}{\sqrt{4^2 + 3^2}} = 0.8\, \mathbf{i} - 0.6\, \mathbf{j}$$

The desired component is

$$(3\,\mathbf{i} + 2\,\mathbf{j} - 5\,\mathbf{k}) \cdot (0.8\,\mathbf{i} - 0.6\,\mathbf{j}) = 2.4 - 1.2 = 1.2 \text{ units}$$

The component vector is, however, given by

$$1.2(0.8\,\mathbf{i} - 0.6\,\mathbf{j}) = 0.96\,\mathbf{i} - 0.72\,\mathbf{j}.$$

***Example* R2.9** Two vectors **A** and **B** are given as

$$\mathbf{A} = 2\,\mathbf{i} + 3\,\mathbf{j}$$

$$\mathbf{B} = 3\,\mathbf{i} - \mathbf{j}$$

Determine (a) the dot product $\mathbf{A} \cdot \mathbf{B}$
(b) the cross product $\mathbf{A} \times \mathbf{B}$
(c) the angle between **A** and **B**

Solution

(a) The dot product is given by

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y$$
$$= 2 \times 3 + 3 \times (-1) = 3 \text{ units}$$

(b) The cross product is expressed as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= [2 \times (-1) - 3 \times 3]\,\mathbf{k} = -11\mathbf{k}$$

This is represented in Fig. Ex. R2.9.

(c) From the definition of the dot product,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta$$

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{3}{\sqrt{2^2 + 3^2} \times \sqrt{3^2 + 1^2}}$$

$$= \frac{3}{3.61 \times 3.16} = 0.264$$

and $\theta = 74.7°$

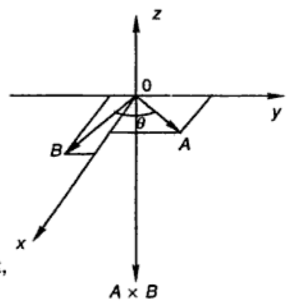


Fig. Ex. R2.9

Alternatively, from the definition of cross product,

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

$$\sin\theta = \frac{|\mathbf{A} \times \mathbf{B}|}{AB} = \frac{11}{3.61 \times 3.16} = 0.964$$

and $\theta = 74.7°$

***Example* R2.10** Two vectors **A** and **B** are given:

$$\mathbf{A} = 2\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$$
$$\mathbf{B} = 3\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$$

Determine (a) their dot product, (b) their cross product and the unit vector along it, and (c) the included angle between vector **A** and the vector resulting from the cross product.

Solution

(a) $\mathbf{A} \cdot \mathbf{B} = 2 \times 3 + 3 \times (-3) + 1 \times 4 = 1$

(b) $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{vmatrix}$$

$$= [3 \times 4 - (-3) \times 1]\,\mathbf{i} - [2 \times 4 - 3 \times 1]\,\mathbf{j} + [2 \times (-3) - 3 \times 3]\,\mathbf{k}$$
$$= 15\,\mathbf{i} - 5\,\mathbf{j} - 15\,\mathbf{k}$$

The magnitude of this product is

$$\sqrt{15^2 + 5^2 + 15^2} = \sqrt{475} = 21.8$$

The unit vector along it is, therefore, given by

$$\frac{15}{21.8}\mathbf{i} - \frac{5}{21.8}\mathbf{j} - \frac{15}{21.8}\mathbf{k} = 0.688\,\mathbf{i} - 0.229\,\mathbf{j} - 0.688\,\mathbf{k}$$

(c) The included angle between the vector resulting from the cross product and either of the constituent vectors must be 90° = $\pi/2$. Examining the same for the present case,

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 15 & -5 & -15 \end{vmatrix}$$

$$= (-45 + 5)\,\mathbf{i} - (-30 - 15)\,\mathbf{j} + (-10 - 45)\,\mathbf{k}$$
$$= -40\,\mathbf{i} + 45\,\mathbf{j} - 55\,\mathbf{k}$$

The magnitude of which is

$$40\sqrt{40^2 + 45^2 + 55^2} = 81.55$$

and

$$\sin\theta = \frac{81.55}{3.74 \times 21.8} = 1$$

whence θ, the angle between **A** and **A × B**, is 90°.

R2.5 DERIVATIVES AND INTEGRALS OF VECTORS

(a) Derivatives of Vectors

The derivative of the vector **A** with respect to a scalar, say time t, is expressed as

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{A}}{\Delta t} = \lim_{\Delta \to 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t} \qquad \text{(R2.20)}$$

where the vector **A** is a function of the scalar t and $\Delta\mathbf{A}$ refers to the changes in **A** during intervals of scalar Δt as shown in Fig. R2.8a.

If $$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

then $$\frac{d\mathbf{A}}{dt} = \frac{dA_x}{dt}\mathbf{i} + \frac{dA_y}{dt}\mathbf{j} + \frac{dA_z}{dt}\mathbf{k} \tag{R2.21}$$

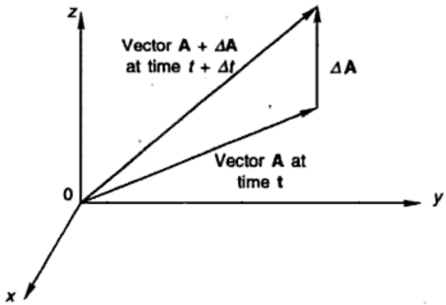


Fig. R2.8(a) ***Change in Vector* A**

The derivative of a given vector with respect to a scalar results in a vector representing the rate of change of the given vector with respect to the scalar.

An extension of the usual rules of differential calculus leads to the following rules:

$$\begin{aligned}
\frac{d}{dt}(n\mathbf{A}) &\equiv n\frac{d\mathbf{A}}{dt} \\
\frac{d}{dt}(\mathbf{A}+\mathbf{B}) &\equiv \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \\
\frac{d}{dt}(\phi\mathbf{A}) &\equiv \phi\frac{d\mathbf{A}}{dt} + \frac{d\phi}{dt}\mathbf{A} \\
\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) &\equiv \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} \\
\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) &\equiv \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B}
\end{aligned} \tag{R2.22}$$

where n is a real number and ϕ is a scalar function of t.

It may be noted that the derivative of a constant vector **C** with respect to any scalar must be zero, i.e.,

$$\frac{d\mathbf{C}}{dt} = 0$$

(b) Derivatives of Unit Vectors

Derivatives of the unit vectors **i**, **j** and **k** along the fixed space coordinates x, y and z respectively must be zero with respect to any scalar.

$$\frac{d\mathbf{i}}{dt} = 0 = \frac{d\mathbf{j}}{dt} = \frac{d\mathbf{k}}{dt} \ldots$$

Let us consider the derivatives of the unit vectors in radial and tangential direction, i.e., of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as shown in Fig. R2.8(b)

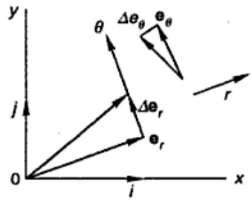


Fig. R2.8(b) *Change in Unit Vectors*

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$\frac{d\mathbf{e}_r}{dt} = (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt}$$

Since $\qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

and $\qquad \omega = \dfrac{d\theta}{dt}$

$$\frac{d\mathbf{e}_r}{dt} = \omega\,\mathbf{e}_\theta$$

Similarly,

$$\frac{d\mathbf{e}_\theta}{dt} = (-\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j})\frac{d\theta}{dt}$$

$$= -\omega\,\mathbf{e}_r$$

(c) Integrals of Vectors

If a vector **A** is a function of a scalar variable, say time t,

$$\mathbf{A}(t) = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

then the integral of **A** over the range t_1 to t_2 is

$$\int_{t_1}^{t_2} \mathbf{A}(t)\,dt = \int_{t_1}^{t_2} (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\,dt$$

$$= \mathbf{i}\int_{t_1}^{t_2} A_x\,dt + \mathbf{j}\int_{t_1}^{t_2} A_y\,dt + \mathbf{k}\int_{t_1}^{t_2} A_z\,dt \qquad \text{(R2.23)}$$

***Example* R2.11**

Assuming $\qquad \mathbf{r} = (r_0 \sin\omega t)\,\mathbf{i} + (r_0 \cos\omega t)\,\mathbf{j}$

evaluate (a) r, the magnitude of the vector at any time t, (b) $\dfrac{d\mathbf{r}}{dt}$, (c) $\dfrac{d^2\mathbf{r}}{dt^2}$ and (d) the integral of **r** from $t = 0$ to $t = \pi/\omega$.

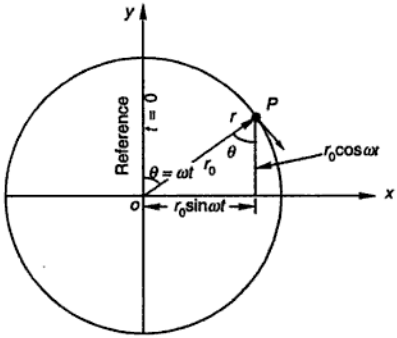


Fig. Ex R2.11

Solution

(a) $r = \sqrt{(r_0 \sin\omega t)^2 + (r_0 \cos\omega t)^2} = \mathbf{r}_0$

(b) $\dfrac{d\mathbf{r}}{dt} = (r_0\omega \cos \omega t)\,\mathbf{i} - (r_0\omega \sin \omega t)\,\mathbf{j}$

the magnitude of which is given by

$$\sqrt{(r_0\omega \cos\omega t)^2 + (r_0\omega \sin\omega t)^2} = \omega \mathrm{r}_0$$

(c) $\dfrac{d^2\mathbf{r}}{dt^2} = -(r_0\omega^2 \sin \omega t)\,\mathbf{i} - (r_0\omega^2 \cos \omega t)\,\mathbf{j} = -\omega^2\,\mathbf{r}$

the magnitude of which is given by $\omega^2\mathbf{r}_0$

(d)
$$\int_0^{\pi/\omega} \mathbf{r}\,dt = \mathbf{i}\int_0^{\pi/\omega} r_0 \sin\omega t\,dt + \mathbf{j}\int_0^{\pi/\omega} r_0 \cos\omega t\,dt$$

$$= -\left|\frac{r_0\cos\omega t}{\omega}\right|_0^{\pi/\omega}\mathbf{i} + \left|\frac{r_0\sin\omega t}{\omega}\right|_0^{\pi/\omega}\mathbf{j}$$

$$= -\frac{r_0}{\omega}(-1-1)\mathbf{i} + \frac{r_0}{\omega}(0-0)\mathbf{j}$$

$$= \frac{2r_0}{\omega}\mathbf{i}$$

***Example* R2.12**

If $\mathbf{A} = 2t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$ and $\mathbf{B} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

evaluate (a) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B})$, (b) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A})$, (c) $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B})$ and (d) $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{A})$

Solution

(a) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{B}$

$= (2t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k})\cdot(\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) + (4t\,\mathbf{i} + \mathbf{j} - 3t^2\,\mathbf{k})\cdot(\sin t\,\mathbf{i} + \cos t\,\mathbf{j})$

$= 3t\sin t + 2t^2\cos t + \cos t$

(b) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\frac{d\mathbf{A}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{A} = 2\mathbf{A}\cdot\frac{d\mathbf{A}}{dt} = 2(2t^2\mathbf{i} + t\,\mathbf{j} - t^3\mathbf{k})\cdot(4t\,\mathbf{i} + \mathbf{j} - 3t^2\,\mathbf{k})$

$= 16t^3 + 2t + 6t^5$

(c) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B}$

$= (2t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k})\times(\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) + (4t\,\mathbf{i} + \mathbf{j} - 3t^2\,\mathbf{k})\times(\sin t\,\mathbf{i} + \cos t\,\mathbf{j})$

$= -t^3\sin t\,\mathbf{i} - t^3\cos t\,\mathbf{j} - (2t^2\sin t + t\cos t)\,\mathbf{k}$

$+ 3t^2\cos t\,\mathbf{i} - 3t^2\sin t\,\mathbf{j} + (4t\cos t - \sin t)\,\mathbf{k}$

$= (-t^3\sin t + 3t^2\cos t)\,\mathbf{i} - (t^3\cos t + 3t^2\sin t)\,\mathbf{j}$

$+ (3t\cos t - (1 + 2t^2)\sin t)\,\mathbf{k}$

(d) $\frac{d}{dt}(\mathbf{A}\times\mathbf{A}) = \frac{d}{dt}(0) = \mathbf{0}$

Alternatively, the bracketed vector operations can be performed first and differentiation done later. For example, in part (b),

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \frac{d}{dt}[(2t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k})\cdot(2t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k})]$$

$$= \frac{d}{dt}(4t^4 + t^2 + t^6)$$

$$= 16t^3 + 2t + 6t^5$$

which is the same as obtained earlier.

R2.6 GRADIENT, DIVERGENCE AND CURL

If a vector **A** and a scalar ϕ are functions of the space coordinates x, y and z then a *vector differential operator* ∇ called '*del*',

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{R2.24}$$

can be operated to define the following:

(a) Gradient of a Scalar ϕ

$$\text{grad}\,\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} \tag{R2.25}$$

Physically, the gradient of a scalar is a vector in the direction of the steepest variation of ϕ with respect to the space coordinates. The component of grad ϕ in any direction is the rate of change of ϕ in that direction.

(b) Divergence of a Vector A

$$\text{div } \mathbf{A} = \nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{R2.26}$$

The divergence of a vector refers to the net efflux of the vector at a point in space.

(c) Curl of a Vector A

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{R2.27}$$

R2.7 SOME VECTOR OPERATIONS

Some of the useful results in vector algebra and calculus commonly referred to by the engineering students are summarized in Table R2.1.

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = \mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = \ldots = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = \mathbf{e}_r \cdot \mathbf{e}_\theta = \mathbf{e}_\theta \cdot \mathbf{e}_\phi = \ldots = 0$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{e}_r \times \mathbf{e}_r = \mathbf{e}_\theta \times \mathbf{e}_\theta = \ldots = 0$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z, \mathbf{e}_\theta \times \mathbf{e}_z = \mathbf{e}_r, \mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\theta$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$$

$$\frac{d\mathbf{A}}{du} = \frac{dA_x}{du}\mathbf{i} + \frac{dA_y}{du}\mathbf{j} + \frac{dA_z}{du}\mathbf{k}$$

If $F = f(x, y)$ where $x = x(t)$, $y = y(t)$

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

Table R2.1 VECTOR OPERATIONS IN DIFFERENT COORDINATE SYSTEMS

	Cartesian	*Cylindrical*	*Spherical*
Vector **A**	$A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$	$A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_z \mathbf{e}_z$	$A_R \mathbf{e}_R + A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi$
$A = \lvert \mathbf{A} \rvert$	$\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$	$\left(A_r^2 + A_\theta^2 + A_z^2\right)^{1/2}$	$\left(A_R^2 + A_\theta^2 + A_\phi^2\right)^{1/2}$
$\mathbf{A} \cdot \mathbf{B} = AB \cos\alpha$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\theta B_\theta + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
$\mathbf{A} \times \mathbf{B} = AB \sin\alpha\, \mathbf{n}$	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \\ A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix}$	$\begin{vmatrix} \mathbf{e}_R & \mathbf{e}_\theta & \mathbf{e}_\phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
$(\mathbf{ABC}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$	$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$	$\begin{vmatrix} A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \\ C_r & C_\theta & C_z \end{vmatrix}$	$\begin{vmatrix} A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \\ C_R & C_\theta & C_\phi \end{vmatrix}$
Operator $\nabla \equiv$	$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$	$\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$	$\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\theta \frac{1}{R}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$
grad $p = \nabla p \equiv$	$\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}$	$\frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z$	$\frac{\partial p}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{R\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi$
div $\cdot\, \mathbf{A} = \nabla \cdot \mathbf{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$	$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{R\sin\theta}\frac{\partial A\phi}{\partial \phi}$
curl $\mathbf{A} = \nabla \times \mathbf{A}$	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$	$\frac{1}{R^2 \sin\theta}\begin{vmatrix} \mathbf{e}_R & R\mathbf{e}_\theta & R\sin\theta\, \mathbf{e}_\phi \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & A_\theta & R\sin\theta\, A_\phi \end{vmatrix}$

(Contd.)

Table R2.1 (*Contd.*) **VECTOR OPERATIONS IN DIFFERENT COORDINATE SYSTEMS**

	Cartesian	*Cylindrical*	*Spherical*
Operator $\nabla^2 \equiv$ (Laplacian)	$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$	$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$
Operator $\frac{D}{Dt}\equiv=(U\cdot\nabla)+\frac{\partial}{\partial t}$	$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$	$u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial \theta}+u_z\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$	$u_R\frac{\partial}{\partial R}+\frac{u_\theta}{R}\frac{\partial}{\partial \theta}+\frac{u_\phi}{R\sin\theta}\frac{\partial}{\partial \phi}+\frac{\partial}{\partial t}$

Line integral $\int \mathbf{A}\cdot d\mathbf{r} = \int (A_x\ dx + A_y\ dy + A_z\ dz)$

Contour or cyclic integral $\oint \mathbf{A}\cdot dr$

Surface integrals $\int_S \mathrm{A}\times ds,\ \int_S \phi\, ds,\ \int_S \mathrm{A}\quad {}^2 ds,\ \int_S \phi\, \mathrm{n}\, ds$

Volume integrals $\int_S \mathbf{A}\, dV$ and $\int_V \phi\, dV$

Divergence Theorem of Gauss

The divergence theorem of Gauss states that if V is the volume bounded by a closed surface S and if $\mathbf{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla\cdot\mathbf{A}\, dV = \iint_S \mathbf{A}\cdot\mathbf{n}\, ds \tag{R2.28}$$

where $\mathbf{n}$ is the unit vector drawn normal to S.

Stokes' Theorem

Stokes' theorem states that if S is an open surface bounded by a simple closed curve C and if $\mathbf{A}$ has continuous derivatives, then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\, ds \tag{R2.29}$$

where C is traversed in the positive (counter clockwise) direction.

Green's Theorem

Green's theorem in a plane states that if R is a closed region in the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous partial derivatives in the region R, then

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy \tag{R2.30}$$

where C is traversed in the positive (counter clockwise) direction.

Green's First Identity

$$\iiint_V (\phi\nabla^2\rho + \nabla\phi\cdot\nabla\rho)\, dV = \iint_S (\phi\nabla\rho)\cdot ds$$

Green's Second Identity

$$\iiint_V (\phi\nabla^2\rho - \rho\nabla^2\phi)\, dV = \int_S (\phi\nabla\rho - \rho\nabla\phi)\, ds$$

R2.8 VECTOR IDENTITIES

Some vector identities commonly referred to by the engineering students are listed below.

Algebraic Identities

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \mathbf{A} + \mathbf{A} = 2\mathbf{A}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \qquad \mathbf{A} \cdot \mathbf{A} = A^2$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad \mathbf{A} \times \mathbf{A} = 0$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad \text{(R2.31)}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{B}[\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})] - \mathbf{A}[\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D})]$$

$$= \mathbf{C}[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D})] - \mathbf{D}[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})] \qquad \text{(R2.32)}$$

Calculus Identities

1. $\nabla^2 \phi = \nabla \cdot \nabla \phi$

2. $\nabla^2 \mathbf{A} = (\nabla \cdot \nabla) \mathbf{A}$

3. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

4. $\nabla \times (\nabla \phi) = 0$

5. $\nabla (\phi \psi) = \phi \nabla \psi + \psi \nabla \phi$

6. $\nabla \cdot (\phi \mathbf{A}) = \phi \nabla \cdot \mathbf{A} + \nabla \phi \cdot \mathbf{A}$

7. $\nabla \times (\phi \mathbf{A}) = \phi \nabla \times \mathbf{A} + \nabla \phi \times \mathbf{A}$

8. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}$

9. $\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \times \nabla) \mathbf{B} + (\mathbf{B} \times \nabla) \mathbf{A}$

10. $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

11. $(\mathbf{A} \cdot \nabla) \mathbf{A} = \nabla \left(\frac{A^2}{2} \right) - \mathbf{A} \times (\nabla \times \mathbf{A})$

12. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - \mathbf{B} (\nabla \cdot \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} - (\mathbf{A} \cdot \nabla) \mathbf{B}$

R2.9 POSITION VECTOR, DISPLACEMENT, VELOCITY AND ACCELERATION

The position of a moving point P in space is described by a *position vector*, a vector directed from an origin to the point. Thus **OP** is the position vector of P in Fig. 1.5 at an instant. At a later instant after an interval of time Δt, the position vector will be **OP′** corresponding to the new position of the moving point. The change in the position from P to P' is called the displacement vector or just the displacement.

$$\text{Displacement} = \mathbf{PP'} = \Delta\mathbf{r} = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} + \Delta z\,\mathbf{k} \tag{R2.33}$$

Since this displacement is brought about in a time interval Δt, the *velocity* of the point is given by

$$\mathbf{V} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k} \tag{R2.34}$$

Similarly, the rate of change of velocity is the *acceleration* given by

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

$$= \dot{\mathbf{V}} = \ddot{\mathbf{r}} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k} \tag{R2.35}$$

It is thus noted that (with respect to time):

The derivative of the position vector is the velocity and the derivative of the velocity is acceleration.

Alternatively (with respect to time),

The integral of the velocity is the position vector and the integral of the acceleration is the velocity

$$\int_{t_1}^{t_2} \mathbf{V}(t)\,dt = \mathbf{r}_2 - \mathbf{r}_1$$

$$= \text{change in position} = \text{displacement} \tag{R2.36}$$

$$\int_{t_1}^{t_2} \mathbf{a}(t)\,dt = \mathbf{V}_2 - \mathbf{V}_1$$

$$= \text{change in velocity in the interval} \tag{R2.37}$$

***Example* R2.13** Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = C$ where C is a constant. Hence find the unit vector normal to the surface $3xz^2 - 3xy - 4x = 7$ at the point (1, 2, –2) and an equation to the tangent plane at this point.

Solution Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on the surface $\phi(x, y, z) = C$.

Vector $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ must, therefore, be along a tangent to the surface at that point.

By differential calculus,

$$\partial\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

or

$$\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\phi z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = 0$$

or

$$\nabla\phi \cdot d\mathbf{r} = 0$$

which implies that $\nabla\phi$ must be perpendicular to $d\mathbf{r}$. Since $d\mathbf{r}$ is tangential to the surface, vector $\nabla\phi$ must be normal to it.

Another interesting point worth noting is that

since $$d\phi = \nabla\phi \cdot d\mathbf{r}$$

$$\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\mathbf{r}}{ds}$$

where s is an arbitrary space direction. For $d\phi/ds$ to be maximum, $\nabla\phi$ and dr/ds must be collinear and dr/ds being unity, $d\phi/ds = \nabla\phi$. In other words, the greatest rate of change of ϕ, i.e., the maximum directional derivative takes place in the direction of, and has the magnitude of the vector $\nabla\phi$.

For the given surface,

$$3xz^2 - 3xy - 4x = 7 = \phi(x, y, z)$$

$$\nabla\phi = \nabla(3xz^2 - 3xy - 4x)$$

$$= (3z^2 - 3y - 4)\,\mathbf{i} - 3x\,\mathbf{j} + 6xz\,\mathbf{k}$$

At $(1, 2, -2), \nabla\phi = 2\,\mathbf{i} - 3\,\mathbf{j} - 12\,\mathbf{k}, |\nabla\phi| = \sqrt{2^2 + (-3)^2 + (-12)^2} = 12.53$

Since $\nabla\phi$ is normal to the surface, the unit vector normal to the surface is given by

$$\frac{2}{12.53}\mathbf{i} - \frac{3}{12.53}\mathbf{j} - \frac{12}{12.53}\mathbf{k} = 0.16\,\mathbf{i} - 0.24\,\mathbf{j} - 0.96\,\mathbf{k}$$

If $d\mathbf{r}$ is tangential to the surface

or $$\nabla\phi \cdot d\mathbf{r} = 0$$

$$(2\,\mathbf{i} - 3\,\mathbf{j} - 12\,\mathbf{k}) \cdot [(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) - (\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k})] = 0$$

which provides $\quad 2(\mathbf{x} - 1) - 3(\mathbf{y} - 2) - 12(\mathbf{z} + 2) = 0$

as the equation of the tangent plane at the point (1, 2, –2).

***Example* R2.14** The motion of a point is expressed as $x = 2t^3$, $y = t^2 + 4t$, $z = 3t - 5$ in terms of the time parameter t. At time $t = 2$, determine (a) the velocity and acceleration, (b) the components of the velocity and acceleration in the direction of $(4\,\mathbf{i} + 3\,\mathbf{j})$, and (c) the unit tangent at the point. Also, determine the displacement of the point from $t = 0$ to $t = 2$.

Solution

(a) The position vector $\mathbf{r}$ is expressed as

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$= 2t^3\,\mathbf{i} + (t^2 + 4t)\,\mathbf{j} + (3t - 5)\,\mathbf{k}$$

$$\text{Velocity } \mathbf{V} = \frac{d\mathbf{r}}{dt} = 6t^2\,\mathbf{i} + (2t + 4)\,\mathbf{j} + 3\mathbf{k}$$

$$= 24\,\mathbf{i} + 8\,\mathbf{j} + 3\,\mathbf{k} \text{ at } t = 2.$$

$$\text{Acceleration } \mathbf{a} = \frac{d\mathbf{V}}{dt} = 12t\mathbf{i} + 2\,\mathbf{j}$$

$$= 24\,\mathbf{i} + 2\,\mathbf{j} \qquad \text{at } t = 2$$

(b) The unit vector along $(4\,\mathbf{i} + 3\,\mathbf{j})$ would be

$$\frac{4}{\sqrt{4^2 + 3^2}}\mathbf{i} + \frac{3}{\sqrt{4^2 + 3^2}}\mathbf{j} = 0.8\mathbf{i} + 0.6\,\mathbf{j}$$

The velocity component at $t = 2$ along this direction is

$$(24\,\mathbf{i} + 8\,\mathbf{j} + 3\,\mathbf{k}) \cdot (0.8\,\mathbf{i} + 0.6\,\mathbf{j}) = 19.2 + 4.8 = 24 \text{ units.}$$

The acceleration component at $t = 2$ along this direction is

$$(24\,\mathbf{i} + 2\,\mathbf{j}) \cdot (0.8\,\mathbf{i} + 0.6\,\mathbf{j}) = 19.2 + 1.2 = 20.4 \text{ units.}$$

(c) The velocity vector must be tangential to the curve at the point and time considered.

Since $\quad V = \sqrt{24^2 + 8^2 + 3^2} = \sqrt{649} = 25.48$

The unit tangent at that instant must be

$$\frac{24}{25.48}\mathbf{i} + \frac{8}{25.48}\mathbf{j} + \frac{3}{25.48}\mathbf{k} = 0.94\mathbf{i} + 0.31\,\mathbf{j} + 0.12\,\mathbf{k}$$

(d) Since $\dfrac{d\mathbf{r}}{dt} = 6t^2\mathbf{i} + (2t + 4)\,\mathbf{j} + 3\,\mathbf{k}$

$$d\mathbf{r} = (6t^2\mathbf{i} + (2t + 4)\,\mathbf{j} + 3\,\mathbf{k})\,dt$$

Displacement $\qquad (\mathbf{r}_2 - \mathbf{r}_1) = \int_{t_1}^{t_2} d\mathbf{r}$

$$= \int_0^2 (6t^2\mathbf{i} + (2t+4)\mathbf{j} + 3\mathbf{k})dt$$

$$= |2t^3\mathbf{i} + (t^2 + 4t)\mathbf{j} + 3t\mathbf{k}|_0^2$$

$$= 16\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}$$

The displacement may alternatively be determined by finding the initial and final positions:

At the initial position, $t = 0$,

$$x_1 = 2t^3 = 0;\ y_1 = t^2 + 4t = 0;\ z_1 = 3t - 5 = -5$$

At the final position, $t = 2$,

$$x_2 = 2t^3 = 16;\ y_2 = t^2 + 4t = 12;\ z_2 = 3t - 5 = 1$$

The displacement is given by

$$\mathbf{r}_2 - \mathbf{r}_1 = (16 - 0)\,\mathbf{i} + (12 - 0)\,\mathbf{j} + (1 - (-5))\,\mathbf{k}$$
$$= 16\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}$$

***Example* R2.15** Determine and sketch the curve traced by a point such that $x = 200\,t$ and $y = 1000 - 4\,t^2$. Also, comment on the salient features of the motion if the distances are in metres and time is in seconds.

Solution Eliminating t from the parametric equations,

$$t = \frac{x}{200}, \quad y = 1000 - 4\left(\frac{x}{200}\right)^2$$

$$y = 1000 - 0.0001x^2$$

This is the equation of the curve to be traced.

x (metre)	0	1000	2000	2236	3162
y (metre)	0	900	600	500	0

The curve is sketched in Fig. Ex. R2.15

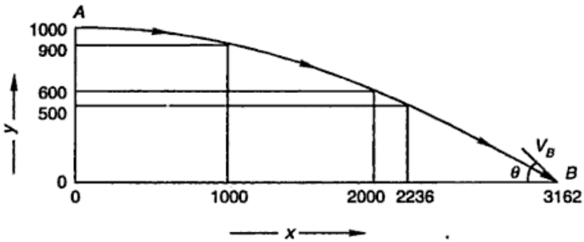


Fig. Ex. R2.15

At $t = 0$, $x = 0$ and $y = 1000$ m correspond to point A where the observation starts. In general, the velocity components are

$$V_x = \frac{dx}{dt} = 200 \text{ m/s}$$

$$V_y = \frac{dy}{dt} = -8\,t \text{ m/s}$$

At $t = 0$, $V_x = 200$ m/s and $V_y = 0$
which shows that the initial velocity at A is wholly horizontal and equals 200 m/s, whereas with the passage of time, the horizontal component stays at 200 m/s and the vertical component increases downwards linearly with time. The acceleration is

$$a_y = \frac{dv_y}{dt} = -8 \text{ m/s}^2$$

which is approximately 80% of the gravitational acceleration due to the earth. The case in hand closely resembles the motion of a bomb released from the low-altitude bomber aircraft flying at 200 m/s parallel to the ground. The bomb, when dropped, travels horizontally and, with the passage of time, acquires a vertical velocity by virtue of the acceleration due to gravity and is resisted by the aerodynamic drag.

It reaches the base at B where $y = 0$ and $x = 3162$ m. The time taken to reach the base is $t = 3162/200 = 15.81$ seconds. The velocity components at the base B are

$$V_x = 200 \text{ m/s}, \quad V_y = -8 \times 15.81 = -126.48 \text{ m/s}$$

or

$$\mathbf{V}_B = 200\,\mathbf{i} - 126.48\,\mathbf{j} \text{ m/s}$$

where $\mathbf{V}_B = \sqrt{200^2 + 126.48^2} = 236.64 \text{ m/s}$

and

$$\theta = \tan^{-1}\left(\frac{126.48}{200}\right) = \tan^{-1} 0.6324 = 32.32°$$

R2.10 SOME OTHER VECTOR QUANTITIES

In addition to certain vector quantities already referred to, a number of other important vector quantities are listed in this section. No attempt is made here to define the quantities but merely to recognise them as vector quantities. It may also be noted that only the *position vector* and the *force* be conferred the vector status axiomatically, rest of the vector quantities can be proved to be vectors by virtue of their definitions.

Force

Force $\mathbf{F}$ is an action exerted on a body which changes or tends to change the state of rest or of rectilinear motion of the body.

Moment

Moment **M** refers to the turning effect of a force **F** about a point O

$$\boxed{\mathbf{M} = \mathbf{r} \times \mathbf{F}}$$

The magnitude of the moment is given by

$$M = rF \sin \theta = F_p \tag{R2.38}$$

where **r** is the position vector of any point on the line of action of the force and p is the perpendicular distance from the point to the line of action of the vector **F**. The moment is directed perpendicular to the plane containing **r** and **F** as shown in Fig. R2.9.

The principle of moments due to Varignon called Varignon theorem, states that *The moment of a force equals the sum of the moments due to its components.*

Consider a force **F** and moment **M** of the force about an origin O. By definition,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

Let **F** be resolved into components $\mathbf{F}_1, \mathbf{F}_2, ..., \mathbf{F}_i, ... \mathbf{F}_n$ such that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + ... + \mathbf{F}_i + ... + \mathbf{F}_n$$

$$= \sum_{i=1}^{n} \mathbf{F}_i$$

and the position vector **r** refers to each of these forces as well.

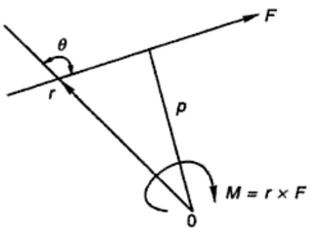


Fig. R2.9 ***Force and Moment***

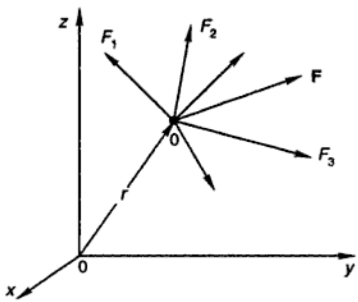


Fig. R2.10 ***Concurrent Forces***

Then,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

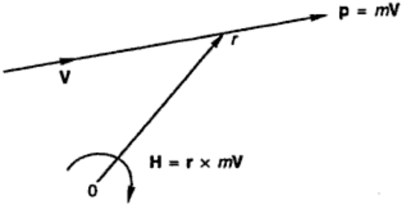
$$= \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + ... + \mathbf{F}_i + ... + \mathbf{F}_n)$$

Therefore,

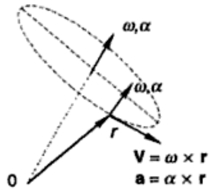
$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + ... + \mathbf{M}_i + ... + \mathbf{M}_n$$

$$= \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^{n} \mathbf{M}_i$$

It will be noticed that this theorem has no outstanding concept to offer. The theorem was proposed by the French mathematician long before the introduction of vector algebra.



(b) Momentum and Angular Momentum



(b) Velocity and Angular Velocity
Acceleration and Angular Acceleration

Fig. R2.11 ***Some Other Vector Quantities***

Linear Momentum

Linear momentum **p** of a body of mass m moving at a velocity **V** at a certain instant is defined as the product of the mass and velocity

$$\boxed{\mathbf{p} = m\mathbf{V}} \qquad \text{(R2.39)}$$

Angular Momentum

Angular momentum or moment of momentum **H** is expressed as the cross product of the position vector and linear momentum or the moment of the linear momentum $m\mathbf{V}$ about a point O as shown in Fig. R2.11.

$$\boxed{\mathbf{H} = \mathbf{r} \times m\mathbf{V}} \qquad \text{(R2.40)}$$

The angular momentum vector is directed perpendicular to the plane of **r** and **V**, i.e., perpendicular to the instantaneous plane of motion.

Infinitesimal Rotation

Infinitesimal rotation $d\theta$, defined as a infinitesimally small amount of the angle of rotation, is a vector quantity. It is important to note here that finite rotation 'θ' is not a vector quantity. The reason is that the finite rotations θ_1, and θ_2, although possessing magnitudes, directions and sense, do not obey the commutative law of addition, i.e.,

$$\theta_1 + \theta_2 \neq \theta_2 + \theta_1 \qquad \text{(R2.41)}$$

This has been demonstrated in Fig. R2.12 where a foot rule is shown subjected to rotations about the x and y axes respectively. An initial rotation $\theta_1 = \pi/2\,\mathbf{i}$ about the x-axis and a subsequent rotation $\theta_2 = \pi/2\,\mathbf{j}$ about the y-axis bring the sheet into positions (Aa) and (Aab). On the other hand, an initial rotation $\theta_2 = \pi/2\,\mathbf{j}$ about the y-axis and a subsequent rotation $\theta_1 = \pi/2\,\mathbf{i}$ about the x-axis result in positions (Ab) and (Aba) as shown. Obviously, the final positions (Aab) and (Aba) are not the same showing that the commutative law for finite rotations fails; admission disqualifying them from being put under the category of vector quantities.

The case of addition of infinitesimal rotations $d\theta_1$ and $d\theta_2$ is also shown in Fig. R2.12. The position of the foot rule after an initial rotation $d\theta_1$ is shown as (Ba) whereas that after an initial rotation $d\theta_2$ is shown as (Bb). The final orientation of the foot rule after both $d\theta_1$ and $d\theta_2$ have been imparted, in either order, is approximately the same in the limiting case shown in the same figure. Hence,

$$\Delta\theta_1 + \Delta\theta_2 \simeq \Delta\theta_2 + \Delta\theta_1$$

and in the limit,

$$d\theta_1 + d\theta_2 = d\theta_2 + d\theta_1 \qquad \text{(R2.42)}$$

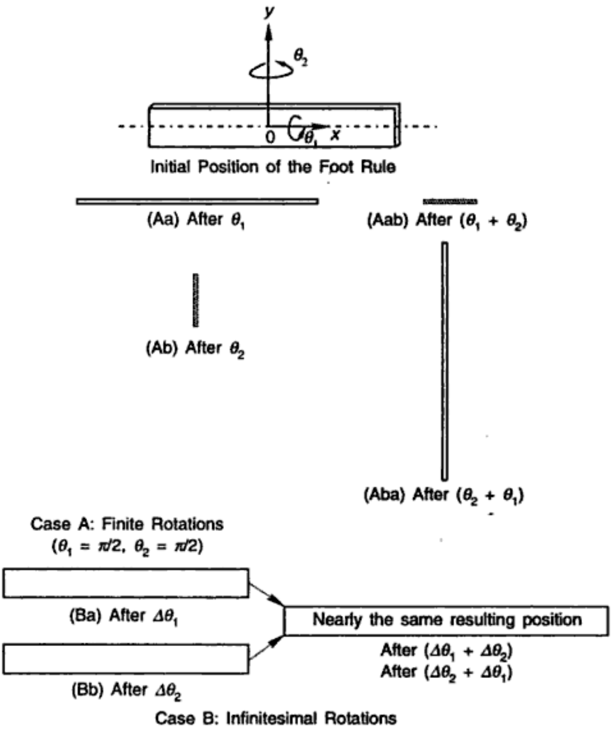


Fig. R2.12 ***Summation of Finite and Infinitesimal Rotations***

Angular Velocity

Angular velocity ω is defined as the rate of change of angular displacement or rotation of a body about an axis

$$\omega = \frac{d\theta}{dt} \qquad \text{(R2.43)}$$

It follows from the vector nature of infinitesimal rotation $d\theta$ and the scalar nature of time interval dt that the angular velocity must be a vector quantity.

If a point is located with a position vector **r** with respect to an origin O on the axis of rotation, the linear velocity **V** of the point is given by

$$\mathbf{V} = \boldsymbol{\omega} \times \mathbf{r} \tag{R2.44}$$

as shown in Fig. R2.11(b).

Angular Acceleration

Angular acceleration α is defined as the rate of change of angular velocity, i.e.,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{R2.45}$$

The linear acceleration **a** of a point with a positive vector **r** with respect to an origin O on the axis of rotation is given by

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} \tag{R2.46}$$

as also shown in Fig. R2.11(b).

***Example* R2.16** A force $\mathbf{F} = 3\,\mathbf{i} + 2\,\mathbf{j}$ passes through a point (0, 2) with respect to an origin O. Determine the moment of the force about the origin and establish its uniqueness with respect to arbitrary position vectors.

Solution The fact that the force passes through a point (0, 2) suggests that the vector $0\,\mathbf{i} + 2\,\mathbf{j} = 2\,\mathbf{j}$ is a position vector.

Hence
$$\mathbf{M} = 2\,\mathbf{j} \times (3\,\mathbf{i} + 2\,\mathbf{j})$$
$$= -6\,\mathbf{k}$$

Alternative position vectors, e.g., $(1.5\,\mathbf{i} + 3\,\mathbf{j})$ and $(-3\,\mathbf{i})$ would result in the same answer

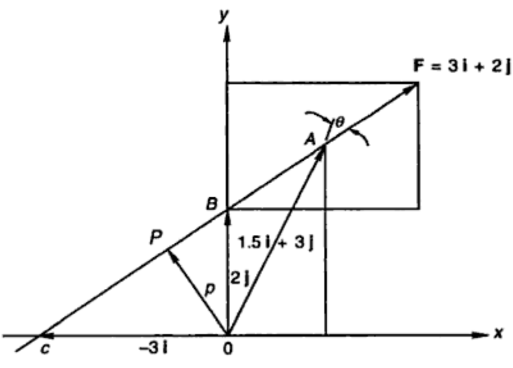


Fig. Ex. R2.16

$$(1.5\,\mathbf{i} + 3\,\mathbf{j}) \times (3\,\mathbf{i} + 2\,\mathbf{j}) = 3\,\mathbf{k} - 9\,\mathbf{k} = -6\,\mathbf{k}$$

$$(-3\,\mathbf{i}) \times (3\,\mathbf{i} + 2\,\mathbf{j}) = -6\,\mathbf{k}$$

If, instead, the perpendicular distance p from O to the line of action of $\mathbf{F}$ is known,

$$\mathbf{M} = \mathbf{F} \cdot p \text{ (clockwise)}$$

$$= \sqrt{3^2 + 2^2} \times 1.66(-\mathbf{k}) = -6\mathbf{k}$$

In fact, $\quad \mathbf{M} = \mathbf{r} \times \mathbf{F} = r\,F \sin\theta\,\mathbf{k}$

$$= F \cdot r \sin\theta\,\mathbf{k}$$
$$= F\,p\,\mathbf{k}$$

which means that the cross product takes care of the included angle between the constituent vectors and allows no discrepancy in the result for different choices of the position vector.

Example **R2.17** A vertical pole is guyed by three cables PA, PB and PC tied at a common point P 10 m above the ground. The base points of the cables are:

$$A\,(-4, -3, 0),\; B\,(5, 1, -1) \text{ and } C\,(-1, 5, 0)$$

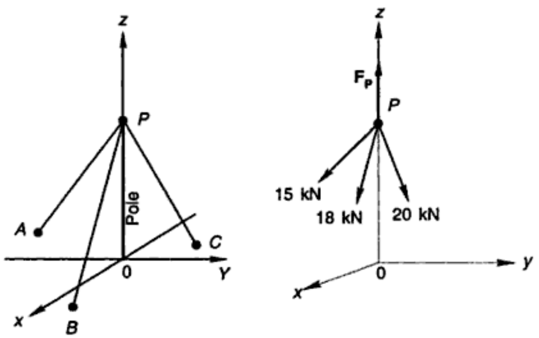
If the tensile forces in the cables are adjusted to be 15, 18 and 20 kN, find the resultant force on the pole at P.

Solution Since P is 10 m above the ground at O, the forces in the cables must be directed along PA, PB and PC such that

$$\mathbf{PA} = -4\,\mathbf{i} - 3\,\mathbf{j} - 10\,\mathbf{k} \qquad PA = \sqrt{4^2 + 3^2 + 10^2} = 11.18$$

$$\mathbf{PB} = 5\,\mathbf{i} + 1\,\mathbf{j} - 11\,\mathbf{k} \qquad PB = \sqrt{5^2 + 1^2 + 11^2} = 12.12$$

$$\mathbf{PC} = -1\,\mathbf{i} + 5\,\mathbf{j} - 10\,\mathbf{k} \qquad PC = \sqrt{1^2 + 5^2 + 10^2} = 11.22$$



Given sketch Free-body Diagram of Point P

Fig. Ex R2.17

The unit vectors along these directions are

$$\mathbf{e}_1 = -\frac{4}{11.18}\mathbf{i} - \frac{3}{11.18}\mathbf{j} - \frac{10}{11.18}\mathbf{k} = -0.358\mathbf{i} - 0.268\mathbf{j} - 0.894\mathbf{k}$$

$$\mathbf{e}_2 = -\frac{5}{12.12}\mathbf{i} + \frac{1}{12.12}\mathbf{j} - \frac{11}{12.12}\mathbf{k} = 0.412\mathbf{i} + 0.082\mathbf{j} - 0.907\mathbf{k}$$

$$\mathbf{e}_3 = \frac{-1}{11.22}\mathbf{i} + \frac{5}{11.22}\mathbf{j} - \frac{10}{11.22}\mathbf{k} = -0.089\mathbf{i} + 0.445\mathbf{j} - 0.891\mathbf{k}$$

The forces in the cables are, therefore, given by

$$15\,\mathbf{e}_1 = -5.37\,\mathbf{i} - 4.02\,\mathbf{j} - 13.41\,\mathbf{k}$$

$$18\,\mathbf{e}_2 = 7.42\,\mathbf{i} + 1.48\,\mathbf{j} - 16.33\,\mathbf{k}$$

$$20\,\mathbf{e}_3 = -1.78\,\mathbf{i} + 8.90\,\mathbf{j} - 17.82\,\mathbf{k}$$

Resultant force at $P = 0.27\,\mathbf{i} + 6.36\,\mathbf{j} - 47.56\,\mathbf{k}$

This is 48 kN in magnitude and acts predominantly downwards to hold the pole in position.

***Example* R2.18** A force of 1000 N in a particular direction must be applied to tow a boat. For some reason, it is not possible to apply the force in that direction but two forces can be applied to 30° and 45° on either side of it in the same plane containing the given force. Determine the magnitudes of the forces required along these directions.

Solution This is an example on the resolution of a force into two components at desired inclinations to it.

$$1000\,\mathbf{e} = \mathbf{F}_1 + \mathbf{F}_2$$

Sum of the components of $\mathbf{F}_1$ and $\mathbf{F}_2$ along $\mathbf{e}$ should add to 1000 N, whereas their components perpendicular to it must cancel.

$$\mathbf{F}_1 \cos 30° + \mathbf{F}_2 \cos 45° = 1000$$

$$\mathbf{F}_1 \sin 30° = \mathbf{F}_2 \sin 45°$$

From these two equations,

$$\mathbf{F}_1 = 732.1 \text{ N and } \mathbf{F}_2 = 517.6 \text{ N}$$

Alternatively, the components can be determined geometrically by realising that 1000 N force must form the diagonal of the parallelogram with adjacent sides $\mathbf{F}_1$ and $\mathbf{F}_2$. Completion of the parallelogram and measurement of $\mathbf{F}_1$ and $\mathbf{F}_2$ on the same scale to which the 1000 N force is drawn yields $\mathbf{F}_1$ and $\mathbf{F}_2$.

Alternatively, vectorially,

$$1000\,\mathbf{e} = F_1\,\mathbf{e}_1 + F_2\,\mathbf{e}_2$$

where $\mathbf{e}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors along the respective forces as shown in Fig. Ex. R2.18(b).

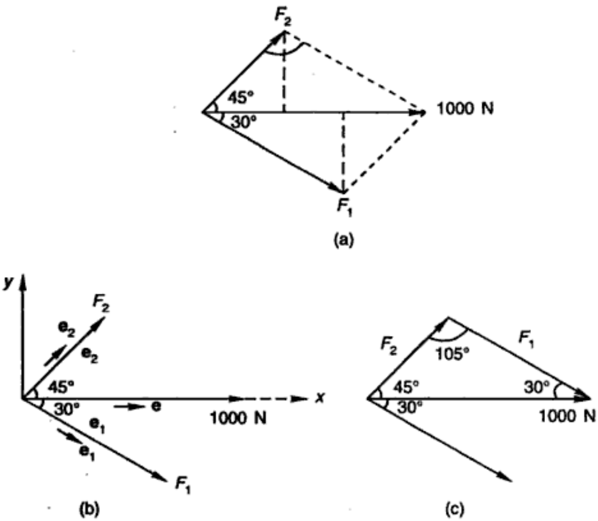


Fig. Ex R2.18

Choosing x-axis along the 1000 N force, and y-axis perpendicular to it as shown,

$$\mathbf{e} = \mathbf{i}$$
$$\mathbf{e}_1 = \cos 30° \, \mathbf{i} - \sin 30° \, \mathbf{j}$$
$$\mathbf{e}_2 = \cos 45° \, \mathbf{i} + \sin 45° \, \mathbf{j}$$

Hence,

$$1000 \, \mathbf{i} = \mathbf{F}_1 (\cos 30° \, \mathbf{i} - \sin 30° \, \mathbf{j}) + \mathbf{F}_2 (\cos 45° \, \mathbf{i} + \sin 45° \, \mathbf{j})$$
$$= (\mathbf{F}_1 \cos 30° + \mathbf{F}_2 \cos 45°) \, \mathbf{i} + (\mathbf{F}_2 \sin 45° - \mathbf{F}_1 \sin 30°) \, \mathbf{j}$$

and the two component equations are

$$1000 = \mathbf{F}_1 \cos 30° + \mathbf{F}_2 \cos 45°$$
$$0 = \mathbf{F}_2 \sin 45° - \mathbf{F}_1 \sin 30°$$

which are indeed the two equations set up in the first method and give

$$\mathbf{F}_1 = 732.1 \text{ N and } \mathbf{F}_2 = 517.6 \text{ N}$$

It may be commented that if the x-axis was chosen in some other direction, say along $\mathbf{F}_1$ then the equations will be different. For x-axis along $\mathbf{F}_1$ the equations are

$$\mathbf{F}_1 + \mathbf{F}_2 \cos 75° = 1000 \cos 30°$$
$$\mathbf{F}_2 \sin 75° = 1000 \sin 30°$$

which again provide the same answer.

Alternatively, the components can be determined by observing that the components $\mathbf{F}_1$ and $\mathbf{F}_2$ add to result in 1000 N by way of a vector triangle as shown in Fig. Ex. R2.16(c). Applying the sine rule,

$$\frac{\mathbf{F}_1}{\sin 45^\circ} = \frac{\mathbf{F}_2}{\sin 30^\circ} = \frac{1000}{\sin 105^\circ}$$

whence, $\mathbf{F}_1 = \dfrac{1000 \sin 45^\circ}{\sin 75^\circ} = 732.1\text{ N}$

$$\mathbf{F}_2 = \frac{1000 \sin 30^\circ}{\sin 75^\circ} = 517.6\text{ N}$$

Another point can be discussed with reference to this example. If the two components desired were not constrained to lie in the plane containing the given force as stated in the question, they will still have to lie in the same plane for equivalence. Hence, the phrase 'in the same plane containing the given force' is indeed superfluous. If, instead, the given force was to be resolved into three or more components in the same plane, then an infinite number of combinations of the magnitudes would be possible even if their directions were specified. On the other hand, if the given force was to be resolved into three space components at given inclinations, it would be possible to determine their magnitudes uniquely. Again, an infinite combination of the magnitudes would be possible for four or more number of space components even in the specified directions.

Concept Review Questions

1. Match the following terms with the statements.

Terms	*Statements*
(a) Sliding vector	(a) Adding a reversed vector to a vector
(b) Bound vector	(b) Division of vector quantities
(c) Undefined	(c) A vector divided by its magnitude
(d) Zero vector	(d) The transmissibility principle of vectors
(e) Unit vector	(e) Equivalence of effect
(f) Equipollent vectors	(f) Application at a unique point

2. If a force $\mathbf{F} = F_r\,\mathbf{e}_r + F_\theta\,\mathbf{e}_\theta$ acts on a body with a position vector $\mathbf{r} = lr$ in the $\mathbf{r} - \theta$ plane, which of the following expressions would result in the moment by the force about the origin.
   (i) $\mathbf{r} \times \mathbf{F}$ (ii) $\mathbf{r} \cdot \mathbf{F}_\theta \mathbf{e}_\theta$
   (iii) $\mathbf{r} \times \mathbf{F}_r \mathbf{r}$ (iv) $\mathbf{r} \cdot \mathbf{F}_r\, \mathbf{e}_r$
3. Discuss why
   (a) the parallelogram law must be obeyed by the vector quantities
   (b) the commutation rule fails for the vector products
   (c) the scalar triple product results in the volume of a parallelopiped
   (d) the position vector and the force must be vector quantities.

4. Show geometrically or otherwise that
   (a) $\mathbf{A} + \mathbf{B} + n\,\mathbf{C} = n\,\mathbf{C} + \mathbf{B} + \mathbf{A}$
   (b) $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i}$
   (c) $\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$
5. A vector may vary in magnitude and in direction with the passage of time. Illustrate the concepts of
   (a) derivative of a vector with respect to time
   (b) integral of a vector with respect to time
   Give one example for each of the applications of differentiation and the integration operations.
6. Define the vector operator del, ∇, and illustrate the physical significance of
   (a) the gradient of a scalar ϕ
   (b) the divergence of a vector **A**
   (c) the curl of a vector **B**.
7. Prove that
   (a) $d\phi \equiv \nabla\phi \cdot d\mathbf{r}$
   (b) $\nabla \times \nabla\phi \equiv 0$
   (c) $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$
   where ϕ is a scalar, **A** is a vector and $d\mathbf{r}$ is a differential displacement vector.

Tutorial Problems

R2.1 A vector quantity of 100 units acts along a line *OP*, terminating at *P*. If the coordinates of *O* and *P* are (3, – 1, 2) and (10, 5, 8) respectively specify the vector quantity in terms of the unit vectors.

(**Ans.** 63.6 **i** + 54.5 **j** + 54.5 **k**)

R2.2 An object must be lifted from the ground point **P**(0, 0) to a point **Q**(0, 10) vertically above **P**. It is feasible to lift the object directed from (0, 0) to (3, 4) as far as necessary and then transport it horizontally to the destination. How long is the feasible path?

(**Ans.** 20 units)

R2.3 A point is located as (–5, 2, 14) with respect to an origin (0, 0, 0). Specify its position vector:
   (a) in terms of the rectangular components,
   (b) in terms of its direction cosines and
   (c) in terms of its unit vector.

(**Ans.** (a) $\mathbf{r} = -5\,\mathbf{i} + 2\,\mathbf{j} + 14\,\mathbf{k}$
(b) $\mathbf{r} = 15\,l\,\mathbf{i} + 15\,m\,\mathbf{j} + 15\,n\,\mathbf{k}$;
$l = -0.33$, $m = 0.13$, $n = 0.93$
(c) $\mathbf{r} = 15\,r$;
$r = -0.33\,\mathbf{i} + 0.13\,\mathbf{j} + 0.93\,\mathbf{k}$)

R2.4 For a triangle with sides of lengths *a*, *b* and *c* and the angles facing these sides *A*, *B* and *C* respectively, prove that
   (a) $c^2 = a^2 + b^2 - 2ab\cos C$ (cosine law) and
   (b) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ (sine law)

R2.5 An object is projected at a velocity of 70 m/s perpendicular to the plane containing vectors $\mathbf{A} = 2\,\mathbf{i} - 6\,\mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{B} = 4\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$. Express the velocity in terms of the unit vectors.

(**Ans.** 30 **i** – 20 **j** + 60 **k** m/s)

R2.6 Two vectors **A** and **B** have an included angle of 60°. If **A** = 4 **i** + 7 **j** + 6 **k** and **B** is recorded as 3 **i** + 3 **j** + ( ) **k** where the coefficient of **k** is missing, calculate this coefficient.

(Ans. – 1.7)

R2.7 If **A** = 2 **i** – 3 **j** – **k**, **B** = **i** + 4 **j** – 2 **k** and **C** = 2 **i**, evaluate the following:

(a) **A** · **B**
(b) **A** × **B**
(c) (**A** + **B**) · (**A** – **B**)
(d) (**A** + **B**) × (**A** – **B**)
(e) (**A** × **B**) · **C**
(f) (**A** + **B**) × **C**

(Ans. (a) –8 (b) 10 **i** + 3 **j** + 11 **k**
(c) –7 (d) –20 **i** – 6 **j** – 22 **k**
(e) 20 (f) –6 **j** – 2 **k**)

R2.8 Show that

(a) (2/3 **i** – 2/3 **j** + 1/3 **k**), (1/3 **i** + 2/3 **j** + 2/3 **k**) and (2/3 **i** + 1/3 **j** – 2/3 **k**) comprise a set of orthogonal unit vectors.
(b) (2 **i** – **j** + **k**), (**i** + 2 **j** – 3 **k**) and (3 **i** – 4 **j** + 5 **k**) are noncoplanar vectors.

R2.9 If **A** = $3t$ **i** + $2t^2$ **j** + 4 t^{-1} **k**, and **B** = $2t^2$ **i** + $3t$ **j** + t^4**k**, compute

(a) $\dfrac{d\mathbf{A}}{dt}$ (b) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B})$ (c) $\int_{t_1}^{t_2} \mathbf{A}(t)\,dt$ and (d) $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B})$

(Ans. (a) 3 **i** + 4 t **j** – $4t^{-2}$ **k** (b) $48t^2$
(c) $3/2(T_2^2 - T_1^2)$ **i** + $2/3(T_2^3 - T_1^2)$ **j** + 4 log (T_2/T_1) **k**
(d) $12t^5$ **i** + $(8 - 15t^4)$ **j** + $(18t - 16t^3)$ **k**)

R2.10 The motion of a point is described by the position vector

$$\mathbf{r} = (2 + 5t^2)\,\mathbf{i} + (4 - 3t^3)\,\mathbf{j}$$

the distance being in metres for the time lapsed in seconds. Compute the velocity and acceleration at the instant t = 2 seconds.

(Ans. V = (20 **i** – 36 **j**) m/s
a = (10 **i** – 36 **j**) m/s^2)

R2.11 A vertical pole is guyed by three cables *PA*, *PB* and *PC* tied at a common point *P* at 8 m above the ground. The base points of the cables are:

A(4, 0, 0), *B*(–1, 4, 0) and *C*(–2, –3, 0) m

If the tension in *PA* is 20 kN, calculate the tensions to be provided in *PB* and *PC* so that the resultant force exerted on the pole is vertical. Find the force exerted on the pole.

(Ans. 22 kN and 28.6 kN, 63 kN)

R2.12 A force 300 N in magnitude acts through a point *P*(1, 6, –5) directed towards another point *Q*(0, 4, –3). Calculate the moment of the force about a point *O*(1, 0, –1), if the distances are in metres.

(Ans. (400 **i** + 400 **j** + 600 **k**) N m)

R2.13 If **r** is the position vector of a point, show that

(a) div **r** = 3
(b) curl **r** = 0
(c) div ($\mathbf{r}^n$ **r**) = $(n + 3)\,\mathbf{r}^n$

R2.14 Interpret the significance of the following relations:

(a) **r** · d**r** = 0
(b) **r** × d**r** = 0
(c) ∇× **r** = 0

where **r** is a vector and d**r** its differential change.

R2.15 In a fluid flow the velocity of a particle is given by

$$\mathbf{V} = 2x\,\mathbf{i} - 2y\,\mathbf{j} + xy\,\mathbf{k} \text{ m/s}$$

where the distances are measured in metres. Referred to the origin (0, 0, 0) compute the cross product $\mathbf{r} \times \mathbf{V}$ for a fluid particle located at (2, 3, 4).

(**Ans.** 42 **i** + 4 **j** – 24 **k**)

R2.16 In a magnetic field the velocity of an electron is given by

$$\mathbf{V} = 100\,\mathbf{i} + 25\,\mathbf{j} \text{ m/s}$$

and the magnetic flux density by

$$\mathbf{B} = 0.01\,\mathbf{i} - 0.001\,\mathbf{j} \text{ Wb/m}^2$$

Compute the cross product $\mathbf{V} \times \mathbf{B}$ for the electron.

(**Ans.** –0.35 **k** Wb/s m)

R2.17 The Indian satellite Aryabhatta is imparted steadiness in its orbit by spinning it about the spin-axis as shown in Fig. Prob. R2.17. If the spin is maintained at 50 revolutions per minute, calculate the linear velocities of the solar cells

$$C_1\ (0.3 \text{ m}, 0.0 \text{ m}, 0.5 \text{ m}) \text{ and } C_2(0.6 \text{ m}, 0.2 \text{ m}, 0.5 \text{ m})$$

with respect to the spin-axis.

(**Ans.** –2.618 **j** m/s, 1.047 **k** – 2.618 **j** m/s)

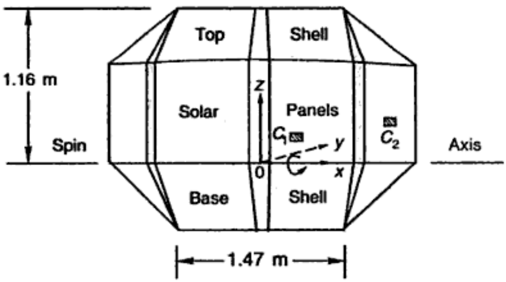


Fig. Prob. R2.17

R2.18 A force of 200 N must be replaced by two forces inclined at right angles to it on one side and at 45° on the other. Show that the magnitude of these forces should be 200 N and 282.6 N respectively.

Also show that, if the components are equally inclined to the given force, the magnitude of each component increases as the inclination with the given force increases until the inclination approaches $\pi/2$.

R2.19 The vertical mast of a flag is positioned by three ropes tied to a common point on the mast with their other ends fixed on the ground. If the angles between the ropes and the mast are 30°, 25° and 30° respectively and they are spaced equally apart in the plan view, determine the magnitudes of the forces in the ropes if the net downward force at the common point is 1000 N. (**Ans.** 422, 357, 357 N)

R2.20 Determine the magnitude of each of the three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ which when put together at a point result in a single force 100 N in a given direction ($\mathbf{e} = 0.6\,\mathbf{i} + 0.8\,\mathbf{j}$). The unit vectors in the respective directions are as follows:

$$\mathbf{e}_1 = 0.5\,\mathbf{i} + 0.5\,\mathbf{j} + 0.707\,\mathbf{k}$$

$$\mathbf{e}_2 = 0.707\ \mathbf{i} + 0.707\ \mathbf{k}$$

$$\mathbf{e}_3 = 0.8\ \mathbf{i} - 0.6\ \mathbf{j}$$

(Ans. –363, 363 and 169 N)

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. Two vectors are equal if
   (a) their magnitudes are equal
   (b) their magnitudes and directions are the same
   (c) they are equal in magnitude and are collinear
   (d) their magnitudes, direction and the sense are the same and they may lie anywhere in space
2. The magnitude of a vector quantity is
   (a) the dot product of the vector with itself
   (b) the cross product of the vector with unit vector along itself
   (c) the dot product of the vector with unit vector along itself
   (d) the cross product of the vector with itself
3. Orthogonality of two vectors demands that
   (a) their dot product equals unity
   (b) the magnitude of the dot and cross products are equal
   (c) their cross product vanishes
   (d) their cross product equals unity
4. If the dot product of two vectors is zero, then
   (a) either of the vectors or both must be zero
   (b) the vectors must be perpendicular to each other
   (c) either (a) or (b) is satisfied
   (d) the vectors must be concurrent
5. If the cross product of two vectors is zero, then
   (a) either of the vectors or both must be zero
   (b) the vectors must be parallel to each other
   (c) the vectors must be perpendicular to each other
   (d) the vectors must be collinear
6. The derivative of a vector with respect to a scalar must be
   (a) in the direction of the given vector
   (b) perpendicular to the given vector
   (c) zero if the vector has a constant magnitude
   (d) zero if the vector is a constant vector
7. The integral of a vector with respect to a scalar
   (a) results in a vector in the direction of the given vector
   (b) is called the line integral
   (c) must be a definite integral
   (d) may have a direction other than that of the given vector.
8. The gradient of a scalar function
   (a) must be a scalar quantity
   (b) must be a vector in the direction of the steepest variation of the scalar

   (c) must be a vector with its magnitude equal to the scalar
   (d) is undefinable
9. The divergence of a vector
   (a) implies the net efflux of the vector at a point in space
   (b) must be a vector quantity
   (c) is less than the curl of the vector
   (d) implies the continuity of the vector space
10. The curl of a vector
   (a) may or may not be a vector
   (b) refers to the rotationality of the vector field
   (c) refers to the efflux of the vector
   (d) vanishes if the vector has a constant magnitude
11. The linear momentum of a particle
   (a) must be directed along the velocity of the particle
   (b) is the dot product of the mass with its velocity
   (c) is the cross product of the mass with its velocity
   (d) is a scalar quantity
12. The angular momentum of a particle is the
   (a) linear momentum per unit angle
   (b) product of the mass with its angular velocity
   (c) moment of the product of the mass and the angular velocity about an origin
   (d) cross product of the position vector and the linear momentum
13. The simplest resultant of a plane force system is always
   (a) a single force
   (b) a wrench
   (c) a single moment
   (d) a single force or a single moment
14. A force acts in the plane of a paper from top to bottom. An anticlockwise moment is applied on it. The line of action of the force will shift parallel to itself to get the simplest resultant
   (a) above the plane of the paper
   (b) under the plane of the paper
   (c) to the right of the given line of action of force in the same plane
   (d) to the left of the given line of action of force in the same plane
15. A plane system of forces has a single force resultant if the sum of the moments
   (a) about the origin is zero
   (b) about any point on the plane is zero
   (c) about any point on a particular line in that plane is zero
   (d) about any point on or outside the plane is zero.
16. For a plane system of forces to have the simplest resultant as a single moment
   (a) the forces must be parallel
   (b) the force system must constitute moments and/or couples only
   (c) the force system cannot have the forces as concurrent
   (d) the force system cannot have an odd number of forces

Answers to Multiple-Choice Questions

1 (d),	2 (c),	3 (e),	4 (c),	5 (b),	6 (d),	7 (d),
8 (b),	9 (a),	10 (b),	11 (a),	12 (d),	13 (d),	14 (c),
15 (c),	16 (b)					

2 FORCES AND FORCE SYSTEMS

Resultant, Equivalence and Origin of Forces

2.1 FORCE AND MOMENT CONCEPTS

It has been stated in Chapter 1 that an 'action' which changes or tends to change the state of rest or of uniform motion of a body must be a force or a moment. A force when exerted on the centre of mass of a body causes or tends to cause a change of state of rest or of uniform rectilinear motion of the body. The action of a moment causes or tends to cause a rotational motion of a body. The moment of a force about a point has been defined as the turning effect of the force about that point

$$\boxed{\mathbf{M} = \mathbf{r} \times \mathbf{F}} \tag{2.1}$$

$$= rF \sin \theta \text{ or } F\,p, \text{ perpendicular to the plane containing } \mathbf{r} \text{ and } \mathbf{F}$$

where $\mathbf{F}$ is the force, $\mathbf{r}$, the position vector of any point on the line of action of the force with the origin at the point about which the moment is desired, p the perpendicular distance from the point to the line of action of the force and θ the angle between $\mathbf{r}$ and $\mathbf{F}$ as shown in Fig. 2.1.

Force and moment concepts are most important in the study of Newtonian mechanics. As a matter of fact, the laws of Newton and Euler lay the foundation of force-based mechanics. An alternative formulation called energy-based mechanics is based on energy concepts where force and moment do not play the primary role. Since we are concerned with the Newtonian formulation of mechanics, we shall have to allot special status to force and moment concepts.

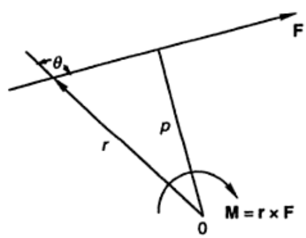


Fig. 2.1 ***Concepts of Moment due to a Force***

A body may be subjected to a single force or a moment or a system of forces under the action of which the body may stay at rest, be in uniform motion or in general motion. It is important, therefore, to recognise the total effect of a system of forces acting on a body. This is represented by the concepts of resultant and equivalent systems which are discussed at length for different force systems acting on a particle or rigid body. The application of the resultant concept for a rigid body is appreciated by using the principle of transmissibility of a force; the principle being applicable to a rigid body but not to a deformable body.

Forces may originate from a variety of circumstances. It is neither possible nor desirable to cover all the possible modes of origin and the nature of forces but it should be helpful to introduce the concepts of gravitational force field, hydrostatic force field and frictional and drag forces. The origin and nature of forces is a fundamental subject and must be conceived before their effect is dealt with in statics and dynamics. There is little scope of solving any numericals on the nature of forces at this stage. Appropriate and adequate number of examples will be taken up in the context of statics and dynamics during our study.

***Example* 2.1** A 50 cm × 30 cm plate is acted on by a 10 kN force at B in the plane of the plate as shown in Fig. Ex. 2.1. Determine the moment of the force about D and about A.

Solution In order to determine the moment about D, fix the origin at D and let the x and y axes be along DC and DA respectively as shown in Fig. Ex. 2.1 (Solution).

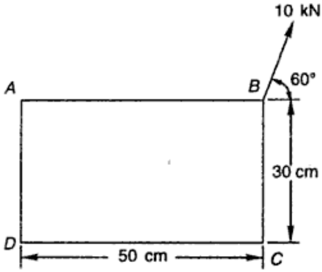


Fig. Ex. 2.1

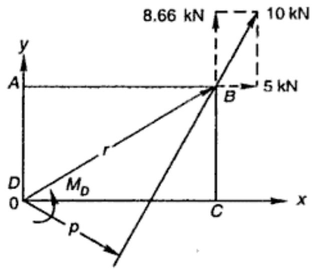


Fig. Ex. 2.1 ***(Solution)***

The force is specified as

$$\mathbf{F} = 10 \cos 60° \, \mathbf{i} + 10 \sin 60° \, \mathbf{j}$$
$$= 5 \, \mathbf{i} + 8.66 \, \mathbf{j} \text{ kN}$$

and the position vector for a point B on the line of action of the force is

$$\mathbf{DB} = 0.5 \, \mathbf{i} + 0.3 \, \mathbf{j} \text{ m}$$

The moment of the force about D is, therefore,

$$\mathbf{M}_D = \mathbf{DB} \times \mathbf{F}$$
$$= (0.5 \, \mathbf{i} + 0.3 \, \mathbf{j}) \times (5 \, \mathbf{i} + 8.66 \, \mathbf{j})$$
$$= 2.83 \, \mathbf{k} \text{ kN m}$$

It may also be noticed that the moment equals

$$M_D = 10 \, p$$

where p is the perpendicular distance from D to the line of action of the force. From the knowledge of this distance by a drawing to scale or by trigonometry, the moment is determined as

$$M_D = 10 \times 0.283 = 2.83 \text{ kN m}$$

directed anticlockwise, i.e., along the positive z direction.

Similarly, the moment about point A may be evaluated by selecting the origin at A and the axes in the same directions as before

$$\mathbf{AB} = 0.5\ \mathbf{i}$$

$$\mathbf{F} = 5\ \mathbf{i} + 8.66\ \mathbf{j}$$

$$\mathbf{M}_A = \mathbf{AB} \times \mathbf{F} = 0.5\ \mathbf{i} \times (5\ \mathbf{i} + 8.66\ \mathbf{j})$$

$$= 4.33\ \mathbf{k} \text{ kN m}$$

2.2 FORCE FIELDS: LINEAR, PLANE AND SPATIAL

A force may be originated by the action of one body on another body in contact with it or by virtue of a force field. A force field implies the existence of a force as a function of the space coordinates and time. In general, in a force field,

$$\mathbf{F} = \mathbf{F}(x, y, z, t) = \mathbf{F}(r, \theta, z, t)$$

A force field is said to be *steady* if, at any point in space, it is independent of time, i.e.,

$$\mathbf{F} = \mathbf{F}(x, y, z) = \mathbf{F}(r, \theta, z)$$

Force fields are classified as linear, plane and spatial as follows:

Linear Force Field

A linear force field is confined to a line only, i.e., the force at any point is a function of one dimension only.

$$\mathbf{F} = \mathbf{F}(x) \text{ or } \mathbf{F} = \mathbf{F}(s)$$

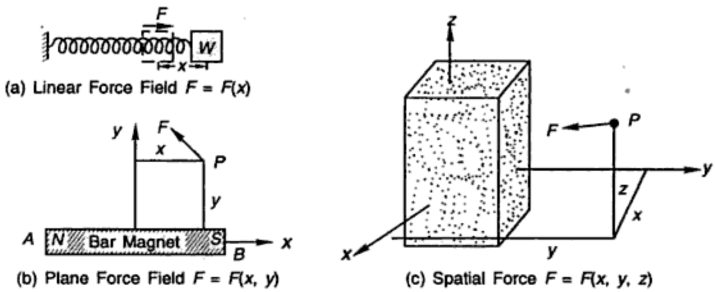
A linear spring is an example of a linear force field. As shown in Fig. 2.2(a), the force $\mathbf{F}$ on the weight W by the spring is a function of the horizontal displacement x of the weight only.

Plane Force Field

A plane force field refers to a field where the force varies with two space coordinates, i.e., x, y or r, θ, etc.

$$\mathbf{F} = \mathbf{F}(x, y) \text{ or } \mathbf{F}(r, \theta)$$

The magnetic field generated by a current-carrying conductor with rectangular cross-section AB is an example of a plane force field. The force exerted on any magnetic particle near it is a function of its two coordinates (x, y) as shown in Fig. 2.2(b).



(a) Linear Force Field $F = F(x)$

(b) Plane Force Field $F = F(x, y)$

(c) Spatial Force $F = F(x, y, z)$

Fig. 2.2 ***Force Fields***

Spatial Force Field

A spatial force field is a general force field in which the force varies with respect to the position of the point in space, i.e., with all the three coordinates x, y, z, or r, θ, z, etc.

$$\mathbf{F} = \mathbf{F}(x, y, z) \text{ or } \mathbf{F} = \mathbf{F}(r, \theta, z)$$

The gravitational force field generated by a body is an example of a spatial force field. The force exerted on any particle P towards the body is a function of the position of this particle with respect to the body in space, i.e., x, y, z as shown in Fig. 2.2(c).

***Example* 2.2** A force field represented by

$$\mathbf{F} = 6\,xy\,\mathbf{i} + 3\,x\,\mathbf{j}$$

acts on a circular plate of 2 m radius placed in the x-y plane with the z-axis passing through the centre of the plate. Determine the force at some salient points on the periphery of the plate.

Solution Let us consider the points A, B, C, D, E, F, G and H on the periphery as shown in Fig. Ex. 2.2. The forces at these points are determined by substituting the

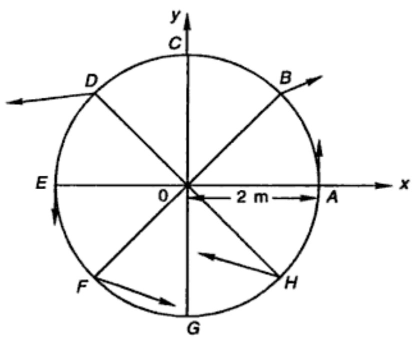


Fig. Ex. 2.2

values of corresponding x and y. For example, at point $A(2, 0)$,

$$\mathbf{F} = 6 \times 2 \times 0\ \mathbf{i} + 3 \times 2\ \mathbf{j} = 6\ \mathbf{j}$$

Point	x	y	*Force* **F**
A	2	0	$6\ \mathbf{j}$
B	1.414	1.414	$12\ \mathbf{i} + 4.242\ \mathbf{j}$
C	0	2	0
D	–1.414	1.414	$-12\ \mathbf{i} - 4.242\ \mathbf{j}$
E	–2	0	$-6\ \mathbf{j}$
F	–1.414	–1.414	$12\ \mathbf{i} - 4.242\ \mathbf{j}$
G	0	–2	0
H	1.414	–1.414	$-12\ \mathbf{i} + 4.242\ \mathbf{j}$

2.3 DISTRIBUTED FORCE FIELDS

A distributed force field is characterised by the action of a continuously distributed force. Such forces may act over a line, a surface or a volume; these are correspondingly denoted as *lineal, surface* and *body* forces.

Lineal Force

A lineal force is one that acts along a line on the body. The force $d\mathbf{F}$ at any small length dl is given by

$$d\mathbf{F} = \omega\ dl$$

where ω is the intensity of loading at dl.

An example of the lineal distributed force is a loaded cable as shown in Fig. 2.3(a).

Surface Force

A surface force acts over a surface and the force $d\mathbf{F}$ at any area dA is given by

$$d\mathbf{F} = p\ dA$$

where p the intensity of force per unit area is generally termed as pressure.

Hydrostatic pressure acting on the surface of a cylinder immersed in water as shown in Fig. 2.3 is an example of the surface force.

Body Force

A body force is essentially the force exerted on the mass or volume content of the body.

Thus the body force $d\mathbf{F}$ on an element of mass dm or volume dV is given as

$$d\mathbf{F} = g\ dm$$

or

$$d\mathbf{F} = \gamma\ dv$$

where g is the force per unit mass on the small mass dm or γ is the force per unit volume on the volume dv.

An example of body force is the force exerted on a body due to gravitational attraction of the earth as shown in Fig. 2.3(c). The force is distributed throughout the volume or mass of the body.

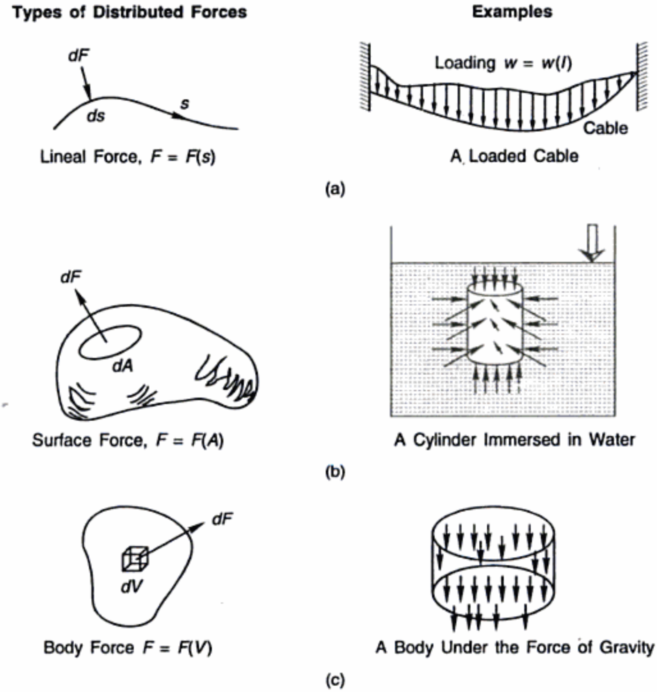


Fig. 2.3 ***Distributed Forces***

2.4 FORCE SYSTEM ACTING ON A BODY

A body may be subjected to a number of forces in a specified manner

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

acting such that the position vectors of any points on their lines of action are

$$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$$

respectively referred to an arbitrary origin O.

Some of the forces may be equal and opposite and may constitute couples. The couples are sometimes bracketed separately from the other forces and are taken into account by way of couple moments. There may, in addition, be moments acting otherwise. Thus, the body may be acted upon by moments

$$\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \ldots$$

The forces acting in a specified manner and the moments acting on the body constitute a force system or a system of forces as represented in Fig. 2.4. A force system is sometimes referred to as an 'external action' on the body.

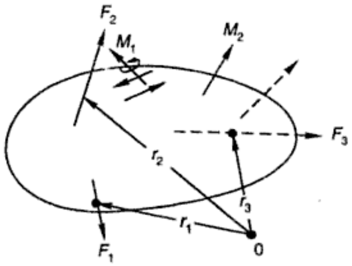


Fig. 2.4 ***A Force System Acting on a Body***

The force system acting on a body may consist of forces which may be qualified in one or more of the following classifications:

1. *Concurrent force systems:* Collinear, planar or spatial
2. *Parallel force systems:* Planar or spatial
3. *Coplanar force systems*: Concurrent and non-concurrent, parallel and non-parallel
4. *Spatial force systems*: Concurrent and non-concurrent, parallel and non-parallel.

It is obvious that a particle may only be subjected to a concurrent force system whereas a rigid body may be under the action of any force system.

2.5 RESULTANT OF A FORCE SYSTEM

The resultant of a force system acting on a body implies the net external action on the body. In other words, the resultant action is a simple equivalent force system which can replace the given force system for an equivalence of effect so far as motion or tendency of motion of the body is concerned. The definition of a rigid body permits no internal dimensional or structural changes within the body; the resultant concept applied to a rigid body stands for complete equivalence of action and is therefore highly meaningful. Similarly, a particle conceived as a relatively small or point object allows the resultant concept to be used to advantage due to complete equivalence of action represented by it. We shall, therefore, confine ourselves to the resultant concept for a particle and a rigid body.

The action of a force is two-fold: first, in its own right as a translational action and second, to generate a moment or rotational action about an arbitrary point or an axis. It follows that the resultant of a force system should, in general, comprise of (a) a force and (b) a moment, as shown in Fig. 2.5.

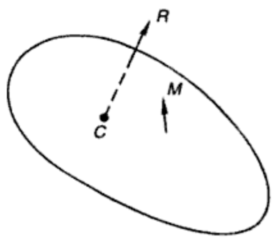


Fig. 2.5 ***Resultant Action for the Force System Acting on a Body***

It is necessary to qualify the point of action or line of action of the force and the direction of the moment as will be shown later. It is also likely that a force system can result in a force only or in a moment only for certain force fields and certain choices of the point of action of the resultant.

Sometimes, the resultant of a force system is referred to as the equivalent or equipollent action. It is, in general, incorrect to term the resultant as the equivalent because the equivalence of an action has a wider implication. The equivalence of action on a body may be desired with different objectives. For example, the objective may be to study the motion of the body, to analyse its internal forces or to compute its deformation or rate of deformation. Thus, by definition, the equivalent system of forces for a given system of forces is such that it produces the same desired effect. In particular, for a rigid body in motion or having a tendency of motion, the analysis of motion can be made with the resultant replacing the given system of forces. The resultant is, therefore, the equivalent action of a system of forces for the dynamic consideration of a rigid body.

It may be seen that the resultant of a plane system of forces must be a force in that plane which may be accompanied by a couple-moment in a direction normal to that plane. Similarly, the resultant of a system of parallel forces should be a force parallel to them which may be accompanied by a moment in a direction normal to the parallel forces. The resultant of a system of concurrent forces should be a single force which must pass through the point of concurrency. These statements cannot be taken for granted; let us discuss the individual cases.

2.6 PRINCIPLE OF PARALLEL TRANSFER OF A FORCE

Consider a force **F** acting through a point p_1 on a rigid body as shown in Fig. 2.6(a). If it is desired that the force be applied through a point p_2 on the body, then the force **F** applied at p_2 must be accompanied by a moment with magnitude

$$M = Fd$$

perpendicular to the plane of transfer of the force for equivalence where d is the perpendicular distance of the parallel transfer of force. This is the principle of translation of a force to a parallel position. The principle can be proved by imagining a pair of equal and opposite forces **F** and –**F** adding to null at point p_2 in the first instance. The perpendicular distance from p_2 to the given force is d. The system of three forces thus constituted as shown in Fig. 2.7(b), may be visualised as a force **F** acting through p_2 and a couple of forces **F** and –**F** acting through points p_1 and p_2 respectively. The couple of forces comprise a couple-moment **M** given by

$$M = Fd \tag{2.2}$$

in magnitude, perpendicular to the plane of transfer of the force **F**.

The sense of the accompanying couple-moment may either be visualised with the help of a neat sketch or observed vectorially as follows:

Let the position vector of a point p_2 on the new line of action of **F**, called $\mathbf{F}_{p_2}$, with reference to a point p_1 on the initial line of action of the force, be $\mathbf{r}_{p_2p_1}$. Then, the moment vector accompanying the transferred force is

$$\mathbf{M} = -\,\mathbf{r}_{p_2p_1} \times \mathbf{F}_{p_2}$$

specified completely in magnitude and direction. If the magnitude of the displace-

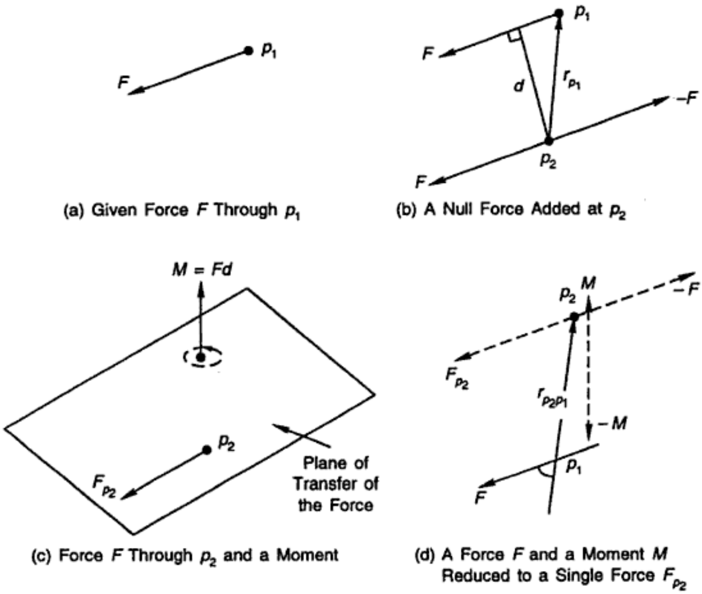


Fig. 2.6 ***Principle of Parallel Transfer of Force***

ment vector is d and it is the perpendicular distance through which the force is transferred, then the direction of the accompanying moment of magnitude

$$M = Fd$$

is equal and opposite to the moment exerted by $\mathbf{F}_{p_2}$ about a point on the initial line of action of the force as shown in Fig. 2.6(c).

Let us now discuss the reverse problem. If a force **F** and a moment **M** act on a rigid body such that the force passes through a point p_1 and the moment has no component along the direction of the force then the given system of **F** and **M** may be replaced by a single parallel force $\mathbf{F}_{p_2}$ passing through the point p_2.

The transfer of **F** so as to pass through p_2 would have required as accompanying moment

$$-\mathbf{r}_{p_2p_1} \times \mathbf{F}_{p_2}$$

which, in this case, must nullify the existing moment **M** normal to the plane of transfer. Hence,

$$-\mathbf{M} = -\mathbf{r}_{p_2p_1} \times \mathbf{F}_{p_2}$$

or

$$\mathbf{M} = \mathbf{r}_{p_2p_1} \times \mathbf{F}_{p_2} \tag{2.3}$$

The point p_2 can be located with respect to p_1 by solving Eq. (2.4) by way of decomposing it into scalar equations as shown in Fig. 2.6(d).

A force acting through a point and a moment can be replaced by a single parallel force for equivalence if the moment has no component in the direction of the force. Let us observe the difficulty when the moment has a component in the direction of the force. Clearly, the transfer of the force parallel to itself involves the application of a moment perpendicular to the plane of transfer of the force; no moment along the direction of the force may creep in. Conversely, any amount of transference of a force parallel to itself cannot involve a moment along the direction of the force so as to nullify the given component along that direction. The simplest system to which an arbitrary force acting at a point and a moment acting on the rigid body may be reduced is the force parallel to itself and a moment along the line of action of the force. The simplest equivalent form, i.e., a force and couple-moment directed along the force is called *wrench*. If the force in the wrench is displaced to any parallel position, the moment directed along the force will remain unaltered and an additional moment arising from the parallel transfer of the force will be required.

The concept of a wrench is explained with reference in Fig. 2.7(a) where an arbitrary force **F** passing through a point p_1 in a rigid body and an arbitrary moment **M** are given. The moment has a component $\mathbf{M}_f$ along the direction of the force and a component $\mathbf{M}_n$ normal to the direction of the force. The force **F** may be transferred to act at another point p_2 in the body such that it is equivalent to **F** at p_1 and the component $\mathbf{M}_n$ of **M**. There is no possibility for the component $\mathbf{M}_f$ in the direction of the force to be taken care of by a parallel transfer of **F**. The simplest equivalent system which remains is, therefore, a force **F** through p_2 and a moment $\mathbf{M}_f$ in the direction of the force. This system, called a wrench, is shown in Fig. 2.7(b).

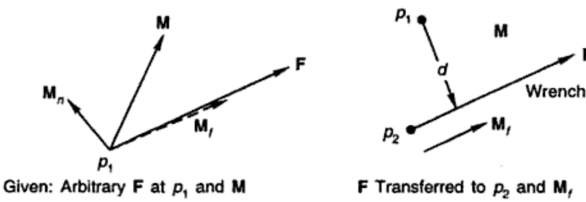


Fig. 2.7 ***Concept of a Wrench on a Rigid Body***

***Example* 2.3** A rigid parallelopiped is made of parallel bars *AB, CD, EF* and *OP* as shown in Fig. Ex. 2.3. A force **F** of 10 kN acting along *CD* is to be replaced by an equivalent action by applying the force along any of the other parallel bars. Determine the equivalent action in each case.

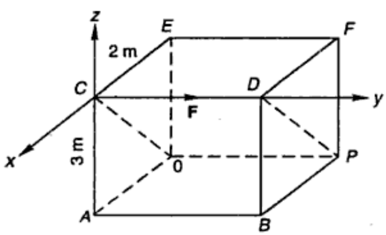


Fig. Ex. 2.3

Solution

If a 10 kN force were to act along

EF, it should be accompanied by a moment

$$M = 10 \times 2 = 20 \text{ kN m}$$

along the positive z-axis or by the moment vector

$$\mathbf{M} = -\,\mathbf{r}_{EC} \times \mathbf{F}_E$$
$$= -\,(-2\,\mathbf{i}) \times 10\,\mathbf{j} = 20\,\mathbf{k} \text{ kN m}$$

If it were to act along *AB*, the accompanying moment would be

$$M = 10 \times 3 = 30 \text{ kN m}$$

along the negative x-axis or by a moment vector

$$\mathbf{M} = -\,\mathbf{r}_{AC} \times \mathbf{F}_A$$
$$= -(-3\,\mathbf{k}) \times 10\,\mathbf{j} = -30\,\mathbf{i} \text{ kN m}$$

Similarly, if the force were transferred to the bar *OP*, the moment accompanying it would be

$$M = 10 \times \sqrt{(3^2 + 2^2)} = 36.06 \text{ kN}$$

along a direction normal to the plane containing *CD* and *OP*, i.e., normal to the plane *OCDP*.

Vectorially, the accompanying moment would be

$$\mathbf{M} = -\mathbf{r}_{OC} \times \mathbf{F}_0$$
$$= -(-2\,\mathbf{i} - 3\,\mathbf{k}) \times 10\,\mathbf{j} = 20\,\mathbf{k} - 30\,\mathbf{i} \text{ kN m}$$

***Example* 2.4** A force **F** acts at a position vector **r**:

$$\mathbf{F} = 5\,\mathbf{i} + 6\,\mathbf{j} + 4\,\mathbf{k} \text{ kN}$$
$$\mathbf{r} = -2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k} \text{ m}$$

A couple-moment **M** also acts:

$$\mathbf{M} = 2\,\mathbf{i} + 3\,\mathbf{j} \text{ kN m}$$

It is desired to replace the system by a wrench. Specify the equivalent wrench.

Solution The unit vector in the direction of the force **F** is

$$\mathbf{f} = \frac{5\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}}{\sqrt{5^2 + 6^2 + 4^2}} = 0.57\,\mathbf{i} + 0.684\,\mathbf{j} + 0.456\,\mathbf{k}$$

The given couple moment may be resolved into a component $\mathbf{M}_f$ in the direction of the force and a component $\mathbf{M}_n$ normal to it.

$$\mathbf{M}_f = \mathbf{M} \cdot \mathbf{f}$$
$$= (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot (0.57\,\mathbf{i} + 0.684\,\mathbf{j} + 0.456\,\mathbf{k})$$
$$= 1.14 + 2.05 = 3.19 \text{ kN m}$$

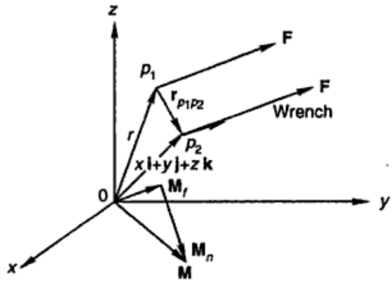


Fig. Ex. 2.4 ***(Solution)***

$$\mathbf{M}_n = \mathbf{M} - \mathbf{M}_f$$

$$= (2\,\mathbf{i} + 3\,\mathbf{j}) - 3.19(0.57\,\mathbf{i} + 0.684\,\mathbf{j} + 0.456\,\mathbf{k})$$

$$= 0.18\,\mathbf{i} + 0.82\,\mathbf{j} - 1.45\,\mathbf{k}$$

The component $\mathbf{M}_f = 3.19(0.57\,\mathbf{i} + 0.684\,\mathbf{j} + 0.456\,\mathbf{k})$

$$= 1.82\,\mathbf{i} + 2.18\,\mathbf{j} + 1.45\,\mathbf{k}$$

is there to stay as a component of the wrench but the normal component $\mathbf{M}_n$ may be eliminated by way of parallel transfer of the force. Let the force be transferred to a new position p_2 defined by a position vector

$$x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The displacement vector for the force is

$$\mathbf{r}_{p_2 p_1} = (x - (-2))\,\mathbf{i} + (y - 3)\,\mathbf{j} + (z - 4)\,\mathbf{k}$$

$$= (x + 2)\,\mathbf{i} + (y - 3)\,\mathbf{j} + (z - 4)\,\mathbf{k}$$

The parallel transfer must be accompanied by a moment vector

$$-\,\mathbf{r}_{p_2 p_1} \times \mathbf{F}_{p_2}$$

$$= -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x+2 & y-3 & z-4 \\ 5 & 6 & 4 \end{vmatrix}$$

which should be negative of $\mathbf{M}_n$ in order to nullify it, i.e., equal to $-\,0.18\,\mathbf{i} - 0.82\,\mathbf{j} + 1.45\,\mathbf{k}$.

The solution of this equation provides x, y and z which implies that the equivalent wrench is such that the force $5\,\mathbf{i} + 6\,\mathbf{j} + 4\,\mathbf{k}$ passes through this point $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and the accompanying moment in the direction of the force is $1.82\,\mathbf{i} + 2.18\,\mathbf{j} + 1.45\,\mathbf{k}$.

***Example* 2.5** Replace the force system consisting of three forces shown in the figure by a wrench passing through a point in the yz plane.

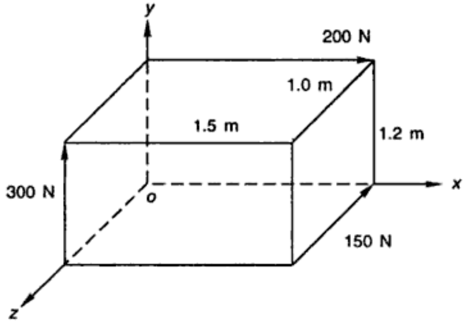


Fig. Ex. 2.5

Solution The resultant force $\mathbf{R} = 200\,\mathbf{i} + 300\,\mathbf{j} - 150\,\mathbf{k}$ N. The moment of the given forces about the origin o is given by

$$\mathbf{M} = 1.2\,\mathbf{j} \times 200\,\mathbf{i} + 1\,\mathbf{k} \times 300\,\mathbf{j} + 1.5\,\mathbf{i} \times (-150\,\mathbf{k})$$
$$= -300\,\mathbf{i} + 225\,\mathbf{j} - 240\,\mathbf{k} \text{ Nm.}$$

Let the wrench be located at point $P(o, y, z)$ located by the position vector $y\,\mathbf{j} + z\,\mathbf{k}$. The moment must be $M\mathbf{i}$, i.e. normal to yz plane.

Then,

$$(y\,\mathbf{j} + z\,\mathbf{k}) \times (200\,\mathbf{i} + 300\,\mathbf{j} - 150\,\mathbf{k}) + M\mathbf{i} = -300\,\mathbf{i} + 225\,\mathbf{j} - 240\,\mathbf{k}.$$

whence,
$$300\,z + 150\,y = M$$
$$200\,z = 225$$
$$200\,y = 240$$
$$y = 1.2 \text{ m},\; z = 1.125 \text{ m},\; M = 217.5 \text{ Nm}$$

The desired wrench, therefore, consists of a force $200\,\mathbf{i} + 300\,\mathbf{j} - 150\,\mathbf{k}$ N and a moment $217.5\,\mathbf{i}$ Nm at the point (0, 1.2, 1.125 m).

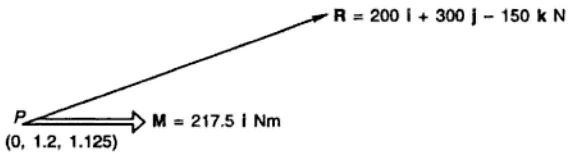
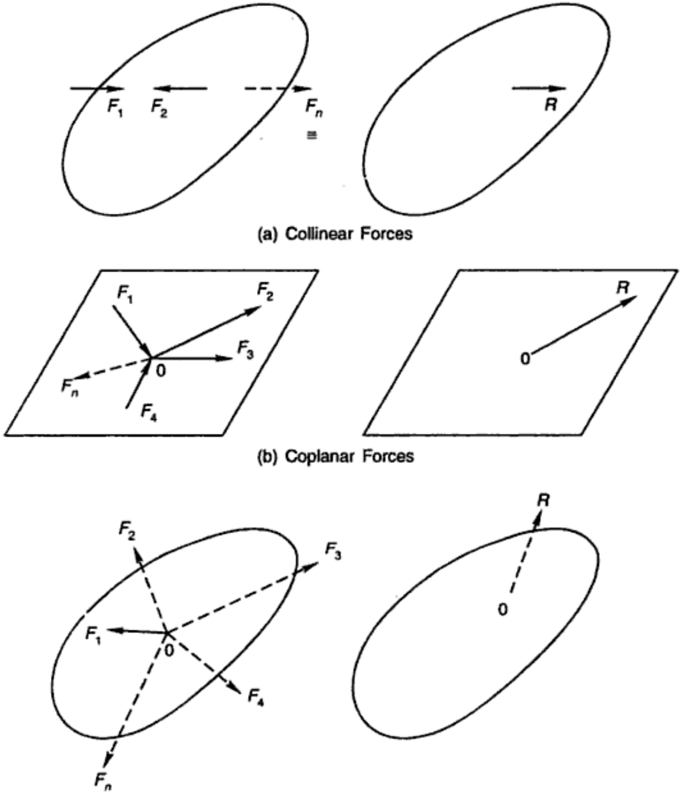


Fig. Ex 2.5 *(Solution)*

2.7 RESULTANT OF A CONCURRENT FORCE SYSTEM

A concurrent force system may be collinear, coplanar or spatial; collinear if the forces have the same line of action, coplanar if the lines of action of the forces lie in a plane and spatial if the lines of action of the forces lie in space as shown in Fig. 2.8.



(c) Spatial Forces

Fig. 2.8 ***Concurrent Force Systems***

(a) Collinear Forces

A system of collinear forces

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

acting on a body may be replaced by a single resultant force **R** acting in the same line of action as the given forces where

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots \tag{2.4}$$

In other words, the sum of the forces in the collinear force system must provide the resultant.

(b) Coplanar Concurrent Forces

A system of coplanar concurrent forces

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

acting on a body may be replaced by a single resultant force **R** passing through the point of concurrency in that plane where

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$$

In terms of rectangular components,

$$\mathbf{F}_1 = F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j}$$

$$\mathbf{F}_2 = F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j}$$

$$\mathbf{F}_3 = F_{3x}\,\mathbf{i} + F_{3y}\,\mathbf{j}$$

and

$$\mathbf{R} = (F_{1x} + F_{2x} + \ldots)\mathbf{i} + (F_{1y} + F_{2y} + \ldots)\,\mathbf{j}$$

$$= R_x\,\mathbf{i} + R_y\,\mathbf{j} \tag{2.5}$$

Geometrically, a polygon of forces can be constructed to add the coplanar concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, ... to result in **R**.

(c) Spatial Concurrent Forces

A system of concurrent forces not confined to a plane,

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

acting on a body may also be replaced by a single resultant force **F** passing through the point of concurrency such that

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$$

$$= (F_{1x} + F_{2x} + \ldots)\,\mathbf{i} + (F_{1y} + F_{2y} + \ldots)\,\mathbf{j} + (F_{1z} + F_{2z} + \ldots)\,\mathbf{k}$$

$$= R_x\mathbf{i} + R_y\,\mathbf{j} + R_z\mathbf{k} \tag{2.6}$$

In view of the space distribution of the forces, the geometrical construction, though possible is not feasible for adding the spatial forces because of the complexity of drawing space diagrams.

***Example* 2.6** At a point *P* on a vertical mast three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act:

$$\mathbf{F}_1 = 50\,\mathbf{i}$$

$$\mathbf{F}_2 = -30\,\mathbf{i} - 15\,\mathbf{j}$$

$$\mathbf{F}_3 = -25\,\mathbf{i} - 10\,\mathbf{j} + 5\,\mathbf{k}$$

Determine the resultant force at *P*.

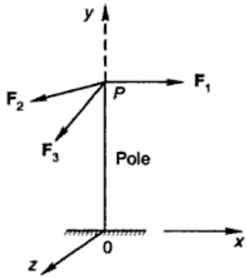


Fig. Ex. 2.6

Solution The resultant of the concurrent forces is given by

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$
$$= 50\,\mathbf{i} - 30\,\mathbf{i} - 15\,\mathbf{j} - 25\,\mathbf{i} - 10\,\mathbf{j} + 5\,\mathbf{k}$$
$$= -5\,\mathbf{i} - 25\,\mathbf{j} + 5\,\mathbf{k}$$

This single force when applied at P results in the same dynamic action as that exerted by the given system of concurrent forces.

2.8 RESULTANT OF A PARALLEL FORCE SYSTEM

A force system consisting of parallel forces may be planar or spatial: *planar*, if the parallel forces lie in a plane and the moments applied externally or couple-moments by the applied forces are directed normal to the plane and *spatial*, if the parallel forces are in space or the moments applied externally are directed in arbitrary directions.

An analysis of parallel force systems such as that of concurrent force systems can be made under the heads of plane and spatial systems but the parallel force systems constitute an important class in engineering and will, therefore, be dealt with separately.

(a) Plane Parallel Force System

Consider a simple case of two parallel forces. Any two parallel forces must be coplanar. Let the two parallel forces be equal in magnitude. If they are in the same sense as in Fig. 2.9(a,i), they add up to result in a single force which has twice the magnitude of each force, directed parallel to them and in the same sense as the constituent forces. The forces also produce a net moment about any point in the plane. The net moment is zero about any point midway between the forces. The net effect of the pair of forces, in this case, is to result in a resultant force **R** as well as an appropriate moment about the point considered. If they are in the opposite sense as in Fig. 2.9(a,ii), they add up to result in a null force. In addition, they have a net turning effect. In fact, they constitute a *couple* and the net turning effect is denoted by the moment of the couple. *The moment of the couple* is determined as

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

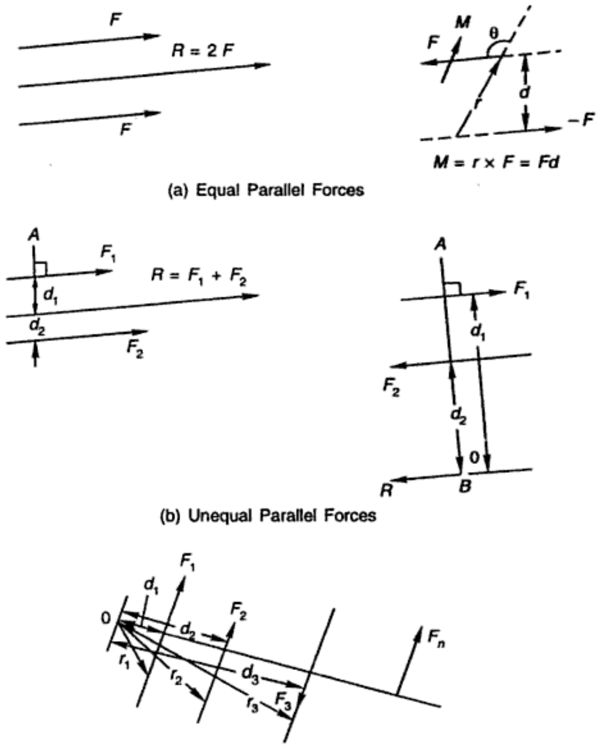
where **r** is the position vector of a point on the line of action of one force with respect to an origin taken on the line of action of the other and **F** is the former force.

Alternatively, the magnitude

$$M = rF \sin\theta$$
$$= Fr \sin\theta = Fd$$

where d is the perpendicular distance between the forces constituting the couple and θ is the angle between **r** and **F**.

It follows that the moment of the couple is directed perpendicular to the plane containing **r** and **F**, i.e., perpendicular to the plane of the couple. It can be concluded,



(a) Equal Parallel Forces

(b) Unequal Parallel Forces

(c) General Coplanar Parallel Forces

Fig. 2.9 ***Resultant of Parallel Forces***

therefore, that a pair of equal parallel forces with opposite sense can be replaced by an equivalent moment. It must be understood that the moment is a free vector; the directions and the lines of action of the forces constituting the couple are of no consequence. Whether the couple is in a particular plane or in any parallel plane is also immaterial. The moment may well be due to an equivalent couple of smaller forces but placed further apart as illustrated in Fig. 2.10.

Consider next the case of a pair of parallel but unequal forces, $\mathbf{F}_1$ and $\mathbf{F}_2$. If they have the same sense, their resultant $\mathbf{R}$ is parallel to them with the same sense and its magnitude

$$R = F_1 + F_2$$

If they have opposite sense, their resultant $\mathbf{R}$ is parallel to them in sense of the larger force and the magnitude

$$R = |F_1 - F_2| \tag{2.7}$$

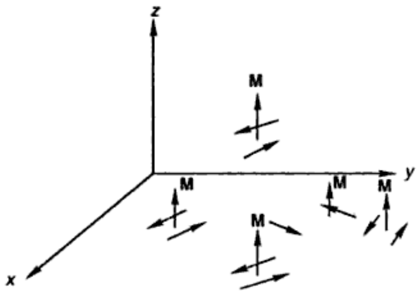


Fig. 2.10 ***Equivalent Couple Moments***

The net effect of the forces is also in producing a turning effect; the moment of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is different at different places. If a line *AB* is drawn perpendicular to the lines of action of the forces, it is easy to observe that the net moment of the force is different both in magnitude and sense about different points. In each case, a point *O* can be located such that the net moment about that point is zero. This point *O* should lie closer to the larger force; on the side of the smaller force if they are in the same sense and on the opposite side of the smaller force if they are in opposite sense. In terms of d_1 and d_2, the distances between the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, and the point *O*,

$$F_1 d_1 = F_2 d_2 \tag{2.8}$$

as is clear in Fig. 2.9(b).

Let us now consider a case of coplanar parallel forces as shown in Fig. 2.9(c).

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

The resultant force is

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 - \mathbf{F}_3 + \ldots \tag{2.9}$$

and the resultant moment about a point *O* through which **R** is passed has the magnitude

$$M = F_1 d_1 + F_2 d_2 - F_3 d_3 + \ldots$$

whereas, in vector notation,

$$\mathbf{M}_0 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 - \mathbf{r}_3 \times \mathbf{F}_3 + \ldots \tag{2.10}$$

is the moment vector about *O* accompanying the resultant force **F** passed through this point.

The general case of coplanar parallel forces can, therefore, be represented by an equivalent system of a single force **R** passing through a point *P* and a moment about an axis perpendicular to the plane of the forces. If **R** is non-zero, it can be made to pass through a point such that the moment vanishes and the single resultant force **R** becomes the equivalent. If **R** is zero, the moment remains as the single equivalent. The single resultant of a system of coplanar parallel forces may, therefore, be a single force if the force is non-zero or a single moment if the resultant force is zero.

The concept of equivalence of coplanar parallel forces may be extended to the case of spatial parallel forces. If the system of parallel forces in space consists of

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

specified by the position vectors

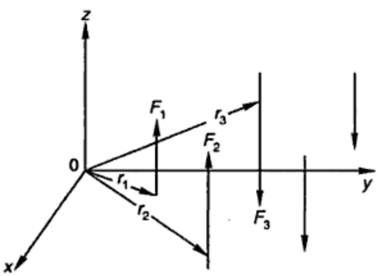
$$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$$

respectively as shown in Fig. 2.11, then the resultant force is given by

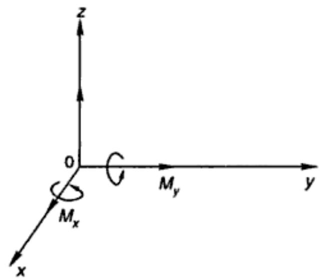
$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$$

and if this force is passing through the origin O, the moment accompanying it for equivalence should be

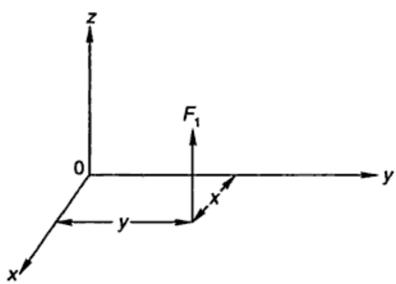
$$\mathbf{M}_0 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \ldots$$



(a) Specified Spatial Force System



(b) Equivalent System at 0



(c) Single Non-Zero Equivalent Force

Fig. 2.11 ***Equivalence of Spatial Parallel Forces***

The system of parallel forces cannot generate a moment in the direction of the parallel forces; the moment $\mathbf{M}_0$ is wholly normal to the lines of action of the parallel forces. It is, therefore, possible to reduce the resultant to

(i) a single equivalent non-zero force by shifting it parallel to itself so as to reduce the accompanying moment to null,

(ii) a single equivalent moment if the resultant force happens to be zero, or
(iii) a null force and null moment in a particular case.

Example 2.7 A rigid bar *AB* is subjected to a system of parallel forces as shown in Fig. Ex. 2.7. Reduce the given system of forces to an equivalent (a) single resultant, (b) force-moment system at *A*, and (c) force-moment system at *D*.

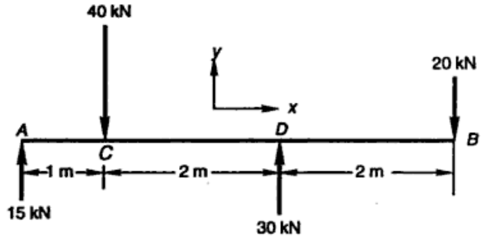


Fig. Ex. 2.7

Solution The resultant force for the coplanar parallel force system is

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$$
$$= 15\,\mathbf{j} - 40\,\mathbf{j} + 30\,\mathbf{j} - 20\,\mathbf{j} = -15\,\mathbf{j}\text{ kN}$$

which implies that it is 15 kN downward. This force may be made to act through any desired point in the bar.

(a) In order that the single force $-15\,\mathbf{j}$ kN be equivalent to the given system of forces, it should be so located that the moment exerted by it about any point should be the same as that exerted by the given system of forces. For example, let us consider the moments about *A* for the given system

$$\mathbf{M}_A = 1\,\mathbf{i} \times (-40\,\mathbf{j}) + 3\,\mathbf{i} \times 30\,\mathbf{j} + 5\,\mathbf{i} \times (-20\,\mathbf{j})$$
$$= -40\,\mathbf{k} + 90\,\mathbf{k} - 100\,\mathbf{k} = -50\,\mathbf{k} \qquad \text{(i)}$$

If the resultant is assumed to act at a distance x from *A*, then

$$\mathbf{M}_A = x\,\mathbf{i} \times (-15\,\mathbf{j}) = -15x\,\mathbf{k} \qquad \text{(ii)}$$

Equating (i) and (ii) as postulated before,

$$-15\,x\,\mathbf{k} = -50\,\mathbf{k}$$
$$x = 3.33\text{ m}$$

(b) If the force of $-15\,\mathbf{j}$ were acted at *A*, the moment accompanying it should be

$$\mathbf{M}_A = -50\,\mathbf{k}\text{ kN m}$$

as determined before.

(c) If the force of $-15\,\mathbf{j}$ were acted at *D*, the moment accompanying it would be

$$\mathbf{M}_D = -3\,\mathbf{i} \times 15\,\mathbf{j} - 2\,\mathbf{i} \times (-40\,\mathbf{j}) + 2\,\mathbf{i} \times (-20\,\mathbf{j})$$

$$= (-45 + 80 - 40)\,\mathbf{k} = -5\,\mathbf{k}\text{ kN m}$$

(d) If the force of $-15\,\mathbf{j}$ were acted at B, the moment accompanying it would be

$$\mathbf{M}_B = 5\,\mathbf{i} \times 15\,\mathbf{j} - 4\,\mathbf{i} \times (-40\,\mathbf{j}) - 2\,\mathbf{i} \times 30\,\mathbf{j}$$

$$= (-75 + 160 - 60)\,\mathbf{k} = 25\,\mathbf{k}\text{ kN m}$$

The results (a), (b), (c) and (d) are shown in Fig. Ex. 2.7 (Solution).

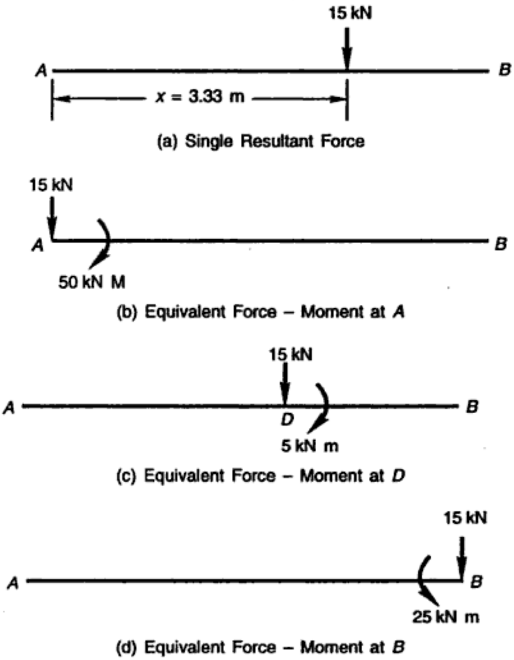


Fig. Ex. 2.7 ***(Solution)***

Example* 2.8** A 4 m × 5 m slab carries four forces normal to it as shown in Fig. Ex. 2.8. Determine the equivalent action which can be applied only at point ***O and determine the single resultant of the force system.

Solution The resultant force **R** for the given force system must be

$$\mathbf{R} = -4\,\mathbf{k} - 3\,\mathbf{k} + 5\,\mathbf{k} - 6\,\mathbf{k} = -8\,\mathbf{k}\text{ kN}$$

which means that it should be 8 kN force acting vertically downward. If this was located to pass through O, the moment accompanying it for equivalence would be

$$\mathbf{M}_0 = 4\,\mathbf{i} \times (-4\,\mathbf{k}) + (1\,\mathbf{i} + 1\,\mathbf{j}) \times (-3\,\mathbf{k}) + (2\,\mathbf{i} + 3\,\mathbf{j}) \times 5\,\mathbf{k} + (1\,\mathbf{i} + 4\,\mathbf{j}) \times (-6\,\mathbf{k})$$

$$= -12\,\mathbf{i} + 15\,\mathbf{j}\text{ kN m}$$

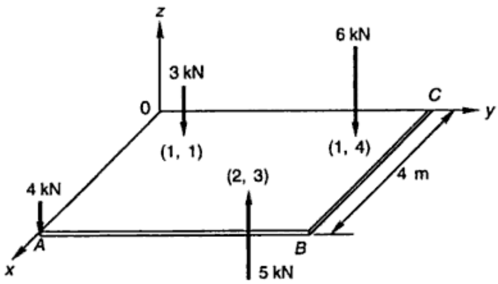
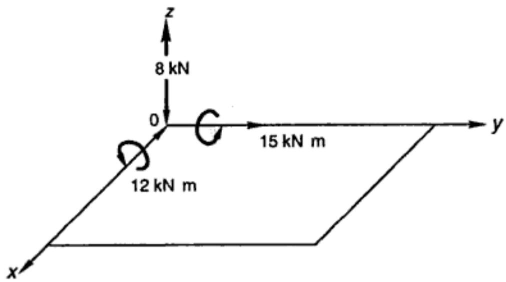
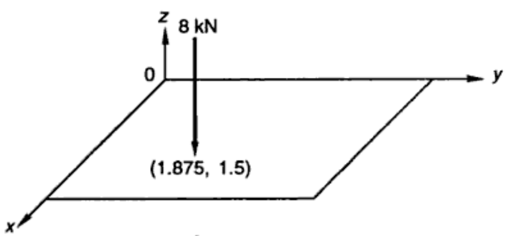


Fig. Ex. 2.8



(a) Equivalent Action Applied at 0



(b) Single Equivalent Force

Fig. Ex. 2.8 *(Solution)*

The moment accompanying the force 8 kN downward applied at O is, therefore, such that it has an x-component of -12 kN m and the y-component of 15 kN m as shown in Fig. Ex. 2.7a (Solution).

In order to locate the point (x, y) through which the force $-8\,\mathbf{k}$ should be passed for complete equivalence, the moment accompanying it should be null. In other words, the moment generated due to the parallel transfer of the force should just cancel the moment $\mathbf{M}_0$, i.e.,

$$-\mathbf{M}_0 = -\mathbf{r} \times \mathbf{F}_{xy}$$

$$-(-12\,\mathbf{i} + 15\,\mathbf{j}) = -(x\,\mathbf{i} + y\,\mathbf{j}) \times (-8\,\mathbf{k})$$

$$= -8x\,\mathbf{j} + 8y\,\mathbf{i}$$

whence $8y = 12; \qquad y = 1.5\text{ m}$

$$-8x = -15; \qquad x = 1.875\text{ m}$$

The single resultant force $-8\,\mathbf{k}$ kN can, therefore, be equivalent to the prescribed parallel force system if it is applied at a point (1.875 m, 1.5 m) as shown in Fig. Ex. 2.7b (Solution).

2.9 RESULTANT OF A COPLANAR FORCE SYSTEM

When the forces constituting a force system lie in a plane and the moments applied externally or generated by the applied forces are directed normal to the plane, the force system is said to be coplanar. Coplanar systems may consist of concurrent or non-concurrent forces, parallel or non-parallel forces. The analyses of plane concurrent and plane parallel forces has been dealt with already. Attention is now focussed on the analysis of general coplanar systems.

For a coplanar force system in the x-y plane, there can be no forces in the z-direction and no moments about the x and y axes.

A system of coplanar forces

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

whose lines of action are specified by the position vectors

$$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$$

confined in the x-y plane and couple moments

$$\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \ldots$$

directed normal to the x-y plane are reducible to a resultant force

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots \tag{2.11}$$

which can be made to pass through any point and a resultant moment $\mathbf{M}$. If the resultant force $\mathbf{R}$ is acted through the origin O, the moment which must accompany it is given by

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \ldots$$

$$= \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \ldots \tag{2.12}$$

This is shown in Figs 2.12(a) and (b).

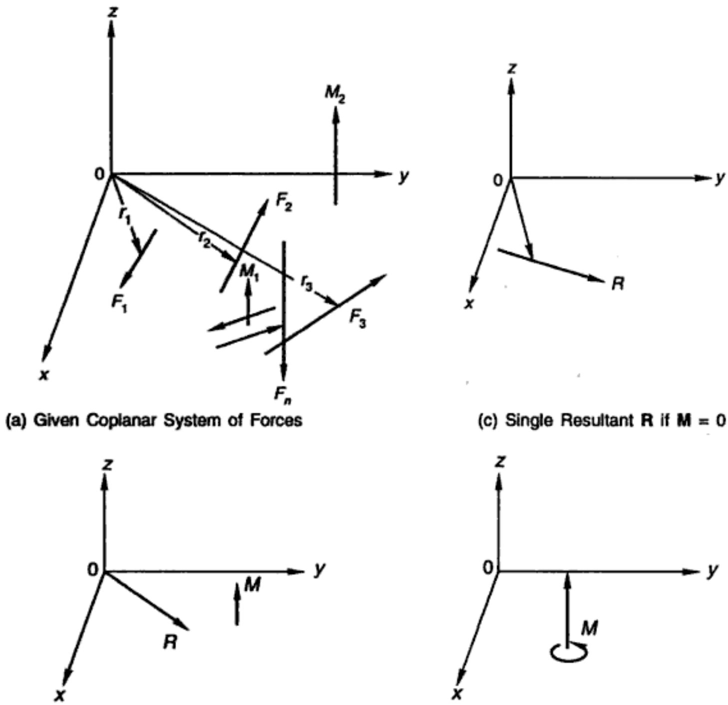
It is possible to shift the resultant force $\mathbf{R}$ parallel to itself until the moment $\mathbf{M}$ is cancelled. The position vector of the line of $\mathbf{R}$ would then be $\mathbf{r}$ such that

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}$$

as shown in Fig. 2.12(c).

In case the resultant force vanishes,

$$\mathbf{R} = 0; \qquad R_x = 0, \qquad R_y = 0$$



(a) Given Coplanar System of Forces

(b) Resultant Force **R** and Moment **M**

(c) Single Resultant **R** if **M** = 0

(d) Resultant Moment **M** if **R** = 0

Fig. 2.12 ***Resultant Force and Moment of a Given Coplanar System of Forces***

the resultant of the coplanar force system may be a couple-moment.

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \ldots$$
$$= \mathbf{M}_1 + \mathbf{M}_2 + \ldots$$

In case the couple-moments add to zero in addition to the resultant force being zero, the resultant is a null force and a null couple-moment.

The discussion on coplanar force systems may be summed up by stating that the force system may be replaced by a non-zero single force **R** in that plane passing through any desired point and an appropriate moment by a non-zero single force **R** appropriately located in that plane, by a single moment normal to that plane if the resultant force **R** is zero or by a zero moment and zero force in some particular case.

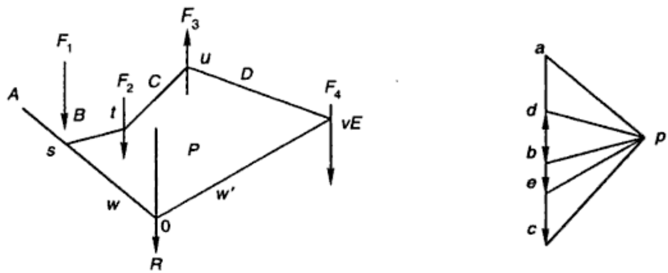
A graphical procedure known as *funicular polygon method* is usually applied to determine the resultant of a plane system of parallel or non-parallel forces.

The forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, ... are drawn and the areas enclosed between their lines of action are marked in capital letters. Such a figure is called *space diagram.* If the space on the left of the line of action of $\mathbf{F}_1$ is called *A* and the other spaces as *B, C, D,* ... as shown in Fig. 2.13(a), the forces are also referred to as *AB, BC, CD,* ... in terms of the space notation.

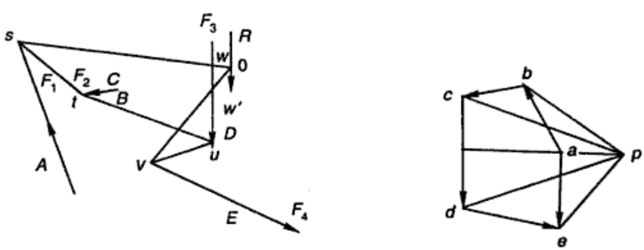
The next step is to draw the *force polygon* by drawing the forces in magnitude and direction and joining them end to end. In the force polygon the forces are named in small letters thus the force between regions *A* and *B* will be named *ab* in the force polygon. The closing side of the polygon in opposite order gives the resultant of the forces in magnitude and direction in accordance with the polygon law of forces.

In order to find the line of action of the resultant, the following procedure is adopted:

Take any arbitrary point *p* inside or outside the force polygon and join all the vertices of the polygon with this point. This arbitrary point is called pole and the lines joining the pole to the vertices of the force polygon are called *rays.* These rays can be imagined as the components of the force, e.g. *cp* and *pd* are components of *cd*; *bp* and *pc* are components of *bc*, and so on. The diagram showing the force polygon, pole and rays is called *ray diagram* as shown in Fig. 2.13(a). Starting with an arbitrary point *s* on the line of action of the force $\mathbf{F}_1$, a line *sb* is drawn parallel to the ray *pb* such that it intersects the line of the action of the next force $\mathbf{F}_0$ as shown in Fig. 2 13(b). From that point, another line is drawn parallel to the next ray *pc* and produced to intersect the line of action of the next force, and so on. Finally, a line



(a) For Parallel Forces



(b) For Coplanar Forces

Fig. 2.13 ***Graphical Methods***

parallel to *sa* is drawn through the starting point *s* and the point of intersection *O* with the last line drawn as above is located. This is the point through which the resultant **R** of the given forces should act. The figure so drawn is called *funicular polygon*. The resultant is, therefore, completely specified by stating that it is a force denoted by *ae* in magnitude and direction as shown in the force polygon and acting through the point *O* located by drawing the funicular polygon.

It may be pointed out that, in general, the ray diagram should be such that it does not coincide with or is parallel to the forces; otherwise it may not be possible to draw the funicular polygon. This fact places a restriction on the choice of the pole *p*, namely, the pole should not lie on the closing side *ae* of the force polygon. In particular, when the given forces are parallel, the force polygon is made up of collinear lines and the pole *p* must not lie on the line.

The graphical procedure is a geometrical manifestation of the theoretical method. The simplicity offered by it is at the expense of accuracy of results. Moreover, the graphical procedure is restricted to the plane system of forces. An insight into the equivalence of the graphical procedure with the theoretical method is provided as follows.

The closing side *ae* of the force polygon *abcde* provides the direction and magnitude of the resultant force in accordance with the polygon law of forces. The construction of the ray diagram enables us to replace each of the given forces by two components, e.g., $\mathbf{F}_1$ by *ap* and *pb*, $\mathbf{F}_2$ by *bp* and *pc* and $\mathbf{F}_3$ by *cp* and *pd*. Summation of the forces shows that the resultant **R** is made up of *ap* and *pe* because the pairs of forces *pb* and *bp*, *pc* and *cp*, etc. cancel off mutually as shown by the arrows on the force diagram. The construction of the funicular diagram by drawing lines parallel to the rays enable us to locate the points on a hypothetical string such that it is in equilibrium. The force $\mathbf{F}_1$ is balanced by the virtual tensions *ap* and *pb* in the string. Similarly, the force $\mathbf{F}_2$ is balanced by *bp* and *pc*, and so on. A point *O* is located where the resultant **R** would be balanced by the components *ap* and *pe*, so that point must be the point of application of the resultant. It may be noted that different choices of the pole result in different rays and hence in different shapes of the funicular polygons. In each case, however, the equilibrium criteria is automatically satisfied and the point *O* finally located must lie on the line of action of the resultant force. It may also be added that a funicular polygon is indeed the shape of a string it would acquire under the application of the given forces at the corresponding points. For the same reason, the method of funicular polygon is also called *string analogy method*.

***Example* 2.9** Three forces of magnitudes 1, 2 and $\sqrt{2}$ N act along the sides of a rigid, triangular frame formed by the lines *AB*, *BC* and *CA* specified by

$$x + y = 1, \quad y - x = 1 \quad \text{and} \quad y = 2$$

in the same order.

Find the resultant and the equation of its line of action.

Solution The sides of the triangle are given by

$$y = -x + 1, \quad y = x + 1 \quad \text{and} \quad y = 2$$

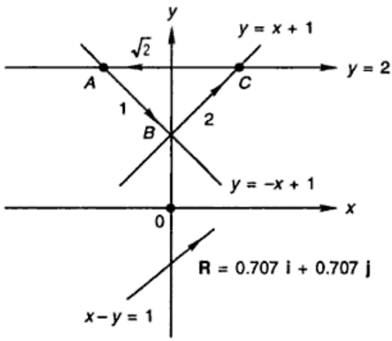


Fig. Ex. 2.9

indicating that their slopes are −1, +1 and 0 respectively. The unit vectors along the sides are, therefore,

$$0.707\,\mathbf{i} - 0.707\,\mathbf{j},\ 0.707\,\mathbf{i} + 0.707\,\mathbf{j} \text{ and } -\mathbf{i} \text{ respectively.}$$

The three forces are, in turn, represented as

$$\mathbf{F}_1 = 0.707\,\mathbf{i} - 0.707\,\mathbf{j} \qquad \text{passing through } (0, 1)$$

$$\mathbf{F}_2 = 1.414\,\mathbf{i} + 1.414\,\mathbf{j} \qquad \text{passing through } (0, 1)$$

$$\mathbf{F}_3 = -1.414\,\mathbf{i} \qquad \text{passing through } (0, 2)$$

The resultant **R** is given by their sum

$$\mathbf{R} = 0.707\,\mathbf{i} + 0.707\,\mathbf{j}$$

The line of action of the resultant is located by considering the equivalence of the given system with the resultant in regard to the moments about the origin *O*;

$$(x\,\mathbf{i} + y\,\mathbf{j}) \times (0.707\,\mathbf{i} + 0.707\,\mathbf{j}) = 1\,\mathbf{j} \times (0.707\,\mathbf{i} - 0.707\,\mathbf{j}) + 1\,\mathbf{j} \times (1.414\,\mathbf{i} + 1.414\,\mathbf{j}) + 2\,\mathbf{j} \times (-1.414\,\mathbf{i})$$

whence $\quad 0.707x - 0.707y = 0.707$

or

$$x - y = 1$$

which is the line of action of the resultant force.

***Example* 2.10** The moments of a given plane system of forces about three points (1, 0), (0, 1) and (1, 2) are +4, +25, +22 units respectively. Find the resultant force and prove that it acts along the line $12x - 9y = 16$.

Solution The plane system of forces may be replaced by a resultant force

$$R_x\,\mathbf{i} + R_y\,\mathbf{j}$$

in that plane acting at a point (x, y) in accordance with the concept of equivalence.

The gravitational force field of the earth is due to its mass M and acts on a body of mass m placed at a height h above the surface of the earth of radius R

$$\boxed{F = GM \frac{m}{(R+h)^2}} \tag{2.16}$$

Denoting the acceleration due to gravity g, and recognising that the weight of the body is given by

$$W = F = m\,g \tag{2.17}$$

$$g = \frac{GM}{(R+h)^2} \tag{2.18}$$

It may, however, be pointed out here that the earth has been assumed as a spherical and stationary body. Since the earth is not a perfect sphere, the radial distance of the surface from the centre of the earth varies with the latitude as well as the altitude. The variation of g has been investigated and expressed as

$$g = 9.806\,16 - 0.025\,928 \cos^2 \lambda + 0.000\,069 \cos^2 2\lambda - 0.000\,003\, h \text{ m/s}^2 \tag{2.19}$$

where λ is the latitude of the place and h is the height in metres above the mean sea level.

It is usual in engineering to consider g, the acceleration due to gravity as constant and the weight force directed perpendicular to the surface of the earth for most earthbound objects. This assumption is known as *assumption of a flat earth*. It is indeed incorrect to allow such an assumption in principle but it is acceptable to admit this assumption for most practical purposes, particularly when the analysis is confined to a region close to the earth.

Another noteworthy fact is that the gravitational force between two bodies of finite sizes is not necessarily along the line joining their centres, nor is it necessary that the magnitude of the force be given by Eq. (2.15). A careful reading of the statement of Newton's law of gravitation would reveal that the law relates to the force between two particles or point bodies. A body of finite size may be thought of as a distributed mass or a distribution of mass elements over the domain of the body. According to the law of gravitation, each element on a body would experience a force from each element of the other body and the total force on a body would require double-volume integration. Only in the special case of two homogeneous solid or thin hollow spheres does the law hold in its stated from because the force of gravitation turns out to be along the line joining their centres and the magnitude is given by Eq. (2.15). It is for this reason that the law holds fairly well for the nearly spherical celestial bodies. In fact, Kepler's laws of planetary motion, formulated before Newton was born, contain Newton's law of gravitation for spherical planetary bodies; Newton's original contribution was to state the laws applicable to particles or infinitesimal elements which may belong to any distributed masses of which spheres are only particular cases.

***Example* 2.11** Determine the gravitational force of attraction of a thin uniform rod of length a and mass M on a concentrated mass m outside the rod but on the same line as the rod and at a distance b from the nearer end.

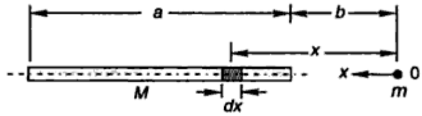


Fig. Ex. 2.11

Solution The force on the concentrated mass *m* can be determined by writing the force due to an element of length *dx* situated at a distance *x* from it and integrating the same for the entire length of the uniform rod. Let the mass density of the rod be ρ per unit length of the rod. The infinitesimal element of length *dx* has a mass ρdx and the force of attraction on the mass *m* due to it is

$$dF = \frac{Gm\rho\, dx}{x^2} \tag{i}$$

The total force of attraction due to the entire length of the rod must be

$$F = \int dF = \int_b^{a+b} \frac{Gm\rho\, dx}{x^2}$$

$$= Gm\rho \left| -\frac{1}{x} \right|_b^{a+b}$$

$$= \frac{Gm\rho a}{b(a+b)} = \frac{GmM}{b(a+b)} \tag{ii}$$

where the mass of the rod is taken as

$$M = \rho a$$

The distance between the concentrated mass and centre of the rod is

$$x_c = b + a/2$$

If the mass of the rod was concentrated at its mid-point, the force of attraction would have been

$$F_c = \frac{GmM}{(b + a/2)^2} \tag{iii}$$

It may be appreciated that **F** and $\mathbf{F}_c$ are of different magnitudes; they are closer if the length of the rod is less,

$$a \to 0, \qquad \mathbf{F} \to \mathbf{F}_c$$

Equation (iii) enables us to interpret that the force of attraction of the rod would be the same if the entire mass of the rod was concentrated at x_s

where $$F = \frac{GmM}{b(a+b)} = \frac{GmM}{x_s^2}$$

whence $$x_s = \sqrt{b(a+b)}$$

Again, x_s coincides with x_c if a tends to zero, as expected.

2.12 HYDROSTATIC FORCE FIELD

A fluid exerts pressure on a surface exposed to it as shown in Fig. 2.15. The pressure distribution due to hydraulic fluids, i.e., liquids is studied under *hydrostatics* and that due to pneumatic fluids is often classified as *aerostatics* although the word hydrostatics is generally used to imply any situation.

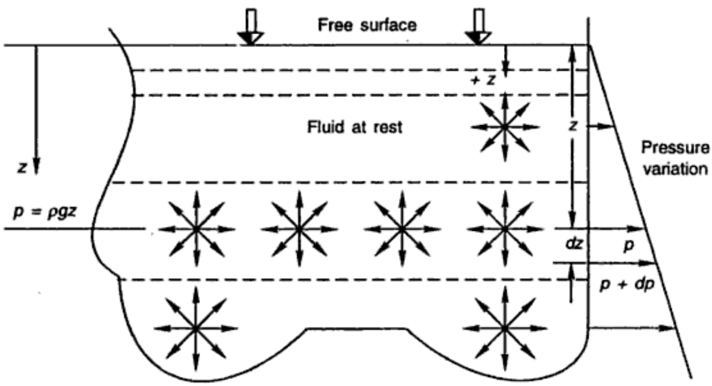


Fig. 2.15 ***Pascal's Law and Hydrostatic Law for Fluids at Rest***

*In a static fluid, the intensity of pressure at a point is the same in all directions.** This statement is due to Pascal and can be proved by considering the equilibrium of a fluid element at a point.

*The rate of change of pressure in a vertically downward direction must equal the local specific weight of the fluid.** This statement comprises the hydrostatic law, Mathematically

$$\boxed{\frac{dp}{dz} = \rho g} \tag{2.20}$$

where dp/dz is the rate of change of pressure in the downward direction and ρg is the specific weight or weight density, ρ being the mass density and g, the acceleration due to gravity.

Alternatively, Eq. (2.20) is written as

$$dp = \rho g \, dz \tag{2.21}$$

implying that the increment of pressure dp in the vertically-downward direction over a distance of dz equals the product of mass density ρ, gravitational acceleration g and the distance dz.

In a fluid of constant density,

$$\int dp = \int \rho g \, dz = \rho g \int dz$$

or

$$p - p_{ref} = \rho g \,(z - z_{ref})$$

Assuming the atmospheric pressure at the free surface of a liquid, as the reference, i.e.,

$$p_{ref} = p_{atm} \text{ at } z_{ref} = 0$$

the pressure at a point with the atmospheric pressure as the reference is

$$\boxed{p = \rho g z} \tag{2.22}$$

In other words, the pressure varies linearly as the depth in an incompressible fluid at rest, the rate of change of pressure being equal to the specific weight of the fluid.

It follows that the pressure at the same depth in different locations of a continuous fluid must be the same and the rate of variation of pressure along an inclined direction depends upon the rate of increment of the vertical coordinate along the incline.

*The pressure on a surface acts normal to that surface.** This statement follows from the fact that, in a static fluid, there can be no shearing stresses; because if the shearing stresses existed, the fluid would flow. The pressure is the intensity of a surface force acting normal to the surface. Hence, the hydrostatic force on a surface element ΔA is given by

$$\mathbf{F} = p\,\Delta A \text{ normal to the area } \Delta A.$$

Concept Review Questions

1. Classify the following statements as true or false and state the reasons:
   (a) The simplest equivalent of a force system is either a force or a couple-moment.
   (b) The force in a force-field must vary monotonically only.
   (c) Force is a transmissible vector or a free vector.
   (d) Equivalent forces are defined on the basis of providing an identical action in a particular capacity.
   (e) Resultant and equivalent are identical concepts.
   (f) A particle may be subjected only to a concurrent force system; collinear, coplanar or spatial.
   (g) A force is a bound vector; its line of action on a rigid body must be specified.
   (h) A moment is a free vector; its line of action on a rigid body need not be specified.
2. (a) Can a system of forces acting on a rigid body be replaced by a wrench at any desired point on it?
   (b) If a rigid slender bar is subjected to a number of forces, which factors would decide whether the resultant will be a single force or not? Is it necessary that the system of forces should not be distributed in space?
3. A number of plane forces act on a simply supported rigid body at two points. Determine the resultant of these forces and appreciate the fact that it is also the equivalent

force so far as the determination of the reactions from the supports is concerned but is not the equivalent force if the body is analysed either for deflection or for the resisting moments and forces developed in the body.

4. Which of the following system of forces may be represented by a single resultant non-zero force
   (a) Concurrent force system
   (b) Parallel force system
   (c) Coplanar force system
   (d) Spatial force system.
5. State Newton's law of gravitation for two point-objects of masses m_1 and m_2 placed a distance r apart. How would you proceed to determine the gravitational force between two bodies which have an arbitrary but specified distribution of mass, e.g., a cricket bat and ball.
6. A plate *AB* of mass *m* and dimensions 2 m × 3 m is to be lifted by a string tied to it and going over a set of pulleys with a suspended mass *M* at the other end as shown in Fig. CRQ 6. Recognised the sources of force in the system.

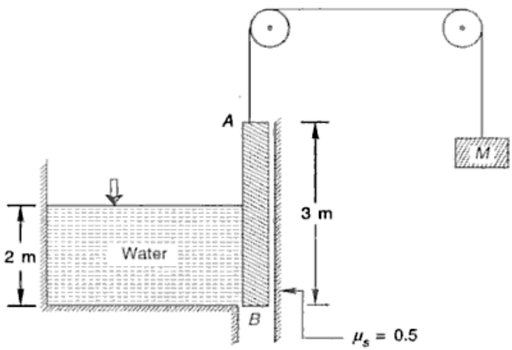


Fig. CRQ 6

7. If the sum of couple-moments and moments of all the forces of a force system about any three noncollinear points is zero, show that the fake system results in a null.

Tutorial Problems

2.1 It is desired to transfer a 50 kN force parallel to itself from a point (2, 1) to a point (1, 2) as shown in Fig. Prob. 2.1. Determine the additional moment, if any, required to maintain equivalence. (**Ans.** 70.7 kN m; 50 **i** + 50 **j**)

2.2 A force of 100 N acting tangential to a drum at *A* must be transferred parallel to itself to its centre *O* or to a diametrically opposite point *B*. Determine the moments which should accompany it for equivalent effect. (**Ans.** 25 N m and 50 N m)

2.3 Determine the resultant of the coplanar concurrent force system shown in Fig. Prob.2.3 (**Ans.** 49 N, $\theta = -26°$)

2.4 Determine the resultant of the concurrent force system acting at a point *O* as shown in Fig. Prob. 2.4. (**Ans.** 19.7 N; $\alpha = 43$; $\beta = 56°$ and $\gamma = 66°$)

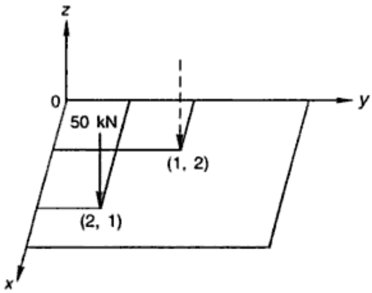


Fig. Prob. 2.1

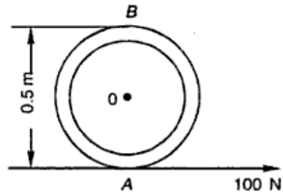


Fig. Prob. 2.2

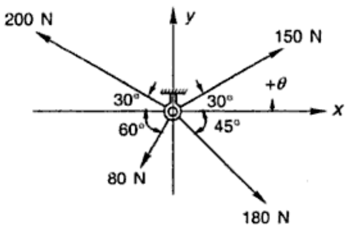


Fig. Prob. 2.3

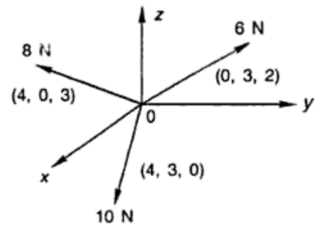


Fig. Prob. 2.4

2.5 Two unlike parallel forces, each 10 N, act 0.3 m apart as shown in Fig. Prob. 2.5(a). What is their resultant action? If one force was 11 N instead of 10 N as shown in Fig. Prob. 2.5 (b), what would the resultant action be? Can these systems be replaced by single resultant forces for equivalence? (**Ans.** 3 Nm; 1N, 3.15 Nm)

2.6 Determine the resultant action of a coplanar parallel force system in Fig. Prob. 2.6. (**Ans.** $M = -30$ N m)

2.7 The resultant of four vertical forces is a couple-moment 30 N m acting counterclockwise. Three of the four forces are shown in Fig. Prob. 2.7. Determine the fourth force. (**Ans.** 33 N upward at 44.5 cm to the right of A on AB)

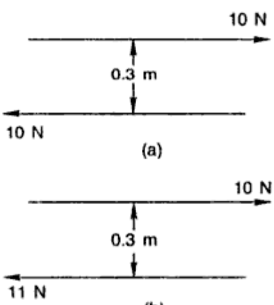


Fig. Prob. 2.5

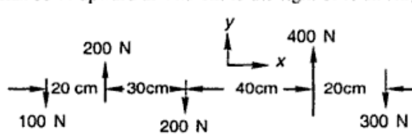


Fig. Prob. 2.6

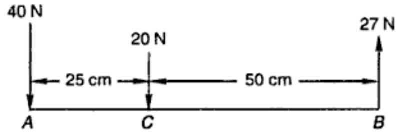


Fig. Prob. 2.7

2.8 The following forces act parallel to the z-axis. Their respective points of application in the x-y plane are also given. Determine the single resultant and locate it in each of the following cases:

Case 1

Force in kN	3	– 4	2	–5
x, y in m	(2, 5)	(1, –5)	(3, 3)	(– 4, – 4)

Case 2

Force in N	100	200	– 300
x, y in cm	(10, 10)	(20, – 50)	(30, – 40)

(**Ans.** Case 1: R = –4 kN at –7.0, –15,3) m
Case 2: M_x = 30 N m, M_y = 40 N m)

2.9 A coplanar parallel force system consisting of three forces acts on a rigid bar AB as shown in Fig. Prob. 2.9. Determine the simplest equivalent action for the force system. If an additional force of 10 kN acts along the bar A to B, what would be the simplest equivalent action?

(**Ans.** –10 kN along 40 kN force; 14.14 kN, – 45°)

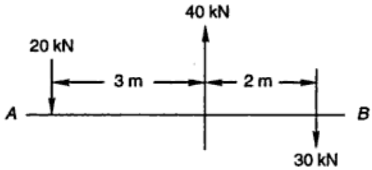


Fig. Prob. 2.9

2.10 A pulley of 1 m diameter is subjected to 2 kN and 1 kN force at A and B respectively as shown in Fig. Prob. 2.10. Its own weight of 0.5 kN acts at the centre O. Determine the resultant force and its line of action with respect to AOB.

(**Ans.** 3.04 kN making an angle of 80.5° with AOB)

2.11 A bell-crank lever AOB is subjected to a horizontal force 10 N at A while a weight of 7.5 N is attached at B. Determine the resultant action on the lever. If the 10 N force were transferred to point P, what would be the change in the resultant action?

(**Ans.** In equilibrium; 1 N m moment and lever turning counterclockwise)

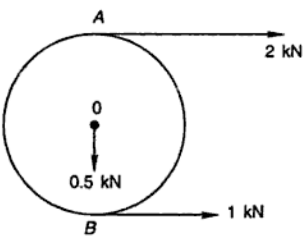


Fig. Prob. 2.10

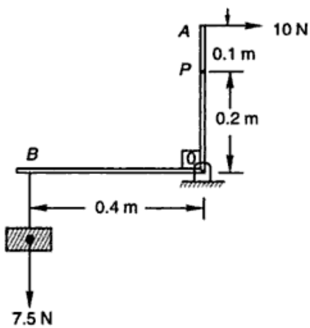


Fig. Prob. 2.11

2.12 A dam is subjected to three forces; 50 kN force on the upstream vertical face AB, 30 kN force on the downstream inclined face and its own weight 120 kN. Determine the single equivalent force and locate its point of intersection with the base AC, assuming all the forces to lie in the same plane. (**Ans.** 137 kN; $y + 5.6\,x = 0$)

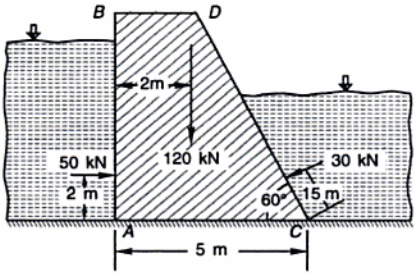


Fig. Prob. 2.12

2.13 A 2 m × 4 m plate is subjected to a system of three coplanar forces as shown in Fig. Prob. 2.13. Determine the equivalent action at *O* which may replace the force system.

(**Ans.** 5.9 kN and 1.7 kN m)

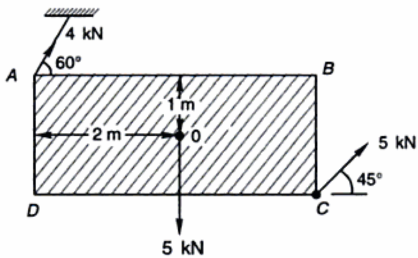


Fig. Prob. 2.13

2.14 A symmetrical truss is loaded by five forces as shown in Fig. Prob. 2.14. Obtain the resultant load and its line of action.

(**Ans.** 15.5 kN along 0.29 **j** – 0.95 **k**)

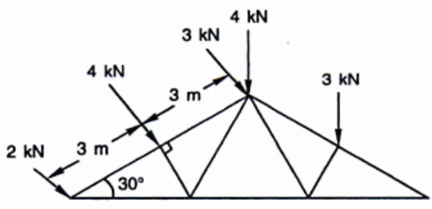


Fig. Prob. 2.14

2.15 Determine the resultant of the forces acting on a bell-crank lever as shown in Fig. Prob. 2.15.

(**Ans.** 4.25 Nm clockwise)

2.16 A bracket is subjected to a coplanar force system shown in Fig. Prob. 2.16. Determine the magnitude and the line of action of the single resultant of the system.

(**Ans.** –200 **i** –154 **j** ; 35 cm from A)

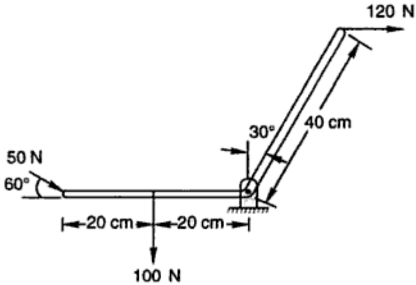


Fig. Prob. 2.15

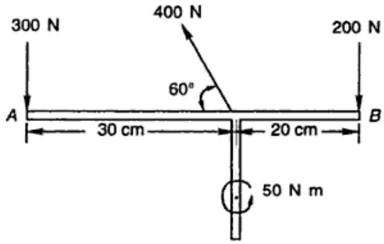


Fig. Prob. 2.16

2.17 Reduce the wrench along the x-axis as shown in Fig. Prob. 2.17 to a system of two forces perpendicular to the y-axis acting at A and B on the y-axis. It is given that $OA = 0.2$ m and $OB = 0.6$ m.

(**Ans.** 100 **i** – 7.5 **k**; – 50 **i** + 7.5 **k**)

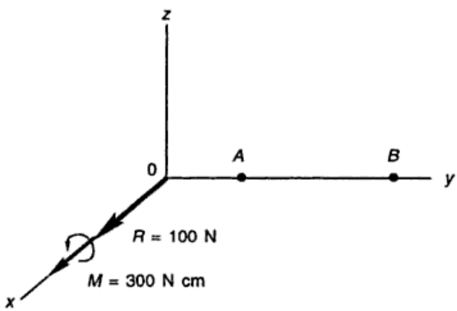


Fig. Prob. 2.17

2.18 A system of three forces acts on a parallelopiped as shown in Fig. Prob. 2.18. Replace the forces by a wrench and specify its point of action on a face.

2.19 Reduce the given force system to an equivalent force plus couple-moment system at corner O as shown in Fig. Prob. 2.19.

(**Ans.** $-56.5\,\mathbf{i} + 17.4\,\mathbf{j} - 17.4\,\mathbf{k}$; $-3.5\,\mathbf{i} - 28.7\,\mathbf{j} + 30\,\mathbf{k}$)

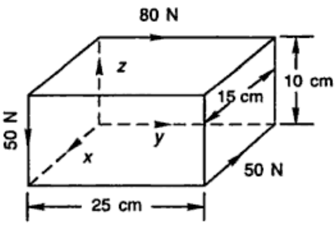


Fig. Prob. 2.18

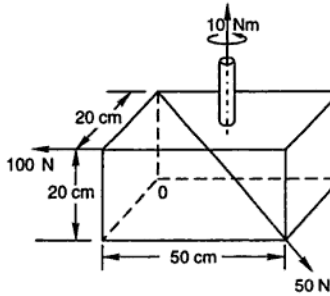


Fig. Prob. 2.19

2.20 Find the force of attraction between two homogeneous solid spheres of radii r_1 and r_2 and masses m_1 and m_2 placed at a distance r between their centres.

$$\left(\textbf{Ans. } \frac{Gm_1 m_2}{r^2} \text{ along the line joining their centres}\right)$$

2.21 Find the force of attraction of a thin uniform rod of length $2a$ and mass M on a particle of mass m placed at a distance b from its midpoint such that the particle is equidistant from its ends.

$$\left(\textbf{Ans. } \frac{GMm}{b\sqrt{a^2+b^2}} \text{ along the line joining the mid-point of the rod with the particle}\right)$$

2.22 Determine the maximum possible friction force which may be developed between a pair of sliding surfaces if the normal reaction between them is 20 kN and the coefficient of static friction is 0.30. What happens to the frictional force if the applied tangential force is increased further and the bodies acquire relative motion?

(**Ans.** 6 kN)

2.23 A stack of plates of different materials are placed one above the other and a horizontal force **F** is applied to one of them in the middle of the stack. Discuss the circumstances under which (a) the plate on which force is applied and all the plates above it slide together (b) the place on which force is applied and some more plates below it slide together and (c) only the plate on which force is applied slides out.

2.24 Three forces act along the three sides of a triangle in the same order with their magnitudes proportional to the sides along which they act. Prove that their simplest resultant is a single moment whose magnitude is proportional to twice the area of the triangle. Hence, extend it for a polygon of any numbers of sides.

2.25 Replace the three forces shown in Fig. Prob. 2.25 by a resultant force **R** passing through the point O and a couple **C** for equivalent effect.

(**Ans.** $\mathbf{R} = 200\,\mathbf{i} - 400\,\mathbf{j} - 500\,\mathbf{k}$ N
$\mathbf{C} = 15\,\mathbf{j} - 24\,\mathbf{k}$ N m)

2.26 Determine the tension in cable BC (Fig. Prob. 2.26). Neglect the weight of AB.

(**Ans.** 5.0 kN)

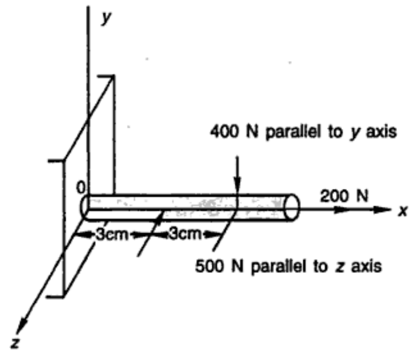


Fig. Prob. 2.25

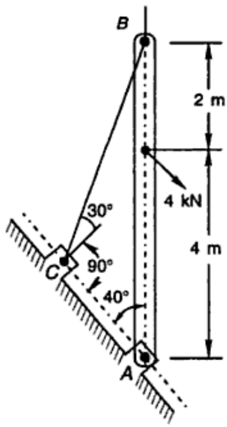


Fig. Prob. 2.26

2.27 Determine the point on the line joining the centres of the earth and moon (Fig. Prob. 2.27) at which the gravitational forces of the earth and moon are equal. It is given that the mass of moon is .0123 times that of earth and distance from earth to moon is 3.8×10^5 km.

(**Ans.** 3.45×10^5 km from earth)

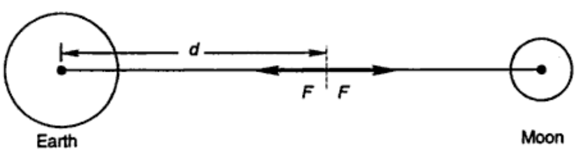


Fig. Prob. 2.27

2.28 In the punch shown in Fig. Prob. 2.28 links *OA* and *AB* have negligible mass and all friction may be neglected. The punch has a mass of 2 kg and moves in a vertical direction only. Find the moment which must be applied to *OA* to maintain its speed constant at 10 rad/s clockwise, with the mechanism in the position shown in Fig. Prob. 2.28 and with **F** zero.

(**Ans.** 4.53 Nm)

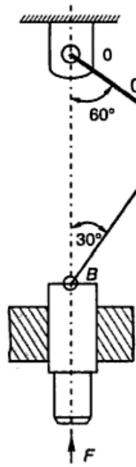


Fig. Prob. 2.28

2.29 Determine the moment of a force 10 kN acting as shown in the figure (a) about the point *C* (b) about the point *H* and (c) about the axis *CF*.

(**Ans.** (a) and (b) 3.71 **j** + 5.94 **k** Nm; (c) 2 **k** Nm)

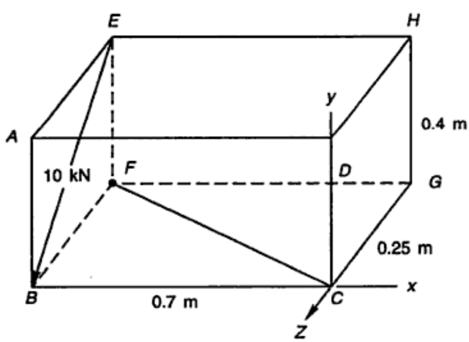


Fig. Prob. 2.29

2.30 (a) Find the resultant of the force system shown in the figure and locate the position of its line of action on the x-axis.

(b) Determine the magnitude and sense of a single vertical force to be applied at point C so as to make the resultant of the entire system pass through A. Also, find that resultant.

(Ans. (a) 300 **i** – 280 **j** N at 107 mm from 0
(b) 72 **j** N; 300 **i** – 208 **j** N at A)

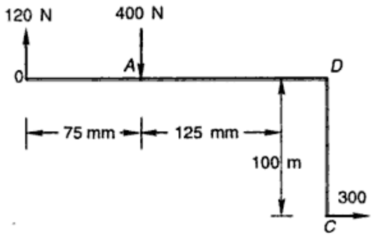


Fig. Pro. 2.30

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

1. A rigid body is acted upon by a force system. It can in general be brought to equilibrium by the application of a force acting
   (a) on a suitable point on the body
   (b) anywhere along a suitable line
   (c) along a suitable line and a moment along the direction of the force
   (d) along a suitable line and a moment in the direction perpendicular to the direction of force
2. The simplest resultant of a spatial parallel force system is always
   (a) a wrench
   (b) a force
   (c) a moment
   (d) a force and moment
3. The force of gravitation between two bodies will be inversely proportional to the square of the distance between their centres of masses if the two are
   (a) of constant densities
   (b) spherical
   (c) of any arbitrary shape
   (d) of the same shape and size
4. A force **F** acting on a rigid body at a point P can be replaced by a force of equal magnitude and in the same direction at a point Q on the body, together with a moment
   (a) equal in magnitude to PQ times F, acting normal to the plane of **F** and **PQ**
   (b) equal in magnitude to F times the distance moved in the lines of actions of the force, acting in the plane of **PQ** and **F**
   (c) given by **F** × **QP**
   (d) given by **F** × **PQ**

Answers to Multiple-Choice Questions

1 (c), 2 (c), 3 (b) 4 (c)

3 Equilibrium Analysis of Static Systems

3.1 EQUILIBRIUM CONCEPT IN MECHANICS

A body is said to be in a state of equilibrium if the body is either at rest or is moving at a constant velocity. The phrase constant velocity implies motion along a straight line at a constant speed. The state of equilibrium, in other words, implies that the body must be at rest with respect to some inertial frame. Equilibrium is a kinematic state of the body; a special state when there is no motion or when the motion is at a constant speed along a straight line. Clearly, all other possible states of motion do not qualify to be categorised as states of equilibrium. Such states are characterised by the presence of the rate of change of momentum, linear and angular. Examples of non-equilibrium are: a particle accelerating along a straight line, a particle going round a curved path and a body rotating about any axis within or outside it.

Let us now try and relate the concept of equilibrium to the action of forces and moments acting on a body. The laws of mechanics governing the motion of a body are:

Newton's Law: $\mathbf{F} = \mathbf{p}$
Euler's Law: $\mathbf{M} = \dot{\mathbf{H}}$

For a body in equilibrium, there should be no rate of change of momentum, linear or angular

$$\dot{\mathbf{p}} = 0 \quad \text{and} \quad \mathbf{H} = 0$$

and hence, the necessary conditions are:

$$\mathbf{F} = 0 \quad \text{and} \quad \mathbf{M} = 0$$

The resultant external force and resultant external moment should, therefore, vanish for a body to be in equilibrium.

Let the forces acting on a body be

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

and the position vectors of any points on the lines of action of the forces be

$$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$$

respectively with reference to an arbitrary origin O as shown in Fig. 3.1. The forces will then tend to compel the body to change its state of rest or of constant velocity in two ways:

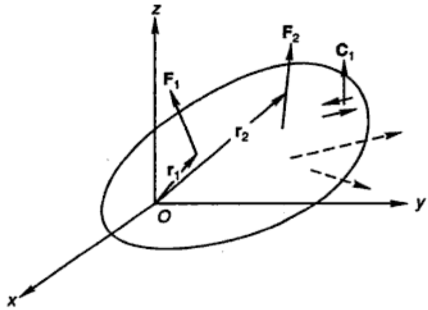


Fig. 3.1 ***Equilibrium under the Action of a System of Forces***

1. By adding to comprise a resultant force

$$\mathbf{R} = \Sigma\,\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots \tag{3.1}$$

   tending to destroy the equilibrium by bringing about a translational acceleration.

2. By generating moment vectors about the reference point O,

$$\mathbf{M}_1 = \mathbf{r}_1 \times \mathbf{F}_1$$

$$\mathbf{M}_2 = \mathbf{r}_2 \times \mathbf{F}_2$$

$$\mathbf{M}_3 = \mathbf{r}_3 \times \mathbf{F}_3$$

— — — — — —

   and the sum of the moments

$$\Sigma\,\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \ldots$$

   tending to destroy the equilibrium by bringing about angular acceleration.

It is possible that the force system acting on a body can as well include some couples with couple-moments

$$\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \ldots$$

in addition to the forces. In that case, the sum of the moments should include the couples provided the forces comprising the couples are not counted in their own right for producing moments.

$$\Sigma\,\mathbf{M} = \mathbf{C}_1 + \mathbf{C}_2 + \ldots + \mathbf{M}_1 + \mathbf{M}_2 + \ldots \tag{3.2}$$

It follows, therefore, that the necessary conditions of equilibrium for a body are:

$$\boxed{\begin{aligned} \Sigma\,\mathbf{F} &= 0 \\ \Sigma\,\mathbf{M} &= 0 \end{aligned}} \tag{3.3}$$

These are, however, not the sufficient conditions for the equilibrium of a body.

In words, the sum of all the forces acting on the body should be zero and the sum of all the moments produced due to the force system about any point should also be zero for equilibrium. It may also be seen that each of these vector equations represents three scalar equations, i.e.,one along each coordinate axis. In the rectangular system of coordinates, the equivalent scalar equations are:

$$\Sigma F_x = 0 \qquad \Sigma M_x = 0$$
$$\Sigma F_y = 0 \quad \text{and} \quad \Sigma M_y = 0$$
$$\Sigma F_z = 0 \qquad \Sigma M_z = 0 \tag{3.3a}$$

If, in a given situation, a body which is acted upon by a system of forces is to be brought to a state of equilibrium it is achieved in two steps:

1. The system of forces is reduced to an equivalent force, its point of application as desired and the accompanying moment.
2. An equilibrant action equal and opposite to the corresponding equivalent action is applied in the desired manner:

$$\mathbf{E}_f = -\mathbf{R} = -\Sigma\,\mathbf{F}$$
$$\mathbf{E}_m = -\Sigma\,\mathbf{M}$$

If the simplest equivalent for a particular case can be reduced to a single force **R** suitably applied, then the simplest equilibrant for that case

$$\mathbf{E} = -\mathbf{R}$$

may also be applied along the same line of action.

***Example* 3.1** The following forces as shown in Fig. Ex. 3.1 are applied to a rigid body initially at rest:

$$\mathbf{F}_1 = 2\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k} \text{ at } (7, 2, 3),\ \mathbf{F}_2 = \mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k} \text{ at } (5, 1, 0)$$
$$\mathbf{F}_3 = -2\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k} \text{ at } (4, 0, -1),\ \mathbf{F}_4 = -\mathbf{i} - \mathbf{j} - \mathbf{k} \text{ at } (2, 2, 1)$$

Show that the body is in equilibrium.

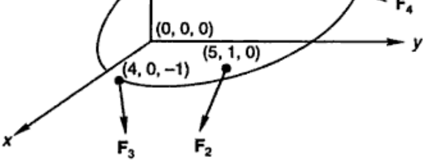


Fig. Ex. 3.1

Solution The necessary conditions of equilibrium of a body are

$\Sigma\,\mathbf{F} = 0$ and $\Sigma\,\mathbf{M} = 0$. In this case,

$$\Sigma \mathbf{F} = (2\,\mathbf{i} + \mathbf{i} - 2\,\mathbf{i} - \mathbf{i}) + (\mathbf{j} - 2\,\mathbf{j} + 2\,\mathbf{j} - \mathbf{j}) + (3\,\mathbf{k} - 4\,\mathbf{k} + 2\,\mathbf{k} - \mathbf{k})$$

= zero, identically.

Taking moments about the origin (0, 0, 0),

$$\Sigma \mathbf{M}_0 = (7\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) \times (2\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}) + (5\,\mathbf{i} + 1\,\mathbf{j} + 0\,\mathbf{k}) \times (\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k})$$

$$+ (4\,\mathbf{i} + 0\,\mathbf{j} - 1\,\mathbf{k}) \times (-2\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}) \times (2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}) \times (-\mathbf{i} - \mathbf{j} - \mathbf{k}) = 0,$$

identically, which shows that the necessary conditions of equilibrium are identically, satisfied. The body should, therefore, be in equilibrium.

It is interesting to observe that for a body in equilibrium, the summation of the moments about any arbitrary point must vanish. Let us take moments about the point (2, 2, 1):

$$\begin{aligned}\Sigma \mathbf{M} &= ((7-2)\mathbf{i} + (2-2)\mathbf{j} + (3-1)\mathbf{k}) \times (2\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}) \\ &\quad + ((5-2)\mathbf{i} + (1-2)\mathbf{j} + (0-1)\mathbf{k}) \times (\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}) \\ &\quad + ((4-2)\mathbf{i} + (0-2)\mathbf{j} + (-1-1)\mathbf{k}) \times (-2\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}) \\ &= (-2 + 2 + 0)\mathbf{i} + (-11 + 11 + 0)\mathbf{j} + (5 - 5 + 0)\mathbf{k} \\ &= \text{zero, identically, as expected}\end{aligned}$$

3.2 FREE-BODY DIAGRAM IN STATICS

The technique of isolating a body from its surroundings or isolating a subsystem from the remaining system by the introduction of 'reaction' forces, etc., referred to as a free-body diagram, is employed extensively in considering equilibrium.

Let us extend the example of a free-body diagram of a book lying flat on a table given in Chapter 1 in view of the concept of friction forces discussed in Chapter 2. Consider the static equilibrium of a book lying flat on a table and a pencil in an inclined position partly on the book and partly on the table. We can consider the system of the table top, book and pencil in equilibrium as shown in Fig. 3.2. A free-body diagram may be drawn for a single body, e.g., the pencil alone, the book alone, a part of the pencil or a page of the book. Free-body diagrams may also be drawn for the subsystems comprising the book and pencil, the book and table top and the pencil and table top. Three such free-body diagrams are shown in Figs. 3.2 (b), (c) and (d).

Some of the interesting points are summarised as follows:

1. A free-body diagram of a system in equilibrium should comprise a force system in equilibrium.
2. If a system of bodies is in equilibrium, then each subsystem and each constituent body must also be in equilibrium.
3. The force due to gravitation, i.e., the weights acting through the centres of mass of the bodies are classified as external forces if the bodies are isolated from the earth, i.e., the table as in Figs. 3.2 (b), (c) and (d) and internal forces if the earth is a part of the system as in Figs. 3.2 (a).

Examples of free-body diagrams of some objects are shown in Fig. 3.3.

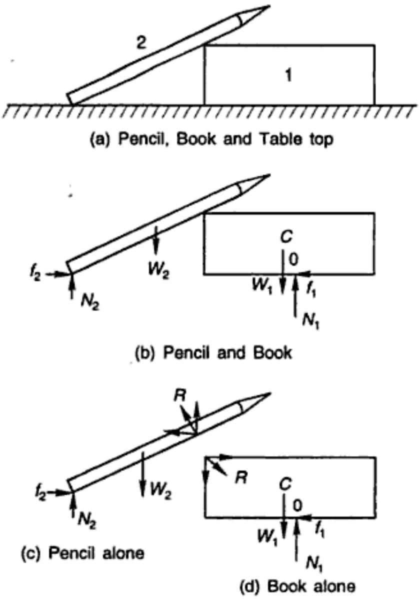


Fig. 3.2 ***Free-body Diagrams***

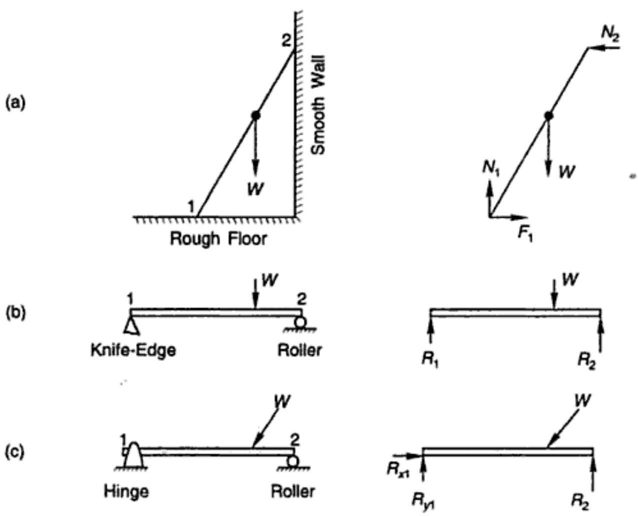


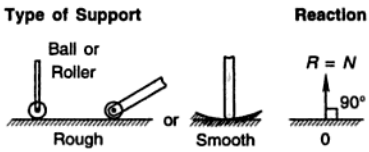
Fig. 3.3 ***Examples of Free-body Diagrams of Objects***

(d)

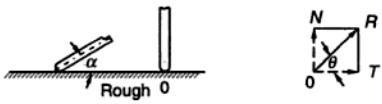
Fig. 3.3 ***Examples of Free-body Diagrams of Objects*** (*Contd.*)

3.3 REACTIONS BY SUPPORTS

Different types of supports are employed to hold structural members and components in motion. The purpose of a support is to provide a desirable reaction. Let us see the nature of reactions offered by different types of supports as shown in Fig. 3.4.

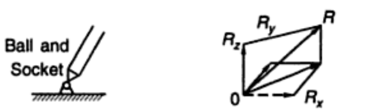


(a) Normal to the surface at that point



(b) With normal and tangential components

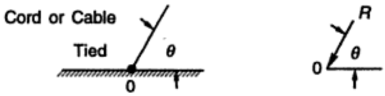
(c) Vertical and horizontal components



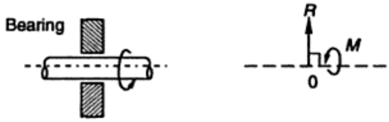
(d) Along the member in space (three components)

(e) A moment and a force depending upon loading

Fig. 3.4 ***Nature of Reactions by Some Typical Supports***



(f) Tension in the Cable



(g) A normal force and a twisting moment

Fig. 3.4 ***Nature of Reactions by Some Typical Supports*** *(Contd.)*

A ball or roller support on a rough surface can roll and hence not provide a tangential reaction. The reaction at a ball or roller support is, therefore, normal to the surface at that point. Similarly, the reaction by a smooth surface to a member in contact with it must be normal to the surface.

Reaction by a rough surface to an element in contact with it can be non-normal, i.e., at an angle θ to it even if the member is inclined at any angle α or normal to it. This is because both normal and tangential components of reaction are possible.

A pin-joint or a hinge gives rise to reaction along the axis of the member.

A ball-and-socket joint provides reaction along the axis of the member. Since a ball-and-socket joint is a universal joint in space, the reaction is also in space.

A fixed end of a member is capable of providing a general reaction **R** and a general moment **M**.

A cord or a cable tied to a surface can be under tension only, the reaction must be along the cord or the cable.

A bearing of a rotating shaft can provide a reaction **R** and a frictional moment **M**.

***Example* 3.2** Draw the free-body diagrams of all the members of a simple-loaded system sketched in Fig. Ex. 3.2 and comment on their usefulness in the analysis of the system.

Solution First, the free-body diagram of the centre system is drawn in Fig. Ex. 3.2(a) Solution. This is necessary to estimate the reactions at the supports A and E. The unknowns R_A and R_E can be determined by considering the equilibrium of the system.

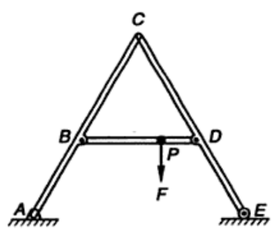


Fig. Ex. 3.2

Free-body diagrams of the members are drawn in Fig. Ex. 3.2 (Solution) (b) and (c).

In Fig. Ex. 3.2 (Solution) (b) all the reactions are taken in the direction of the positive coordinate axes. By doing so we get 15 reaction unknowns, viz. A_x, A_y, B_x, B'_x, B_y, B'_y, etc. These 15 unknowns can be solved by using three equilibrium equations for each member and 2 action-reaction equations at each joint of two members. (Total number of equations = $3 \times 3 + 2 \times 3 = 15$.)

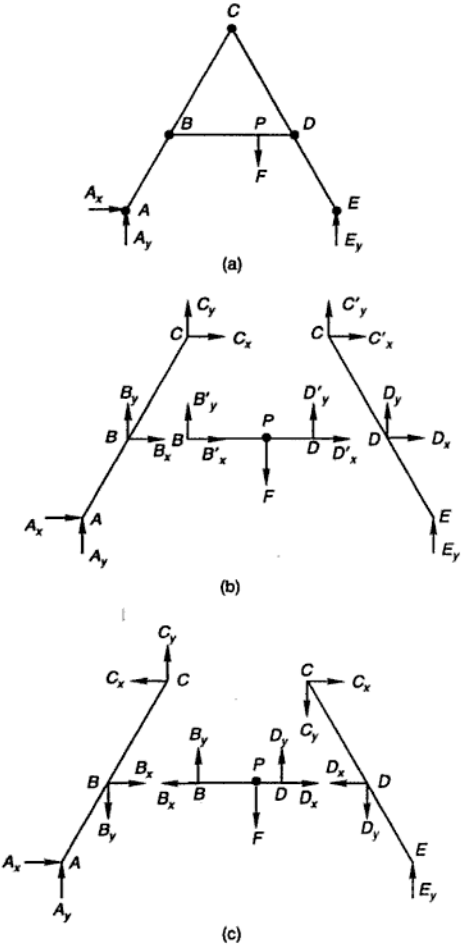


Fig. Ex. 3.2 ***(Solution)***

Though such a procedure leads to the correct solution of the structure, it is not a normal practice to assume all the reactions to be initially positive in a free body diagram.

While drawing the direction of these forces in free-body diagram, the following points are considered:

1. The action-reaction equations are applied directly while drawing the free-body diagram itself. For example in Fig. (c), the action-reaction principle

gives that at joint C, $C_x = -C_x'$, $C_y = -C_y'$. These two equations are eliminated from the calculations and C_x', C_y' are shown equal and opposite to C_x, C_y respectively in the free-body diagram itself. By doing this in the present case at every point we are left to analyse the structure for 9 equilibrium equations only.

2. It is a usual practice to assume the direction of the reactions intuitively rather than taking them arbitrarily. For example, in Fig. (c) on the member *BD*, B_y and D_y should act in upward direction in order to balance *F*. Similarly intuitively we can see that $\mathbf{A}_y$, $\mathbf{E}_y$ should act upward and $\mathbf{C}_x$ should act to the left on *CA*. However, a later check by the equilibrium equations decides the validity of these assumptions.

It must also be noted that free-body diagrams of the members are not mutually independent and hence it is, in general, not possible to determine the unknowns on a member by considering the equilibrium of that member alone. For example, suppose after finding R_A and R_E, we wish to determine the reactions at *B*, *C* and *D*, i.e., B_x, B_y, C_x, C_y, D_x and D_y. There are four unknowns on the members *ABC* considered alone and also four unknowns on the member *CDE* considered alone, but taken together as *ABCDE*, there are again only four unknowns. Similarly, on member *BD* alone, there are four unknowns. It may not be possible to solve for the unknowns by considering the equilibrium of each member one by one, but it must be possible to determine all the unknowns by writing all the equations of equilibrium for all the members.

3.4 EQUILIBRIUM OF A PARTICLE

A particle, by definition, has negligible dimension in comparison with the coordinates describing its motion. It is an idealisation of a real body when its mass can be considered to be concentrated at a central point. A particle can, therefore, experience a system of forces which must be concurrent at the particle. The concurrent system of forces can act along a common line to constitute collinear forces or they can belong to a plane or a spatial system as shown in Fig. 3.5. In each case, the condition of equilibrium must be

$$\Sigma \mathbf{F} = 0$$

Since there is a point through which all the forces pass, it follows that the summation of the moments of all the forces about that point vanishes. The summation of the moments of the concurrent forces about any other point in space must also be zero. The other condition

$$\Sigma \mathbf{M} = 0$$

is, therefore, automatically satisfied by the state of concurrency of the forces. As a matter of fact, the conditions of equilibrium, namely

$$\mathbf{F} = 0 \qquad \text{and} \qquad \mathbf{M} = 0$$

are alternative conditions for a concurrent force system; one implies the other also. Hence, it is a matter of choice whether to use one or the other.

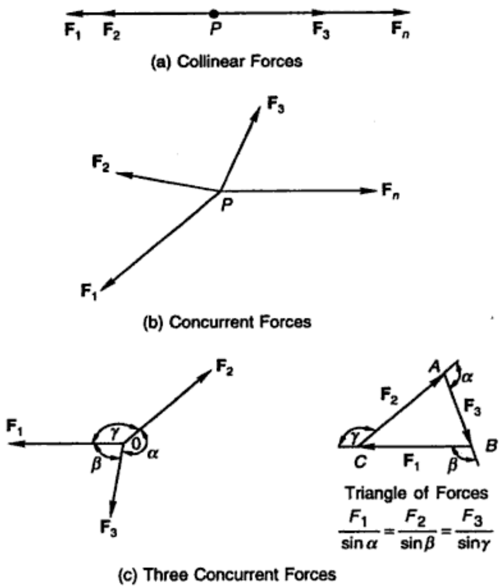


Fig. 3.5 ***Equilibrium of a Particle***

For equilibrium under the action of coplanar concurrent forces,

$$\Sigma \mathbf{F} = 0$$

which implies that

$$F_{1x} + F_{2x} + \ldots = 0; \qquad F_{1y} + F_{2y} + \ldots 0$$

In words, the sums of the components of the forces along any two mutually perpendicular directions in the plane of the forces must be zero for the equilibrium of a particle under the action of coplanar concurrent forces. The polygon of forces should automatically close to provide the zero resultant force for equilibrium.

It is interesting to observe that *two concurrent forces must be coplanar* since a plane can always be passed through two intersecting straight lines. Further, *three concurrent forces in equilibrium must be coplanar*, for if the third force was not in the plane of the other two it would result in a component normal to the plane and upset the equilibrium. However, four concurrent forces in equilibrium may be coplanar or spatial. Three concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ which maintain the particle in equilibrium as shown in Fig. 3.5(c) are related by the sine law.

Lami's Theorem

Lami's theorem states that, if three forces acting on a particle keep it in equilibrium, then each force is proportional to the sine of the angle between the other two and the constant of proportionality is the same. Symbolically,

$$\boxed{\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} = k} \tag{3.4}$$

where α, β and γ are the angles between $\mathbf{F}_2$, $\mathbf{F}_3$; $\mathbf{F}_1$, $\mathbf{F}_3$ and $\mathbf{F}_1$, $\mathbf{F}_2$ respectively. The triangle of forces, in this case, should close to provide the resultant force zero for equilibrium and by applying the sine-rule for a triangle, Lami's theorem may be obtained as follows:

For the triangle *ABC* shown in Fig. 3.5(c) corresponding to the forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, acting at a point *O*,

$$\angle CAB = 180° - \alpha$$

$$\angle ABC = 180° - \beta$$

$$\angle BCA = 180° - \gamma$$

From the sine rule for the triangle,

$$\frac{F_1}{\sin(180° - \alpha)} = \frac{F_2}{\sin(180° - \beta)} = \frac{F_3}{\sin(180° - \gamma)}$$

and from the fact that $\sin(180° - \alpha) = \sin\alpha$, etc., it reduces to the Lami's theorem

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

For equilibrium, under the action of spatial concurrent forces,

$$\Sigma\,\mathbf{F} = 0$$

which implies that

$$F_{1x} + F_{2x} + \dots = 0$$

$$F_{1y} + F_{2y} + \dots = 0$$

$$F_{1z} + F_{2z} + \dots = 0$$

In words, the sums of the components of the forces along the three coordinate directions must be zero for the equilibrium of a particle under the action of spatial concurrent forces.

Let us discuss another aspect of the state of a particle. If a particle is subjected to a system of concurrent forces

$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \dots$$

then it may or may not be in equilibrium. If it is in equilibrium, then

$$\Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \dots = 0$$

If it is not in equilibrium, then the system reduces to a resultant force

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \dots$$

which represents the net external force on the particle. In case it is desired to bring the particle in equilibrium, provision must be made to apply a force equal and opposite to **R**. The force which, when applied on it brings it to a state of equilibrium, is called the *equilibrant force* denoted by **E**

$$\mathbf{E} = -\mathbf{R} \tag{3.5}$$

***Example* 3.3** A body weighing 800 N is hung from a horizontal ring 6 m in diameter by means of three cords, each 5 m long. On the ring, two of the cords are placed 90° apart and the point of attachment of the third cord bisects the remaining arc of the ring. Find the tension in each cord.

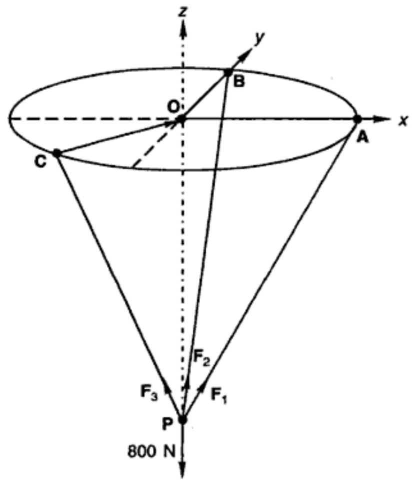


Fig. Ex. 3.3

Solution With reference to Fig. Ex. 3.3 where the body is represented by a particle *P*,

$$\mathbf{OA} = 3\,\mathbf{i},\ \mathbf{OB} = 3\,\mathbf{j},\ \mathbf{OC} = -3/\sqrt{2}\,\mathbf{i} - 3/\sqrt{2}\,\mathbf{j},\ \mathbf{OP} = -5\,\mathbf{k}$$

From these vectors

$$\mathbf{PA} = 3\,\mathbf{i} + 5\,\mathbf{k},\ \mathbf{PB} = 3\,\mathbf{j} + 5\,\mathbf{k},\ \mathbf{PC} = -2.12\,\mathbf{i} - 2.12\,\mathbf{j} + 5\,\mathbf{k}$$

and the unit vectors along these cords are given by

$$\mathbf{e}_{PA} = (3\,\mathbf{i} + 5\,\mathbf{k})/\sqrt{3^2 + 5^2} = 0.51\,\mathbf{i} + 0.86\,\mathbf{k}$$

$$\mathbf{e}_{PB} = 0.51\,\mathbf{j} + 0.86\,\mathbf{k}$$

$$\mathbf{e}_{PC} = -0.36\,\mathbf{i} - 0.36\,\mathbf{j} + 0.86\,\mathbf{k}$$

Let the tension in these cords be F_1, F_2 and F_3 respectively in magnitude. Vectorially,

$$\mathbf{F}_1 = F_1(0.51\,\mathbf{i} + 0.86\,\mathbf{k})$$

$$\mathbf{F}_2 = F_2(0.51\,\mathbf{j} + 0.86\,\mathbf{k})$$

$$\mathbf{F}_3 = F_3(-0.36\,\mathbf{i} - 0.36\,\mathbf{j} + 0.86\,\mathbf{k})$$

For equilibrium of the particle P,

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 - 800\,\mathbf{k} = 0$$

whence

$$0.51F_1 - 0.36F_3 = 0$$

$$0.51F_2 - 0.36F_3 = 0$$

$$F_1 + F_2 + F_3 = 800/0.86 = 930$$

and solving these equations,

$$F_1 = 272 \text{ N}, F_2 = 272 \text{ N and } F_3 = 386 \text{ N}$$

Experiment E1

Equilibrium under Coplanar Forces

OBJECTIVE

To study the equilibrium of a particle under the action of forces in a plane.

APPARATUS

A horizontal circular force table, also called *universal force table*, as shown in Fig. E 1.1(a), or a vertical rectangular force table, also called *Gravesand's apparatus*, as shown in Fig. E1.1(b), standard weights and metre rod.

BACKGROUND INFORMATION

The state of equilibrium of a particle refers to a state of uniform velocity of rest. In the present case, it is intended to study the forces acting on a particle when it is at rest.

A particle cannot be in equilibrium when a single force is applied on it. It would be in equilibrium under the action of two or more forces if the vectorial summation of forces is zero

$$\Sigma\,\mathbf{F} = 0 \tag{E1.1}$$

In particular, if a particle is subjected to only two forces, the forces must be equal and opposite, i.e., equal in magnitude, in the same line of action but opposite in sense in order to keep it in equilibrium. If it is subjected to only three forces, the three forces must be coplanar for equilibrium. Conversely, three non-coplanar forces cannot keep a particle in equilibrium.

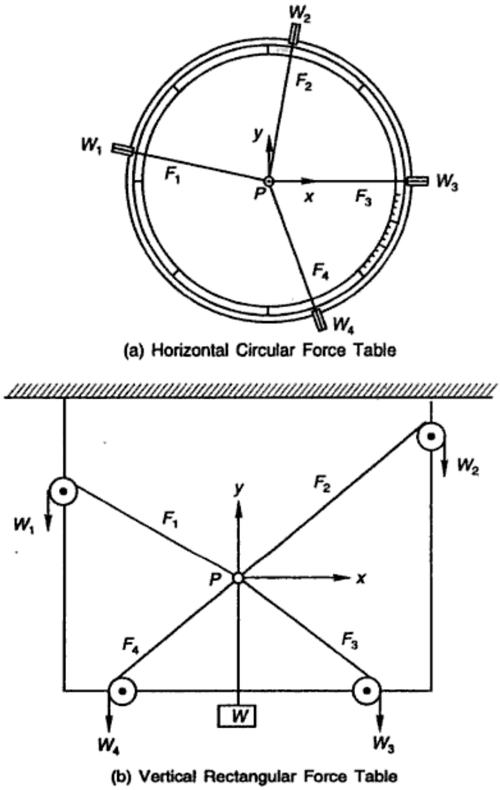


Fig. E1.1 ***Force Table***

This fact can be appreciated by recognising that the resultant of any two must be equal and opposite to the third force for equilibrium, and this cannot happen unless the three forces lie in one and the same plane. If a particle is in equilibrium under the action of four or more forces, the forces may be spatial, i.e., not necessarily confined to act along the same line or in the same plane.

When a particle is in equilibrium under the action of three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, as shown in Fig. E1.2, the condition of equilibrium, i.e.,

$$\Sigma \mathbf{F} = 0; \ \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$$

may alternatively be expressed as Lami's theorem or Triangle Law of Equilibrium.

Lami's Theorem *If a particle is in equilibrium under the action of three forces, each force must bear the same proportionality with the sine of the angle between the other two forces.*

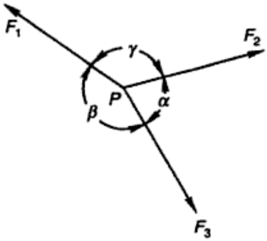


Fig. E1.2 ***A Particle in Equilibrium***

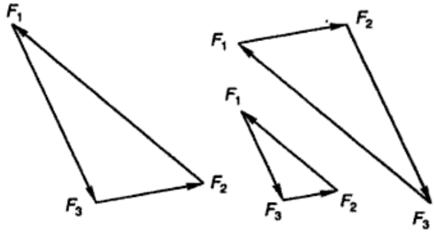


Fig. E1.3 ***Possible Triangles to Represent Force F_1 F_2 and F_3***

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

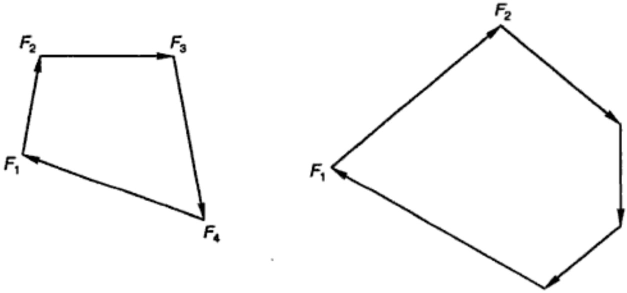
Triangle Law of Equilibrium *If a particle is in equilibrium under the action of three forces, the forces must be represented in magnitude, direction and sense by the sides of a triangle, taken in order, in the same sense.*

When a particle is in equilibrium under the action of more than three coplanar forces, the condition of equilibrium i.e.,

$$\Sigma \mathbf{F} = 0;\ \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 + \ldots = 0$$

may alternatively be stated in terms of the *polygon law of equilibrium.*

If a particle is in equilibrium under the action of n coplanar forces, the forces must be represented in magnitude, direction and sense by the n sides of a polygon, taken in order, in the same sense.



(a) Polygon of Forces for Fig. E.1.1(a)

(b) Polygon of Forces for Fig. E1.1(b)

Fig. E1.4 ***Polygon Law of Equilibrium***

Observations and Calculations

The table of observations and calculations depends precisely on the motivation of the experiment which may be one or more of the following:

(a) *To demonstrate the triangle law or polygon law of equilibrium*
In this case, a particle may be subjected to as many forces as desired on either force table and the positions of the strings together with the loads suspended at the corresponding endpoints are noted. The forces, in magnitude and direction, are drawn end to end, in order to verify whether a closed triangle or polygon is formed. If so, within the allowable limits of error, the corresponding law is verified; if not, the sources of error must be recognised and minimised to allow a closer prediction by the law.

(b) *To demonstrate the application of Lami's theorem*
In this case, only three forces must be made to act on the particle on either force table. The three forces must automatically be coplanar for maintaining equilibrium.

Measurements must be made for the directions of the three strings and the loads applied at the ends. The included angles between every pair of forces are calculated and, as per Fig. E1.2, the following constants are calculated:

$$k_1 = \frac{F_1}{\sin \alpha}; \quad k_2 = \frac{F_2}{\sin \beta}; \quad k_3 = \frac{F_3}{\sin \gamma}$$

If k_1, k_2 and k_3 are equal within the allowable limits of error, the validity of the Lami's theorem is upheld; if not efforts must be made to recognise the sources of error and to improve upon the result.

(c) *To determine the two unknown loads hanging at the ends of two strings.*
In this case, Lami's or triangle law/polygon law of equilibrium is taken for granted to be valid and, from the knowledge of the other forces, the unknown forces are determined. The procedure is to record all the directions of the strings and all the known loads. Then apply Lami's theorem or use the triangle law if the number of total forces is three or use the polygon law if the number of forces exceed three. It will be seen that, with all the directions known and with magnitudes for two less than all the forces known, it is possible to complete the polygon. The unknown forces are then read off from the corresponding sides of the closed polygon to the same scale as the other forces.

(d) *To demonstrate the validity of the condition* **SF** $= 0$ *for a given case of equilibrium*
In this case, the measurements of the inclinations of the strings may be made with reference to *x*- and *y*-axis selected arbitrarily. The corresponding forces in the strings are known from the loads applied at the ends. The forces are then resolved into *x* and *y* components. Summation of the *x* components and of the *y* components are made separately and observed whether the results are close to zero or not, i.e.,

$$S_x = F_{x1} + F_{x2} + F_{x3} + \ldots$$

$$S_y = F_{y1} + F_{y2} + F_{y3} + \ldots$$

If the summations S_x and S_y are each close to zero, the experimental values obey the condition of equilibrium $\Sigma \mathbf{F} = 0$, if not the sources of error must be minimised.

RESULTS

The results, whether or not a theorem or law holds for the situations examined, should be recorded. Discrepancies, if any, may be mentioned.

POINTS FOR DISCUSSION

1. State the sources of error in the experiment. Notice if the pulleys have friction, if the threads have knots, if the central ring and the strings are coplanar, if the strings are tied to the ring radially, if the weights are standard, if the graduations are uniform, etc.
2. The condition of equilibrium $\Sigma\mathbf{F} = 0$ is valid in general. The graphical conditions, i.e., the triangle law and polygon law of equilibrium apply only when the forces are coplanar. Can you suggest how the graphical method may be extended to be employed for spatial forces?
3. When the experimental values differ from the theoretical and graphical values, which of them must be in error and why?
4. Should the condition $\Sigma\mathbf{F} = 0$ hold good for the equilibrium of a rigid body if it is subjected to a system of forces which are
   (a) concurrent
   (b) non-concurrent
   (c) coplanar and
   (d) spatial
   Discuss why the answer is "yes" in all these cases.
5. What happens to the point *P* if a weight, say W_1, is increased by 10%? What needs to be done to bring it back to equilibrium?

***Example* 3.4** Four pieces of string knotted at *A* support two equal masses in equilibrium in a vertical plane as shown. Determine the tensions in the strings *AB* and *AC* and the angle θ between *AB* and *AE* for minimum tension in *AB*.

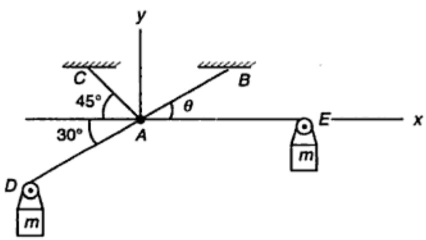


Fig. Ex. 3.4

Solution As shown in Fig. Ex. 3.4 the tension in *AE* must be *mg*; so also the tension in *AD* must be *mg*, since the pulleys are assumed to be frictionless.

Let the *x*-axis be along *AE* and *y*-axis perpendicular to it. Then,

$$\mathbf{F}_{AE} = mg\,\mathbf{i}$$

$$\mathbf{F}_{AD} = -\,mg\cos 30\,\mathbf{i} - mg\sin 30\,\mathbf{j}$$

$$= -\frac{\sqrt{3}}{2}\, mg\, \mathbf{i} - mg/2\, \mathbf{j}$$

$$\mathbf{F}_{AB} = F_{AB}\,(\cos\theta\ \mathbf{i} + \sin\theta\,\mathbf{j})$$

$$\mathbf{F}_{AC} = F_{AC}\,(-\cos 45\ \mathbf{i} + \sin 45\ \mathbf{j})$$

$$= -\frac{F_{AC}}{\sqrt{2}}\,\mathbf{i} + \frac{F_{AC}}{\sqrt{2}}\,\mathbf{j}$$

For equilibrium at A, $\Sigma\mathbf{F} = 0$

or
$$\mathbf{F}_{AB} + \mathbf{F}_{AC} + \mathbf{F}_{AD} + \mathbf{F}_{AE} = 0$$

$$mg - \frac{\sqrt{3}}{2}\, mg + \mathbf{F}_{AB}\cos\theta - \frac{\mathbf{F}_{AC}}{\sqrt{2}} = 0 \qquad \text{(i)}$$

$$-mg/2 + \mathbf{F}_{AB}\sin\theta + \frac{\mathbf{F}_{AC}}{\sqrt{2}} = 0 \qquad \text{(ii)}$$

These are 2 equations for the three unknowns F_{AB}, F_{AC} and θ. The third equation comes up from the physical constraint, i.e., the force in AB must be the minimum.

From (i),
$$\mathbf{F}_{AB}\cos\theta - \frac{\mathbf{F}_{AC}}{\sqrt{2}} = \left(\frac{\sqrt{3}}{2} - 1\right) mg$$

and from (ii),
$$\mathbf{F}_{AB}\sin\theta + \frac{\mathbf{F}_{AC}}{\sqrt{2}} = \frac{mg}{2}$$

Adding the two,

$$\mathbf{F}_{AB}\,(\cos\theta + \sin\theta) = \frac{\left(\sqrt{3} - 1\right)}{2}\, mg$$

Minimum $\mathbf{F}_{AB}$ corresponds to maximum $(\cos\theta + \sin\theta)$

i.e.,
$$\frac{d}{d\theta}\,(\cos\theta + \sin\theta) = 0$$

$$-\sin\theta + \cos\theta = 0$$

or
$$\sin\theta = \cos\theta, \text{ i.e., } \underline{\theta = 45^\circ}$$

Now,
$$\mathbf{F}_{AB}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{\left(\sqrt{3} - 1\right)}{2}\, mg$$

whence,
$$\mathbf{F}_{AB} = \frac{\left(\sqrt{3} - 1\right)}{2\sqrt{2}}\, mg \cong \underline{0.26\, mg}$$

Now, $$0.26\,mg\cdot\frac{1}{\sqrt{2}}+\frac{\mathbf{F}_{AC}}{\sqrt{2}}=\frac{mg}{2}$$

$$\mathbf{F}_{AC}=\underline{0.45\,mg}$$

3.5 EQUILIBRIUM OF A RIGID BODY

A rigid body may be subjected to one of the force systems classified as follows:

(a) Concurrent force system: Collinear, plane or spatial
(b) Parallel force system: plane or spatial
(c) Coplanar force system: concurrent and non-concurrent, parallel and non-parallel
(d) Spatial force system: concurrent and non-concurrent, parallel and non-parallel.

(a) Concurrent Force Systems

The analysis of the static equilibrium of a rigid body under the action of concurrent forces is quite similar to that of a particle. Concurrent forces may be collinear, coplanar or spatial and the vector method or the algebraic method can be employed with advantage.

If there are two forces acting at a point in equilibrium, the forces must be collinear. If there are three forces acting at a point in equilibrium, the forces must lie in a plane, i.e., the force system must be coplanar. This follows from the fact that any two lines of forces acting at a point must constitute a plane and the third force cannot have a component normal to that plane; otherwise that unbalanced component would upset the equilibrium. If there are four or more forces acting at a point in equilibrium, these may be coplanar or spatial.

The condition of equilibrium for a rigid body under the action of concurrent forces

$$\mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3,\ \ldots$$

is that their resultant

$$\mathbf{R}=\Sigma\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\ldots$$

must vanish, i.e.,

$$\mathbf{R}=\Sigma\mathbf{F}=0$$

This vector equation for equilibrium stands for three scalar equations for a general concurrent force system:

$$R_x=\Sigma F_x=0=F_{1x}+F_{2x}+F_{3x}+\ldots$$

$$R_y=\Sigma F_y=0=F_{1y}+F_{2y}+F_{3y}+\ldots$$

$$R_z=\Sigma F_z=0=F_{1z}+F_{2z}+F_{3z}+\ldots$$

where x, y and z are the coordinate axes arbitrarily drawn through the point of concurrency as shown in Fig. 3.6(a).

If, on the other hand, there is a rigid body subjected to a system of concurrent forces resulting in **R**, then an equilibrant force **E** given by

$$\mathbf{E} = -\mathbf{R}$$

must be applied passing through the point of concurrency in order to bring the body to equilibrium.

(b) Parallel Force System

If a rigid body is subjected to a parallel force system, the resultant of the forces may be a non-zero force or a zero force, unaccompanied or accompanied by a couple-moment. Equilibrium of the rigid body demands that the resultant force and resultant moment must vanish:

$$\Sigma\mathbf{F} = 0; \qquad \Sigma\mathbf{M} = 0 \tag{3.6}$$

It may be understood that the moment about any arbitrary point O as shown in Fig. 3.6(b) may be considered and should be equated to zero.

For a plane parallel-force system, the moment of the forces about any point in the plane of the forces must be perpendicular to the plane of the forces. It is interesting to note that moment summation about different points provide different equations, such as

$$\Sigma\mathbf{M}_1 = 0, \qquad \Sigma\mathbf{M}_2 = 0, \qquad \Sigma\mathbf{M}_3 = 0, \qquad \ldots$$

In fact, there can be only two of these equations mutually independent; one in its own right and the other in lieu of

$$\Sigma\mathbf{F} = 0$$

It follows that for a system of plane parallel forces acting on a body, the conditions of equilibrium may alternatively be stated as

$$\Sigma\mathbf{M}_1 = 0 \quad \text{and} \quad \Sigma\mathbf{M}_2 = 0 \tag{3.7}$$

where 1 and 2 are two suitably chosen points.

(c) Coplanar Force System

If the forces applied on a rigid body are such that their lines of action lie in the same plane and the moments due to the couples or otherwise are directed perpendicular to the plane, the body is said to be subjected to a coplanar force system. It is usual to choose the x-y plane in the plane of the coplanar force system and the x- and y-axes are chosen conveniently in regard to the directions of the forces as shown in Fig. 3.6(c). The necessary conditions of equilibrium reduce to a set of three equations:

$$\Sigma F_x = 0 \quad \text{and} \quad \Sigma\mathbf{M} = 0 \tag{3.8}$$
$$\Sigma F_y = 0$$

The moment summation referred to above is about any point in the plane of the forces; it is about the z-axis through the chosen point.

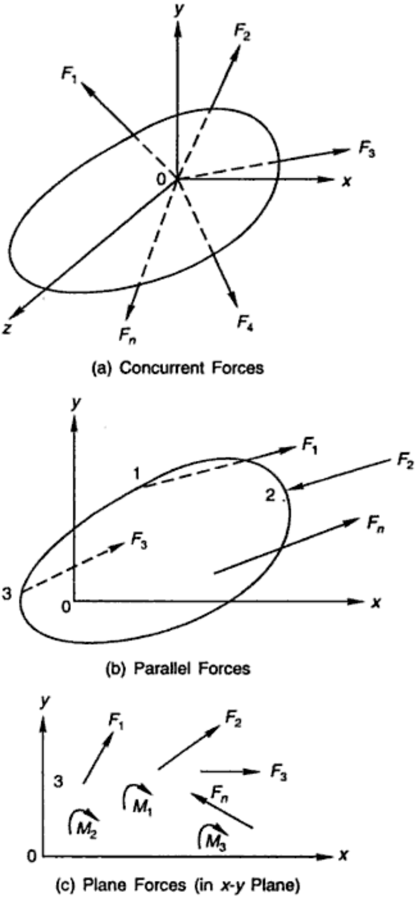


Fig. 3.6 ***Different Force Systems Acting on a Rigid Body***

An interesting and extremely useful point is to express the force summations of equilibrium in terms of equivalent moment summations. The advantage of doing so is that the moments can be taken about the line of action of a force which is unknown and needs to be eliminated at least temporarily. For example, if there are three unknown coplanar forces in a system, then moments about the line of action of each force, in turn, will yield three moment equations.

$$\begin{aligned} \Sigma M_1 &= 0 \\ \Sigma M_2 &= 0 \\ \Sigma M_3 &= 0 \end{aligned} \tag{3.9}$$

which comprise a set equivalent to Eq. (3.8).

It may as well be decided to consider the equivalent equations of equilibrium as

$$\begin{aligned} \Sigma M_1 &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_2 &= 0 \end{aligned} \tag{3.10}$$

In each case, there are, in essence, two force equations and one moment equation; the apparent difference is only in the embodiment of these equations. It may so happen that the three equations set up in a particular case may not form an independent or a complete set, e.g., when a moment is taken about the point of intersection of two or more lines of forces. In such cases further equations for moment summations would provide an answer. It may also be noted that it is often advisable to set up a redundant equation to provide a check on the solution of the problem.

(d) Spatial Force System

The necessary conditions of equilibrium for a rigid body subjected to a general force system are those specified earlier by general equilibrium considerations

$$\begin{aligned} \Sigma \mathbf{F} &= 0 \\ \Sigma \mathbf{M} &= 0 \end{aligned}$$

These two vector equations are equivalent to a set of six scalar equations:

$$\begin{aligned} \Sigma F_x &= 0 & & \Sigma M_x = 0 \\ \Sigma F_y &= 0 & \text{and} \quad & \Sigma M_y = 0 \\ \Sigma F_z &= 0 & & \Sigma M_z = 0 \end{aligned} \tag{3.11}$$

where the x, y and z axes are chosen arbitrarily but with due regard to convenience of handling the force system. For example, it may be preferable to have an axis in a direction in which a number of forces act and it may be advantageous to choose the x-y plane as the plane in which a number of forces lie.

3.6 EQUILIBRIUM OF A SYSTEM OF PARTICLES

Consider a system consisting of three particles P_1, P_2 and P_3 as shown in Fig. 3.7. The system is subjected to net external forces $\mathbf{F}_1$ at P_1, $\mathbf{F}_2$ at P_2 and $\mathbf{F}_3$ at P_3 as shown.

The resultant action on the system of particles consists of a single force $\mathbf{F}$ equal to the sum of the external forces.

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

The system will be in equilibrium if the resultant of the external forces vanishes, i.e.,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$$

or $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ constitute a closed triangle.

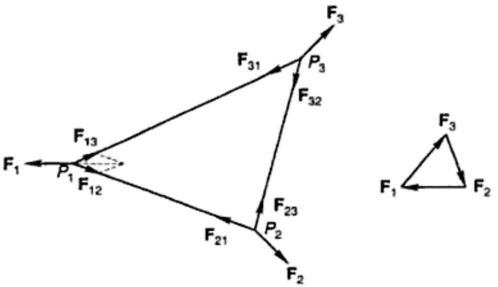


Fig. 3.7 ***A System of Three Particles***

It is essential that each constituent particle of a system must also be in equilibrium. Take, for example, particle P_1. It is subjected to the external force $\mathbf{F}_1$ as well as the internal forces, $\mathbf{F}_{13}$ due to particle P_3 and $\mathbf{F}_{12}$ due to particle P_2. Obviously, $\mathbf{F}_1$, $\mathbf{F}_{12}$ and $\mathbf{F}_{13}$ must keep the particle in equilibrium.

Since action and reaction must be equal and opposite

$$\mathbf{F}_{12} = -\mathbf{F}_{21};\ \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

$$\mathbf{F}_{13} = -\mathbf{F}_{31};\ \mathbf{F}_{13} + \mathbf{F}_{31} = 0$$

$$\mathbf{F}_{23} = -\mathbf{F}_{32};\ \mathbf{F}_{23} + \mathbf{F}_{32} = 0$$

It follows that the sum of all internal forces in a system must be zero.

Extending the argument to a system of a number of particles, say n, the equilibrium demands that

1. *The vector sum of all external forces is zero*

$$\Sigma\mathbf{F}_e = 0$$

2. *The vector sum of all external plus all internal forces must be zero*

$$\Sigma\mathbf{F}_e + \Sigma\mathbf{F}_i = 0$$

in order that each particle be in equilibrium separately.

It is interesting to note that a system of three particles must lie in a plane and the forces on the particles must constitute a coplanar force system for equilibrium. A system of four or more particles may not be coplanar and may constitute spatial force systems for equilibrium.

General Comments

A number of rigid bodies may be interconnected to comprise a system. In such a case, if the total system of rigid bodies is in equilibrium, every subsystem and every component of the system must also be in equilibrium.

It follows, therefore, that the necessary conditions for equilibrium may be written for the total system as well as for any desired subsystem or a component of the system. For example, if two rigid bodies are connected by an inextensible string which is kept taut, then the net external forces and moments should satisfy the conditions of equilibrium as well as the net external forces and moment, as observed from the free-body diagram of each body, should also satisfy the conditions of equilibrium separately.

***Example* 3.5** Three homogeneous cylinders of the same material having masses m, $2m$ and m are placed in a container with a curved base as shown in Fig. Ex. 3.5. Determine the reaction on cylinder P from the left wall, upper cylinder and curved base. Assume that the cylinders are of equal length and that they are placed such that their centres of gravity lie in the same vertical plane. The radius of the curved base is three times the radius of cylinder P.

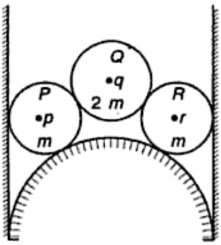


Fig. Ex. 3.5

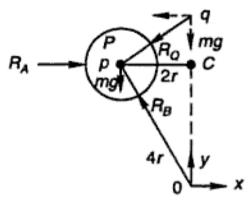


Fig. Ex. 3.5 (*Solution*)

Solution By geometry of the system,

$$op = 3r + r = 4r$$

$$pc = 3r - r = 2r$$

$$oc = \sqrt{(4r)^2 - (2r)^2} = 3.464r$$

$$pq = r + \sqrt{2}\ r = 2.414r$$

and
$$cq = \sqrt{2.414^2 - 2^2}\ r = 1.35r$$

because the radius of the top cylinder is $\sqrt{2}\ r$ from the ratio of their masses, i.e., given by the expression

$$\sqrt{\frac{2 \cdot \pi r^2 l\rho}{\pi l\rho}}$$

The free-body diagram of the cylinder P shows the following forces.acting on it:

Reaction $\mathbf{R}_A$ by the left vertical wall; $\mathbf{R}_A$ must be horizontal.

Reaction $\mathbf{R}_B$ from the supporting curved surface; $\mathbf{R}_B$ must be directed along op.

Reaction $\mathbf{R}_Q$ by the top cylinder Q; $\mathbf{R}_Q$ is directed along qp.

Weight **mg** of the cylinder itself; acting downwards.

From the equilibrium consideration of the cylinder P,

$$\mathbf{R}_A + \mathbf{R}_B + \mathbf{R}_Q + mg\,\mathbf{j} = 0$$

or $$R_A\mathbf{i} - R_B\frac{2r}{4r}\mathbf{i} + R_B\frac{3.464r}{4r}\mathbf{j} - R_Q\frac{2r}{2.414r}\mathbf{i} - R_Q\frac{1.35r}{2.414r}\mathbf{j} - mg\,\mathbf{j} = 0$$

whence,

$$R_A - 0.5\,R_B - 0.829\,R_Q = 0 \quad \text{(i)}$$

and $$0.866R_B - 0.56\,R_Q - mg = 0 \quad \text{(ii)}$$

It is also known that the vertical component of $\mathbf{R}_Q$ must be equal to half the weight of the top cylinder;

$$0.56\,R_Q = mg$$

or $$R_Q = 1.79\,mg \quad \text{(iii)}$$

Substituting R_Q from (iii) in (ii),

$$R_B = \frac{2\,mg}{0.866} = 2.31\,mg$$

and substituting these values in (i),

$$R_A = (0.5 \times 2.31 + 0.829 \times 1.79)\,mg$$
$$= 2.64\,mg$$

The reactions $\mathbf{R}_A$, $\mathbf{R}_B$ and $\mathbf{R}_Q$ are, therefore, given by

$$\mathbf{R}_A = 2.64\,mg\ \mathbf{i}$$
$$\mathbf{R}_B = (-0.5\,\mathbf{i} + 0.866\,\mathbf{j}) \times 2.31\,mg$$
$$= -1.155\,mg\,\mathbf{i} + 2\,mg\,\mathbf{j}$$
$$\mathbf{R}_Q = (-0.829\,\mathbf{i} - 0.56\,\mathbf{j}) \times 1.79\,mg$$
$$= -1.484\,mg\,\mathbf{i} - mg\,\mathbf{j}$$

***Example* 3.6** Three identical cylinders, each weighing W, are stacked, as shown in Fig. Ex. 3.6, on smooth inclined surfaces, each inclined at an angle θ with the horizontal. Determine the smallest angle θ to prevent the stack from collapsing.

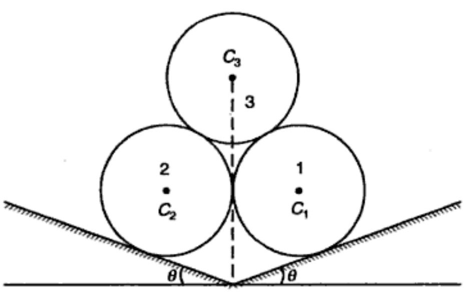


Fig. Ex. 3.6

Solution The free-body diagrams are drawn (Fig. Ex. 3.6 (Solution)) for cylinders 1 and 3 in order to understand the forces acting on each of them. The limiting case of collapse of the stack implies that the reaction between cylinder 2 and cylinder 1 becomes zero

$$R_{21} = 0$$

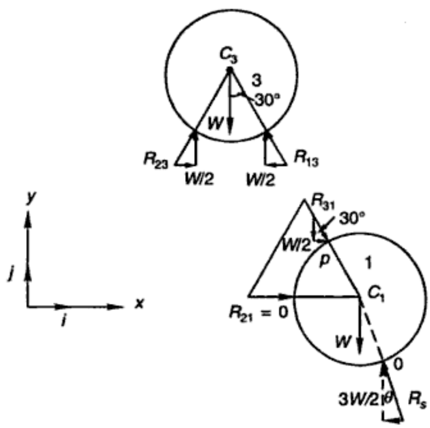


Fig. Ex. 3.6 ***(Solution)***

From the equilibrium of the cylinder 3 with regard to its free-body diagram,

$$\Sigma F_y = 0; \; R_{13} \cos 30° + R_{23} \cos 30° - W = 0$$

$$\Sigma F_x = 0; \quad R_{23} \sin 30° - R_{13} \sin 30° = 0$$

whence $$R_{13} = R_{23} = \frac{W}{2 \cos 30°}$$

From the equilibrium of cylinder 1 with regard to its free-body diagram,

$$\Sigma F_y = 0; \quad R_s \cos \theta - W - R_{31} \cos 30° = 0$$

$$\Sigma F_x = 0; \quad R_{31} \sin 30° - R_s \sin \theta = 0$$

Substituting

$$R_{31} = R_{13} = \frac{W}{2 \cos 30°}$$

Simplifying and solving for θ,

$$\tan \theta = \frac{\tan 30°}{3} = \frac{0.577}{3} = 0.1924$$

and $$\theta = 10.9°$$

It shows that if the inclined surfaces are frictionless, the angles between the surfaces and the horizontal line should not be less than 10.9°. If it is less, the reaction R_s will not be able to provide a horizontal resisting component to balance the horizontal component of R_{31} and the cylinders will fall apart. On the other hand, if the angle exceeds 10.9°, R_s will provide a horizontal resisting component in excess of that required by R_{31} and then the reaction from cylinder 2 will also act to guarantee equilibrium.

***Example* 3.7** A painter's scaffold 10 m long and weighing 0.75 kN is supported in a horizontal position by vertical ropes attached at equal distances from the ends of the scaffold as shown in Fig. Ex. 3.7. Find the greatest distance from the ends that the ropes may be attached to permit a 1 kN painter to stand safely at one end of the scaffold.

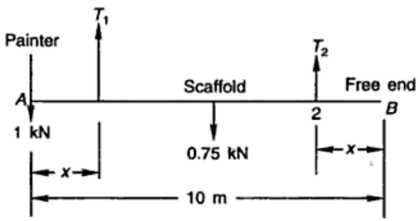


Fig. Ex. 3.7

Solution If the painter stands at A then the free-body diagram of the scaffold is as shown in Fig. Ex. 3.7. The tensions in the strings are T_1 and T_2. For equilibrium of the scaffold,

$$\Sigma \mathbf{F} = 0;\ T_1 + T_2 = 1 + 0.75 = 1.75 \text{ kN} \quad \text{(i)}$$

$$\Sigma \mathbf{M}_A = 0;\ T_{1x} + T_2\,(10 - x) - 0.75 \times 5 = 0 \quad \text{(ii)}$$

For x to be as large as possible, the tension T_2 in the string 2 reduces to zero. Then,

$$T_1 = 1.75 \text{ kN}$$

and

$$x = \frac{0.75 \times 5}{1.75} = 2.14 \text{ m}$$

If x is more than 2.14 m, i.e., if the ropes are attached closer to each other, then the solution of (i) and (ii) would show negative tension T_2; a state of compression which is not possible in a rope. On the other hand, if x is less than 2.14 m, i.e., if the ropes are attached farther, T_2 remains positive, permitting safe operation of the painter.

As a digression, let us demonstrate what we meant by the equivalence of the set of conditions, Eq. (3.7) with the set of conditions, Eq. (3.6) for equilibrium. In this example, the conditions chosen to provide Eqs. (i) and (ii) came from Eq. (3.6). Instead, Eq. (3.7) would provide, say,

$$\Sigma \mathbf{M}_1 = 0; \quad 1x - 0.75(5 - x) + T_2(10 - 2x) = 0$$

$$\Sigma \mathbf{M}_2 = 0; \quad 1(10 - x) - T_1(10 - 2x) + 0.75(5 - x) = 0$$

or

$$1.75x + 10T_2 - 2T_2x = 3.75 \quad \text{(iii)}$$

$$-1.75x - 10T_1 + 2T_1x = -13.75 \quad \text{(iv)}$$

which are equivalent to Eqs. (i) and (ii). For x to be as large as possible $T_2 = 0$ which when substituted in Eq. (iii), provides

$$x = \frac{3.75}{1.75} = 2.14 \text{ m}$$

and from (iv)

$$T_1 = 1.75 \text{ kN, as before}$$

***Example* 3.8** The boom of a crane is shown in Fig. Ex. 3.8. If the weight of the boom is negligible compared with the load $W = 60$ kN, find the compression in the boom and also the limiting value of the tension T when the boom approaches the vertical position.

Solution By drawing the free-body diagram of the boom, we observe that it is under the action of three forces:

(i) 60 kN acting downwards at B

(ii) Tension T along the string inclined at an angle α with the horizontal. It can be resolved into $T \sin \alpha$ and $T \cos \alpha$ components acting vertically and horizontally respectively at B.

(iii) Reaction **A** at A acting along AB. it can be resolved into $A \sin \theta$ and $A \cos \theta$ components as shown in Fig. Ex. 3.8 (Solution).

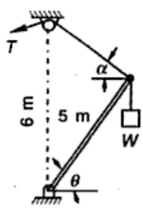


Fig. Ex 3.8

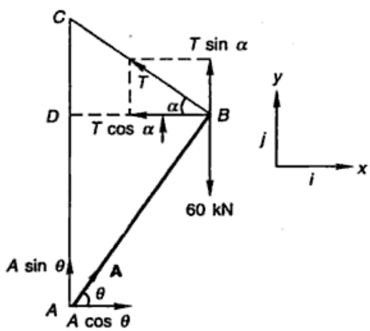


Fig. Ex. 3.8 *(Solution)*

For equilibrium of the boom,

$$\Sigma \mathbf{F} = 0; \ \Sigma F_x = 0, \ \Sigma F_y = 0$$

$$A \cos \theta - T \cos \alpha = 0; \ \cos \alpha = \frac{A \cos \theta}{T}$$

$$T \sin \alpha + A \sin \theta - 60 = 0; \ \sin \alpha = \frac{60 - A \sin \theta}{T}$$

whence $$\tan \alpha = \frac{60 - A \sin \theta}{A \cos \theta} \quad \text{(i)}$$

By the geometrical configuration,

$$\tan \alpha = \frac{CD}{BD} = \frac{AC - AD}{BD} = \frac{6 - 5 \sin \theta}{5 \cos \theta} \quad \text{(ii)}$$

From Eqs (i) and (ii), by comparison, A must be 50 kN

and $$T^2 (\cos^2 \alpha + \sin^2 \alpha) = A^2 \cos^2 \theta + (60 - A \sin \theta)^2$$

or $$T^2 = A^2 + 3600 - 120 A \sin \theta$$

For vertical position of the boom,

$$\theta = 90°, \sin \theta = 1$$

$$T^2 = A^2 + 3600 - 120A$$

$$= 2500 + 3600 - 6000 = 100$$

and $$T = 10 \text{ kN}$$

It is interesting to observe the implication of this answer. The reaction at A remains 50 kN for all values of θ, i.e., for all inclinations of the boom. The load of 60 kN, therefore, requires only 10 kN to be shared by the string.

***Example* 3.9** A three-bar pendulum *ABCD* has three bars each 2 m in length and weighing 2.5 kN as shown in Fig. Ex. 3.9. It is held in equilibrium by applying a horizontal force of 3 kN at the free end. Determine the angles θ_1, θ_2 and θ_3 made with the vertical.

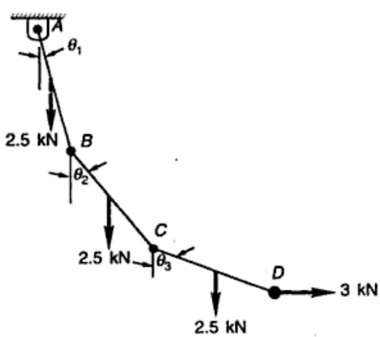


Fig. Ex. 3.9

Solution

$$AB = BC = CD = 2 \text{ m}$$

Weight of each bar assumed to act at their respective midpoints = 2.5 kN

Consider the equilibrium of bar *CD*:

$$\Sigma \mathbf{M}_c = 0;\ 3\ CD \cos \theta_3 - 2.5\ CD/2 \sin \theta_3 = 0$$

whence $\tan \theta_3 = \dfrac{6}{2.5} = 2.4$

and $\theta_3 = 67.4°$

Also, by the equilibrium of bar *CD*,

$$\Sigma \mathbf{F} = 0;\ \mathbf{R}_1 + 3\ \mathbf{i} - 2.5\ \mathbf{j} = 0$$

whence $R_{1x} = -3$ kN and $R_{1y} = 2.5$ kN

Consider now the equilibrium of bar *BC*:

$$\Sigma \mathbf{M}_B = 0;\ -2.5 \times 2/2 \sin \theta_2\ \mathbf{k} + 2\ (\sin \theta_2\ \mathbf{i} - \cos \theta_2 \mathbf{j}) \times (3\ \mathbf{i} - 2.5\ \mathbf{j}) = 0$$

or $-2.5 \sin \theta_2 - 5 \sin \theta_2 + 6 \cos \theta_2 = 0$

$$\tan \theta_2 = \frac{6}{7.5} = 0.8$$

$$\theta_2 = 38.66°$$

$$\Sigma\ \mathbf{F} = 0;\ \mathbf{R}_2 + 3\ \mathbf{i} - 2.5\ \mathbf{j} - 2.5\ \mathbf{j} = 0$$

and $\mathbf{R}_2 = 5\ \mathbf{j} - 3\ \mathbf{i}$

and the free-body of bar *AB* is drawn. For the equilibrium of *AB*,

$$\Sigma \mathbf{M}_A = 0;\ -2.5 \times 2/2 \sin \theta_1\ \mathbf{k} + 2\ (\sin \theta_1\ \mathbf{i} - \cos \theta_1\ \mathbf{j}) \times (3\ \mathbf{i} - 5\ \mathbf{j}) = 0$$

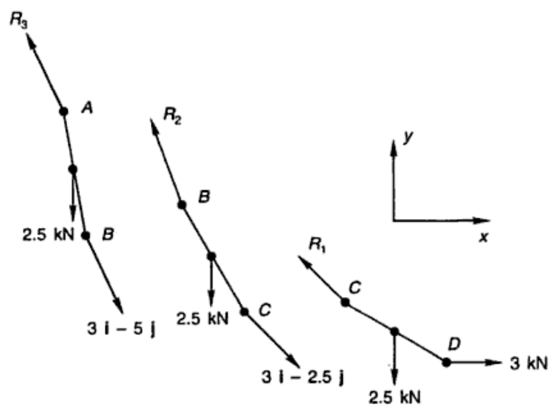


Fig. Ex. 3.9 *(Solution)*

or $$-2.5 \sin \theta_1 - 10 \sin \theta_1 + 6 \cos \theta_1 = 0$$

$$\tan \theta_1 = \frac{6}{12.5} = 0.48$$

$$\theta_1 = 25.64°$$

It may be checked that

$$\mathbf{R}_3 = -3\,\mathbf{i} + 7.5\,\mathbf{j}$$

by applying the condition of $\Sigma\mathbf{F} = 0$ for this bar. This value of $\mathbf{R}_3$ was expected from the equilibrium consideration of the entire system:

$$\mathbf{R}_3 + 3\,\mathbf{i} - 2.5\,\mathbf{j} - 2.5\,\mathbf{j} - 2.5\,\mathbf{j} = 0$$

$$\mathbf{R}_3 = -3\,\mathbf{i} + 7.5\,\mathbf{j}$$

***Example* 3.10** A uniform bar AB hinged at A, is kept horizontal by supporting and settling a 40 kN weight with the help of a string tied at B and passing over a smooth peg C as shown in Fig. Ex. 3.10. The bar weighs 20 kN. Determine the reactions at the supports A and C as well as the tension in the string.

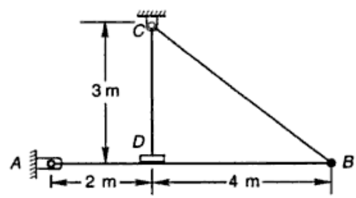


Fig. Ex. 3.10

Solution Consider the equilibrium of the bar AB with reference to its free-body diagram as shown in Fig. Ex. 3.10 (Solution). It may be noted that the weight of 40 kN is partly resting on the bar and is partly supported by the string to the extent of tension T.

$$\Sigma\mathbf{M}_A = 0$$

From this relation

$$-(40 - T) \times 2\,\mathbf{k} - 20 \times 3\,\mathbf{k} + 6\,\mathbf{i} \times T(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = 0$$

or $$-140 + 2T + 6T \sin\theta = 0$$

Taking $\sin\theta = \frac{3}{5} = 0.6$,

$$T = 25 \text{ kN}$$

$$\Sigma F_x = 0;\ A_x - T\cos\theta = 0$$

$$A_x = 25 \times \frac{4}{5} = 20 \text{ kN}$$

$$\Sigma F_y = 0;\ A_y + T\sin\theta - (40 - T) - 20 = 0$$

$$A_y = 20 \text{ kN}$$

Hence the tension in the string is 25 kN and the reaction at the hinge A is (20 **i** + 20 **j**) kN which is 28.28 kN inclined at 45° upwards with the bar. Consider now the equilibrium of the entire system with reference to the free-body diagram:

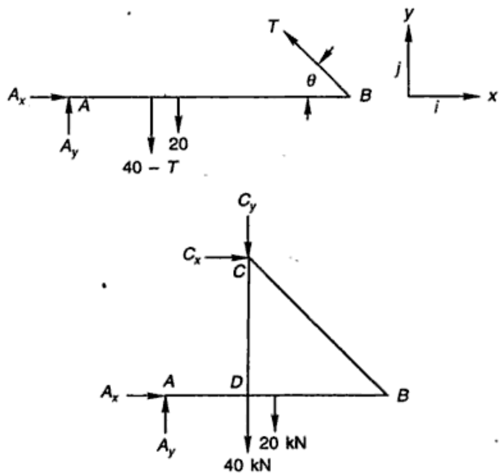


Fig. Ex. 3.10 *(Solution)*

$$\Sigma F_x = 0; \qquad A_x + C_x = 0$$

$$\Sigma F_y = 0; \; A_y - 40 - 20 - C_y = 0$$

whence

$$C_x = -A_x = -20 \text{ kN}$$

$$C_y = A_y - 60 = 20 - 60 = -40 \text{ kN}$$

which shows that the reaction at C must be $(-20\,\mathbf{i} - 40\,\mathbf{j})$ kN

***Example* 3.11** A rectangular table 1 m × 2 m is mounted on three equal supports. The table weighs 2 kN which acts through its centre of gravity C. If two vertical loads 1 kN and 4 kN are applied on the surface of the table as shown in Fig. Ex. 3.11, calculate the reaction at the supports.

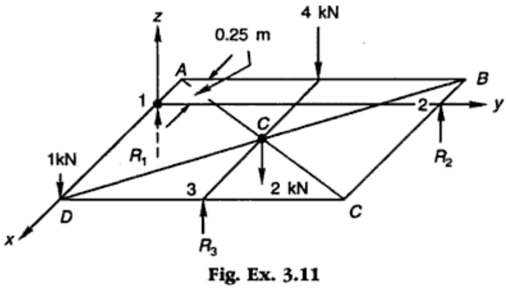


Fig. Ex. 3.11

Solution Choosing the plane of the table as the x-y plane, the origin O at 1 and coinciding the y-axis with the line joining the reactions 1 and 2 as shown, the loads and reactions are specified as follows:

Description	*Load*	*Position Vector*
1 kN downward	$-1\,\mathbf{k}$	$0.75\,\mathbf{i}$
2 kN downward	$-2\,\mathbf{k}$	$0.25\,\mathbf{i} + 1\,\mathbf{j}$
4 kN downward	$-4\,\mathbf{k}$	$-0.25\,\mathbf{i} + 1\,\mathbf{j}$
Reaction R_1	$R_1\,\mathbf{k}$	$0\,\mathbf{i} + 0\,\mathbf{j}$
Reaction R_2	$R_2\,\mathbf{k}$	$2\,\mathbf{j}$
Reaction R_3	$R_3\,\mathbf{k}$	$0.75\,\mathbf{i} + 1\,\mathbf{j}$

For equilibrium, the necessary conditions are:

$$\Sigma\mathbf{F} = 0 \qquad \text{and} \qquad \Sigma\mathbf{M}_0 = 0$$

From the former,

$$(-1 - 2 - 4 + R_1 + R_2 + R_3)\,\mathbf{k} = 0$$

or

$$R_1 + R_2 + R_3 = 7 \qquad \text{(i)}$$

From the latter,

$$0.75\,\mathbf{i} \times (-1\,\mathbf{k}) + (0.25\,\mathbf{i} + 1\,\mathbf{j}) \times (-2\,\mathbf{k}) + (-0.25\,\mathbf{i} + 1\,\mathbf{j})$$
$$\times (-4\,\mathbf{k}) + 0 + 2\,\mathbf{j} \times R_2\,\mathbf{k} + (0.75\,\mathbf{i} + 1\,\mathbf{j}) \times R_3\,\mathbf{k} = 0$$

or

$$0.75\,\mathbf{j} + 0.5\,\mathbf{j} - 2\,\mathbf{i} - 1.00\,\mathbf{j} - 4\,\mathbf{i} + 2\,R_2\,\mathbf{i} - 0.75\,R_3\,\mathbf{j} + R_3\,\mathbf{i} = 0$$

whence

$$(-6 + 2\,R_2 + R_3)\,\mathbf{i} = 0$$

or

$$2\,R_2 + R_3 = 6 \qquad \text{(ii)}$$

and

$$(0.25 + 0.75\,R_3) = 0$$

$$0.75\,R_3 = 0.25 \qquad \text{(iii)}$$

or

From (iii), $R_3 = 0.33$ kN

From (ii), $R_2 = 2.835$ kN

and

From (i), $R_1 = 3.835$ kN

***Example* 3.12** A horizontal rigid bar *AB* weighing 1 kN/m carries a load of 2 kN at its free end *A*. It is supported by the ball-and-socket joint at *B* and the cables *PQ* and *RS* are taut. Determine the tensions in the cables and the reaction at *B*.

Solution With reference to the *x*, *y* and *z*-axes, as shown in Fig. Ex. 3.12, the forces on the bar *AB* are shown and tabulated below. The unit vector $\mathbf{e}_{QP}$ along *QP* may be determined by locating *P* and *Q*;

$$P(-1, 1, 0) \qquad \text{and} \qquad Q(0, 0, 2)$$

$$\mathbf{e}_{QP} = \frac{(-1-0)\mathbf{i} + (1-0)\mathbf{j} + (0-2)\mathbf{k}}{\sqrt{(-1)^2 + 1^2 + (-2)^2}}$$

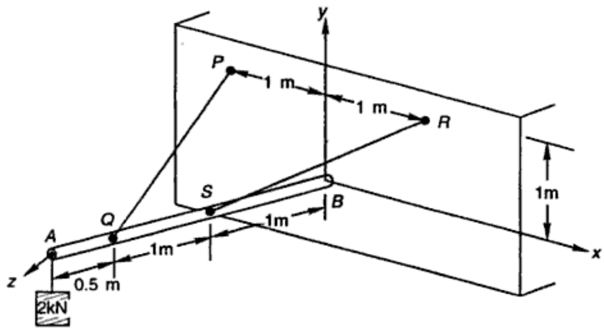


Fig. Ex. 3.12

$$= -0.41\ \mathbf{i} + 0.41\ \mathbf{j} - 0.82\ \mathbf{k}$$

Similarly, the unit vector $\mathbf{e}_{SR}$ along SR is expressed as

$$\mathbf{e}_{SR} = \frac{(1-0)\mathbf{i} + (1-0)\mathbf{j} + (0-1)\mathbf{k}}{\sqrt{(1^2 + 1^2 + 1^2)}}$$

$$= 0.58\ \mathbf{i} + 0.58\ \mathbf{j} - 0.58\ \mathbf{k}$$

Description	*Force*	*Position Vector*
1. Weight at the free end	$-2\ \mathbf{j}$	$2.5\ \mathbf{k}$
2. F_Q, the force in cable QP	$F_Q(-0.41\ \mathbf{i} + 0.41\ \mathbf{j} - 0.82\ \mathbf{k})$	$2\ \mathbf{k}$
3. F_s, the force in cable SR	$F_s\ (0.58\ \mathbf{i} + 0.58\ \mathbf{j} - 0.58\ \mathbf{k})$	$1\ \mathbf{k}$
4. Weight of the bar	$-1\ \mathbf{j}$	$1.25\ \mathbf{k}$
5. Reaction at B	$B_x\ \mathbf{i} + B_y\ \mathbf{j} + B_z\ \mathbf{k}$	0

For equilibrium, $\Sigma\mathbf{F} = 0$ and $\Sigma\mathbf{M} = 0$.
From the summation of the forces on AB,

$$-0.41\ F_Q + 0.58\ F_s + B_x = 0 \quad \text{(i)}$$

$$-2 + 0.41\ F_Q + 0.58\ F_s - 1 + B_y = 0 \quad \text{(ii)}$$

$$-0.82\ F_Q - 0.58\ F_s + B_z = 0 \quad \text{(iii)}$$

The moments of the forces about the origin at B are calculated as follows:

1. $2.5\ \mathbf{k} \times (-2\ \mathbf{j}) = 5\ \mathbf{i}$
2. $2\ \mathbf{k} \times F_Q\ (-0.41\ \mathbf{i} + 0.41\ \mathbf{j} - 0.82\ \mathbf{k}) = (-0.82\ \mathbf{j} - 0.82\ \mathbf{i})\ F_Q$
3. $1\ \mathbf{k} \times F_s\ (0.58\ \mathbf{i} + 0.58\ \mathbf{j} - 0.58\ \mathbf{k}) = (0.58\ \mathbf{j} - 0.58\ \mathbf{i})\ F_s$
4. $1.25\ \mathbf{k} \times (-1\ \mathbf{j}) = 1.25\ \mathbf{i}$
5. 0

Summation of the moments about the origin yields

$$\Sigma \mathbf{M} = 5\,\mathbf{i} + (-0.82\,\mathbf{j} - 0.82\,\mathbf{i})\,F_Q + (0.58\,\mathbf{j} - 0.58\,\mathbf{i})\,F_s + 1.25\,\mathbf{i} = 0$$

whence,

$$6.25 - 0.82\,F_Q - 0.58\,F_s = 0 \qquad \text{(iv)}$$

$$-0.82\,F_Q + 0.58\,F_s = 0 \qquad \text{(v)}$$

Now, we have a set of five equations (i) to (v) for the five unknowns

$$F_Q,\ F_s,\ B_x,\ B_y \text{ and } B_z$$

which may be solved to provide

$$F_Q = 3.80 \text{ kN along } QP$$

$$F_s = 5.37 \text{ kN along } SR$$

$$B_x = 1.56 \text{ kN}$$

$$B_y = -1.67 \text{ kN}$$

$$B_z = 6.23 \text{ kN}$$

Experiment E2

Equilibrium under Spatial Forces

OBJECTIVE

To study the equilibrium of a particle under the action of forces in space.

APPARATUS

A skeleton space frame consisting of bars with provisions to pass the strings over frictionless pulleys at the desired points. Standard weights and metre rod.

BACKGROUND INFORMATION

The equilibrium of a particle under the action of forces in space is essentially governed by the condition

$$\Sigma \mathbf{F} = 0$$

i.e., the vector sum of all the forces must vanish.

In the space frame shown in figure a load *W* is supported by three strings *AP, BP* and *CP* in order to keep the knot *P* in equilibrium at rest. Taking the origin at a corner point *O*, the coordinates of *A, B, C* and *P* are measured and are used to find the unit vectors **ap, bp** and **cp** along *AP*, *BP* and *CP* respectively.

For example,

$$\mathbf{r}_A = x_A\,\mathbf{i} + z_A\,\mathbf{k}$$

$$\mathbf{r}_P = x_P\,\mathbf{i} + y_P\,\mathbf{j} + z_P\,\mathbf{k}$$

$$\mathbf{r}_{AP} = (x_A - x_P)\,\mathbf{i} + y_P\,\mathbf{j} + (z_A - z_P)\,\mathbf{k}$$

whence,
$$AP = \sqrt{(x_A - x_P)^2 + y_P^2 + (z_A - z_P)^2}$$

and
$$\mathbf{ap} = \frac{(x_A - x_P)}{AP}\mathbf{i} + \frac{y_P}{AP}\mathbf{j} + \frac{(z_A - z_P)}{AP}\mathbf{k}$$

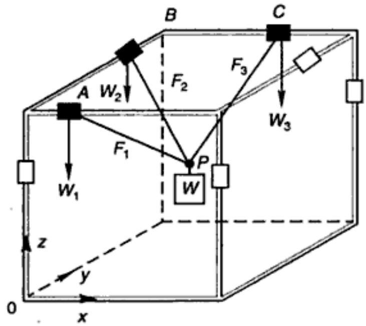
The forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ in *AP*, *BP* and *CP* respectively are such that

$$\Sigma\mathbf{F} = 0;\ \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 - W\,\mathbf{k} = 0$$

or $F_1\,\mathbf{ap} + F_2\,\mathbf{bp} + F_3\,\mathbf{cp} - W\,\mathbf{k} = 0$

From a knowledge of the unit vectors **ap, bp** and **cp** in terms of **i, j** and **k**, this vector equation is rewritten as three scalar equations and the three unknowns F_1, F_2 and F_3 are obtained.

In the apparatus the strings are kept taut under the application of hanging loads at *A*, *B* and *C*. Neglecting friction at the pulleys, the tensions in the strings must equal the corresponding applied loads.



Skeleton Space Frame

Observations and Calculations

It is advisable to record the coordinates and to formulate the equations in a tabular form. For example, the table of force F_1 is as follows:

Point P			*Point A*			*For AP*		
x_P	y_P	z_P	x_A	y_A	z_A	$(x_A - x_P)$	$(y_A - y_P)$	$(z_A - z_P)$

$AP =$

ap $=$

Table of Results

Member	*Forces in members*		*Difference*
	Theoretical (N)	*Experimental (N)*	*Abs. %*
AP			
BP			
CP			

POINTS FOR DISCUSSION

1. State the necessary and sufficient conditions of equilibrium for a rigid body. In particular, is the condition

$$\Sigma \mathbf{F} = 0$$

just necessary or necessary and sufficient as far as the equilibrium of a particle is concerned?
2. If a load W hangs from P maintained in equilibrium under the action of F_1, F_2 and F_3 as done in the experiment and the load W is increased by 10%, what will happen to the position of the strings and why? Would the strings stay unaltered if the tension in each string were increased by 10%?
3. Enumerate the sources of error in the experiment and suggest how each can be minimised.
4. Would you suggest some other design of the skeleton frame to be able to perform experiments on equilibrium under the action of spatial and coplanar forces?

Concept Review Questions

1. (a) If a system is in equilibrium, is it necessary that it should be static? Relate the concept of equilibrium with the laws of motion.
   (b) If a rigid body is rotating at a constant angular velocity about some axis, is the body said to be in equilibrium or not?
2. Draw the free-body diagrams for the following:
   (a) a nut-cracker in action
   (b) a ladder leaning against a wall
   (c) a bullock-cart in motion
   (d) an aeroplane in flight
   (e) a body floating in a liquid.
3. (a) State the Lami's theorem for the equilibrium of a body under the action of three coplanar forces.
   (b) Prove that a body must be in equilibrium if Lami's theorem is obeyed.
4. Examine the truth in the following statements and rewrite them after necessary corrections.
   (a) A rigid body must be in equilibrium if the resultant of the force system acting on it vanishes.
   (b) A rigid body subjected to a couple may be brought to equilibrium by a force placed suitably in the plane of the couple.
   (c) A pin joint and a hinge are identical supports so far as the reaction is concerned.
   (d) If three concurrent forces keep a particle or a rigid-body in equilibrium, then the forces must be coplanar.
   (e) The equilibrant action required to bring a rigid-body into equilibrium should be equal and opposite to any equivalent force system acting on the body.
5. Illustrate why the necessary conditions of equilibrium

$$\Sigma \mathbf{F} = 0 \quad \text{and} \quad \Sigma \mathbf{M} = 0$$

can be replaced by the equivalent moment equations

$$\Sigma \mathbf{M}_1 = 0 \qquad \text{and} \qquad \Sigma \mathbf{M}_2 = 0$$

for a plane force system acting on a body.

Tutorial Problems

3.1 A small boat is held static in a river by means of three inextensible taut ropes *OA*, *OB* and *OC*. The water in the river exerts a force on the boat in the direction of the flow. If the tensions in *OA* and *OB* are 1 kN and 0.6 kN respectively, as shown in Fig. Prob. 3.1, determine the force exerted by the flow on the boat and the tension in rope *OC*.
(a) Will the boat remain in equilibrium if rope *OC* breaks?
(b) What would be the tension in *OA* after *OC* breaks?

(**Ans.** 0.116, 0.808 kN; yes, 0.067, 0.133 kN)

3.2 A force **F** applied to a stretched elastic string at *O* stretches it to a position *AOB* as shown in Fig. Prob. 3.2. If the tension in each part of the string is 50 N, determine the magnitude and direction of the force applied. (**Ans.** 61 N, 7.5°)

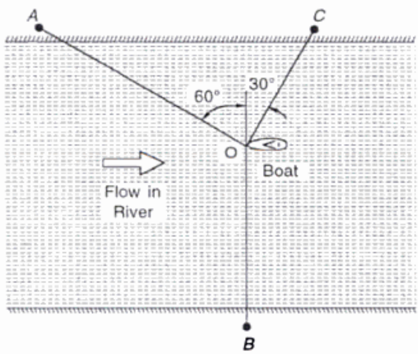


Fig. Prob. 3.1

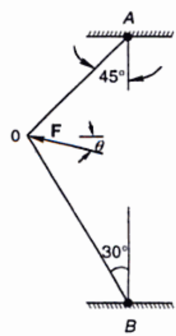


Fig. Prob. 3.2

3.3 In an old drawing, four forces are shown acting at a point but the line of action of the fifth force has been disfigured. However, it is known that the moments exerted by the forces about some origin in space are:
$2\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$, $3\,\mathbf{i} - 4\,\mathbf{j} + 7\,\mathbf{k}$, $2\,\mathbf{i} + 3\,\mathbf{j} - 4\,\mathbf{k}$, $-8\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$ and $1\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$ respectively. Determine whether the fifth force passes through the same point or not.

(**Ans.** Yes, since $\Sigma M = 0$)

3.4 A 4 m × 5 m slab carries four forces normal to it as shown in Fig. Prob. 3.4.
(a) Determine the magnitude and point of application on it of a single force equivalent to the given system of forces.
(b) If the slab is to be in equilibrium, determine the magnitude and location of the equivalent.
(c) If the slab must be brought into equilibrium by holding it at the origin *O*, determine the reaction necessary from the device to hold it.
(d) Is it possible to hold the slab in equilibrium by applying suitable forces at the three free corners *A*, *B* and *C*? (**Ans.** – 8 kN; at 1.9 m, 1.5 m point)

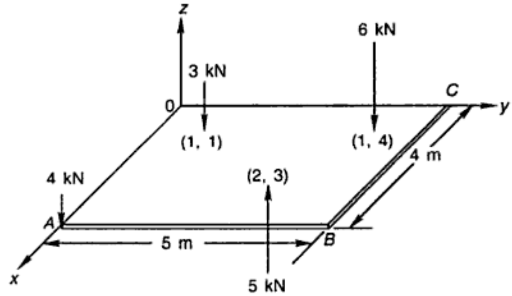


Fig. Prob. 3.4

3.5 A simple stone-crushing mechanism consists of a piston on which a force of 15 kN acts and three rigid weightless links *OA*, *OB* and *OC* hinged at *O*, *A*, *B* and *C* as shown in Fig. Prob. 3.5. At the given orientation, what is the force exerted on the stone *S* trapped between the jaw and the fixed wall. (**Ans.** 28 kN)

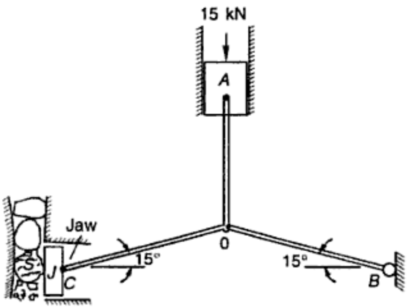


Fig. Prob. 3.5

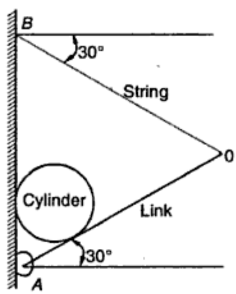


Fig. Prob. 3.6

3.6 A cylinder having a diameter of 0.5 m and mass 50 kg is supported on a uniform rigid link *AO* 2 m long and of mass 10 kg by means of a string *OB* held taut as shown in Fig. Prob. 3.6. Assuming the surfaces of contact of the cylinder as frictionless, calculate the tension in the string and the reaction at the hinge *A*.

(**Ans.** $T = 335$ N, $R_A = 424$ N at 89.4° to horizontal)

3.7 (a) A weight *W* tied to the lower end of a suspended cord of length *l* is pulled by a horizontal force *F* so as to displace it by a distance *d* away from its vertical position. Express the force *F* and tension *T* in the cord as a function of the horizontal displacement of the weight.

(b) Determine the horizontal distance to which a 10 m long inextensible nylon thread holding a mass of 1000 N can be drawn before it breaks. The thread can withstand a maximum tension of 10 kN.

(**Ans.** $F = W \cot(\cos^{-1} d/l)$, $T = W \operatorname{cosec}(\cos^{-1} d/l)$, $d = 10$ m)

3.8 A uniform metre rod *AB*, assumed rigid, of mass 0.5 kg is suspended from its ends in an inclined position and a mass of 1 kg is suspended from a point *D*. Determine the tension in each string. Where should the suspended mass be placed in order to get equal tension in the strings? (**Ans.** $F_1 = 4.9$ N and $F_2 = 9.81$ N; at C)

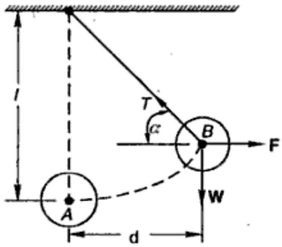


Fig. Prob. 3.7

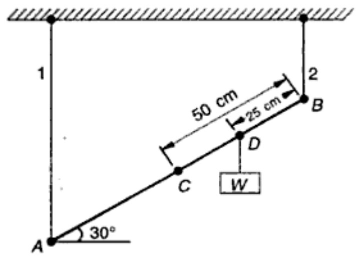


Fig. Prob. 3.8

3.9 A crane is idealised by a uniform rigid arm of weight W_a supported on a knife edge of the rest and held by a cable together with a counter weight of 5 kN as shown in Fig. Prob. 3.9. The 10 kN load held by it is moved outward on the arm with a constant velocity v of 0.2 m/s. Assuming the system to be in equilibrium when $\theta = 30°$ find the rate of change of the tension T in the cable.

$$\left(\textbf{Ans.}\ \frac{dT}{dt} = 5\frac{V}{\cos\theta} = 1.155 \text{ kN/s}\right)$$

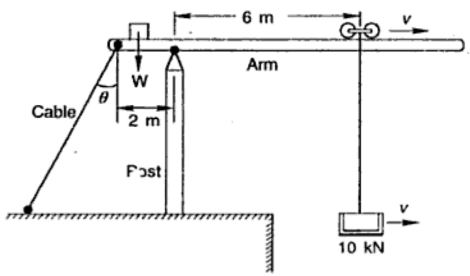


Fig. Prob. 3.9

3.10 Two cylinders 1 and 2 are connected by a rigid bar of negligible weight hinged at each cylinder and are left to rest in equilibrium in the position shown under the application of a force P applied at the centre of cylinder 2. Determine the magnitude of force P if the masses of the cylinders are $m_1 = 100$ kg and $m_2 = 50$ kg.

(**Ans.** 263 N)

3.11 Three identical balls rest on a smooth horizontal surface touching one another as shown in Fig. Prob. 3.11. A fourth ball of the same size and weight is placed on top of these three to form a pyramid and the three lower balls are now held together by an encircling string as shown in the plane view. Determine the tension in the string if the mass of each ball is 900 kg. Assume all surfaces of contact to be smooth.

(**Ans.** 1200 N)

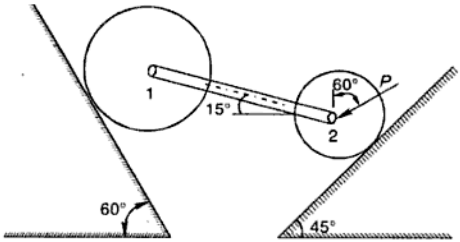


Fig. Prob. 3.10

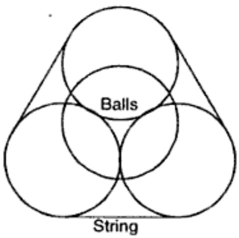


Fig. Prob. 3.11

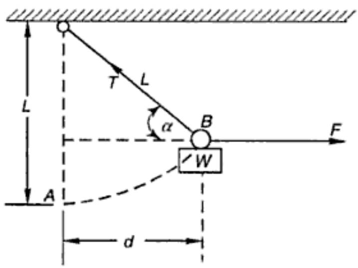


Fig. Prob. 3.12

3.12 Fig. Prob. 3.12 shows a weight W tied to the end of a cord of length L. Find the magnitude of the force **F** required to pull the weight to an angle α from its vertical position and the tension in the cord. (**Ans.** $W \cot \alpha$, $W \operatorname{cosec} \alpha$)

3.13 A three-wheeler scooter rickshaw with weight 2 kN acting at its centre of gravity C is shown schematically in Fig. Prob. 3.13. The driver D weighing 0.5 kN and the passenger P weighing 0.8 kN are located as shown. Calculate the reactions at the wheels 1, 2 and 3 for equilibrium on a horizontal road.

(**Ans.** 1.00, 1.23, 1.07 kN)

3.14 A vertical tower of height h is subjected to a horizontal force F at its top and it is anchored by two equal guy wires symmetrically as shown in Fig. Prob. 3.14. Determine the tension T in the guy wires if $h = 20$ m, $a = 3$ m, $b = 4$ m and (b) the horizontal force $F = 10$ kN. (**Ans.** 34.5 kN)

3.15 A vertical mast AB is supported in a ball-and-socket joint at A and by cables BC and DE as shown in Fig. Prob. 3.15. A force

$$\mathbf{F} = 500\,\mathbf{i} + 400\,\mathbf{j} - 300\,\mathbf{k}$$

is applied at B. Calculate the reaction provided by the ground at A.

(**Ans.** 5100 N)

3.16 Two cables BG and BH are attached to hold the boom ABC 1.4 m long, hinged at A, horizontally as shown in Fig. Prob. 3.16. Determine the tensions in the cables if a load of 1 kN acts at C. (**Ans.** 1230 N and 486 N)

Look up Hints to Tutorial Problems!

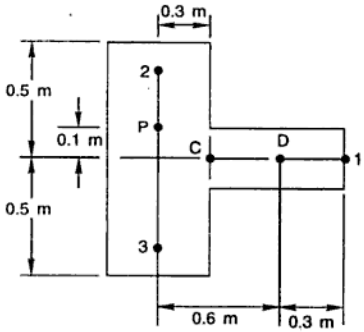


Fig. Prob. 3.13

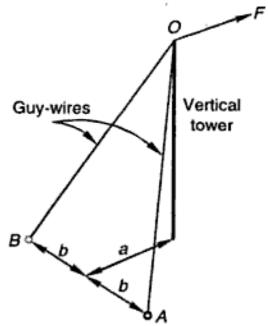


Fig. Prob. 3.14

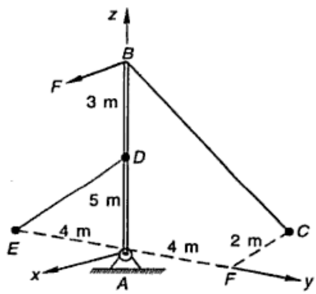


Fig. Prob. 3.15

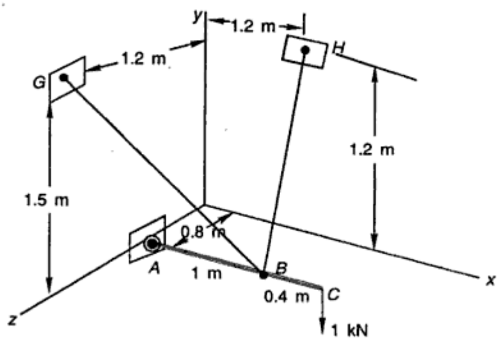


Fig. Prob. 3.16

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. A rigid body is in equilibrium under the action of three forces. It implies that the forces must
   (a) be concurrent
   (b) be coplanar
   (c) either be concurrent or coplanar
   (d) pass through the centre of mass
2. A rigid body is in equilibrium. Given that the moment of all the forces acting on the body about some axis is zero and also given that forces are concurrent, implies that
   (a) the resultant force is zero
   (b) the forces have a line of action passing through the axis
   (c) the resultant forces have a line of action parallel to the axis
   (d) any of (a), (b), (c) can be true
3. A body is acted upon by a force system. It can in general be brought to equilibrium by the application of
   (a) a force acting on a suitable point on the body
   (b) a force acting anywhere along a suitable line
   (c) a force acting along a suitable line and a moment along the direction of the force
   (d) a wrench acting anywhere on the body.
4. Lami's theorem
   (a) relates the forces with the sines of angles
   (b) state that, for equilibrium under the action of three concurrent forces, there is a unique constant of proportionality between a force and the angle between the other two forces
   (c) may be applied to consider a relationship between forces and angles of a polygon representation of forces
   (d) may be applied for a body which may or may not be in equilibrium
5. If the sum of all the forces acting on a body is zero, it may be concluded that the body
   (a) must be in equilibrium
   (b) cannot be in equilibrium
   (c) may be in equilibrium provided the forces are concurrent
   (d) may be in equilibrium provided the forces are parallel

Answers to Multiple-Choice Questions

1(b), 2(d), 3(c), 4(b), 5(c)

S

APPLICATIONS IN STATICS

This section consists of some salient applications in Statics under the following three chapters:

- ❑ **S1 SIMPLE STRUCTURES**
- ❑ **S2 THIN RIGID BEAMS**
- ❑ **S3 FRICTION**

S1 SIMPLE STRUCTURES

S1.1 TYPES OF STRUCTURES

A structure may consist of a truss or a frame—pin-connected or rigidly secured. A truss is an assemblage of slender bars fastened together at their ends by smooth bolts or ball-and-socket joints acting as hinges. A truss, by definition, is a pin-connected structure. The bar members, therefore, act as two-force members which can either be in tension or in compression; there can be no transverse force in a member of a truss. A frame structure, on the other hand, consists of members which may be subjected to a transverse load in addition to the axial load. We shall again limit our discussion to pin-connected frames. The reason for leaving out rigidly-secured structures, such as welded trusses is that the members may then be subjected to initial loads, axial or transverse, the estimation of which is a task by itself.

A simple structure is thus a pin-connected frame or truss. A truss consists of slender-bar members which can carry no transverse loads. It follows that the loading in a truss must be at the joints only. A truss consisting of members which lie in a plane and are loaded in the same plane is called *plane truss*. If a truss is made of non-coplanar members, it is referred to as *space truss*. Similarly, a frame may be a plane frame or a space frame depending upon its structure.

Let us now examine trusses with regard to their rigidity. Trusses are classified as *just-rigid, over-rigid* and *non-rigid mechanisms*. If the members are allowed any relative movement, then the assemblage of members is called a non-rigid truss or mechanism and if the members are not allowed any relative movement, then it is called a rigid truss. A just-rigid truss is that which, on the removal of any single member, becomes non-rigid. An over-rigid truss is the one that has redundant members which may be removed to render the truss just-rigid. Examples of such trusses are shown in Fig. S1.1. We shall confine our study to the just-rigid simple trusses and frames.

The number of joints j in a truss is related to the number of members m. A necessary relationship between the number of joints j and the number of members m for a just-rigid plane truss is

$$m = 2j - 3$$

This is, however, not a sufficient relationship. In other words, a truss cannot be just-rigid if m and j are related otherwise but a truss may not be just-rigid even if m and j are related as before.

Consider, for example a simple truss *ABCDEF* as shown in Fig. S1.2. A just-rigid truss requires the number of members to be given by

$$m = 2j - 3$$

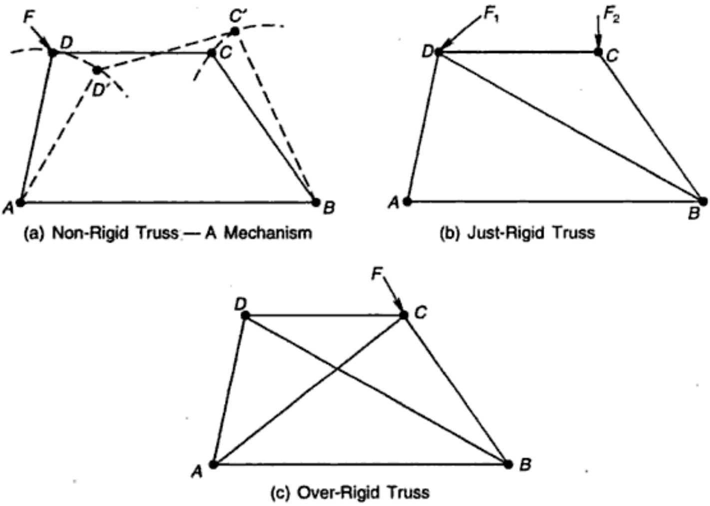


Fig. S1.1 ***Types of Trusses***

According to the law, the number of members must be $(2j - 3)$ for a truss to be just-rigid but if the number is different from $(2j - 3)$ the following may happen:

$m < (2j - 3)$ the truss cannot be just-rigid; parts of which must be under-rigid and a part may be just-rigid or over-rigid as shown in Fig. S1.2(b)

$m > (2j - 3)$ the truss cannot be just-rigid; parts of which must be over-rigid and a part may be under-rigid or just-rigid as shown in Fig. S1.2(c).

It may be appreciated that $m = (2j - 3)$ is no guarantee for a truss to be just-rigid. Some parts of such a truss may be over-rigid and some other parts under-rigid. For example, in Fig. S1.2(d), parts *ABCF* are under-rigid and *FCDE* over-rigid.

It may be appreciated that simple or just-rigid trusses are generated from the basic triangular truss by successively adding a pair of new members to the existing joints and by generating a new joint by connecting the new members. Now, for a basic triangular truss which is just-rigid, the number of joints $j = 3$. For each additional joint, two members must be added to keep it just-rigid. If we wish to visualise a truss of j joints, then $(j - 3)$ joints must be added to the basic triangular truss. The number of members which will be added are $2(j - 3)$ and the total number of members become

$$m = 2(j - 3) + 3$$

whence

$$m = 2j - 3$$

It may also be observed from the equation that the number of members in a simple just-rigid truss must be odd.

There are some standard types of trusses known after the names of the originators or their shape. Some of them, *Warren truss, Pratt truss, Howe truss* and *K-truss* are shown in Figs. S1.3(a), (b), (c) and (d). Let us check the just-rigidity of the trusses:

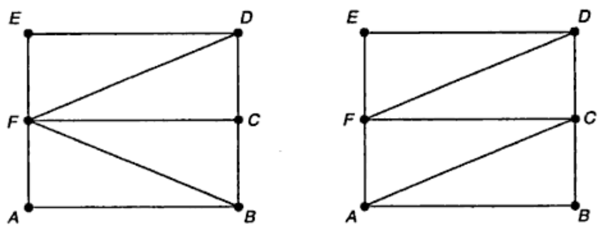
Table S1.1 Typical Just-Rigid Trusses

Types of Truss	*Joints j*	*Members m*	*Condition* $m = 2j - 3$
(a) Warren Truss	7	11	Satisfied
(b) Pratt Truss	12	21	Satisfied
(c) Howe Truss	12	21	Satisfied
(d) K-Truss	16	29	Satisfied

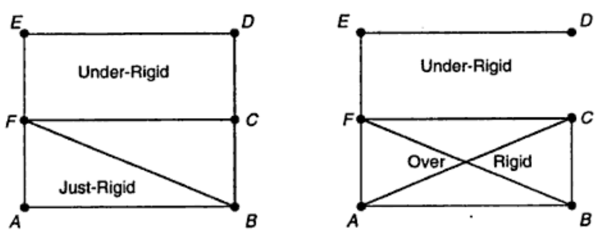
Just-rigid trusses are *statically determinate* and the over-rigid trusses are statically indeterminate. This statement follows from the fact that the number of members in a just-rigid truss are in accordance with the necessary conditions of equilibrium for the number of joints. We shall, therefore, confine ourselves to the analysis of just-rigid trusses. The task of determining the reactions at the supports and the forces in the members of simple plane trusses is achieved by three standard techniques known as *method of joints*, *method of sections* and *graphical method* with the help of the *Maxwell's diagram*. These methods are discussed in the following sections.

A space truss (or frame) consists of members which do not lie in a single plane. If the non-coplanar members are pin jointed, it is called a *simple space truss*. A necessary relationship between the number of joints j and the number of members m for a just rigid simple space truss is

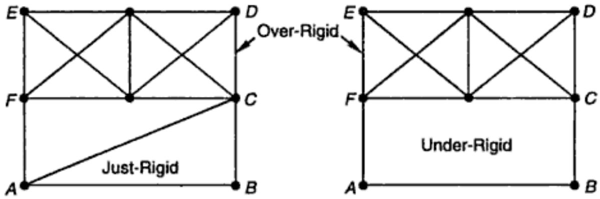
$$m = 3j - 6$$



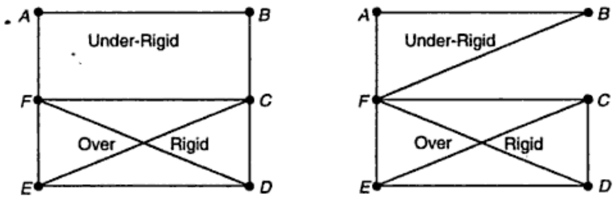
(a) Just-Rigid Trusses $m = (2j - 3)$



(b) $m < (2j - 3)$, Non-Rigid Trusses

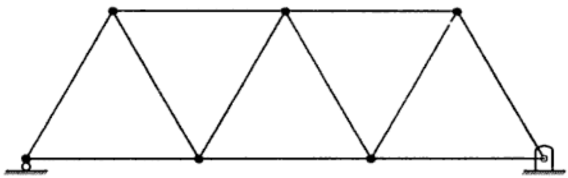


(c) $m > (2j-3)$ Part Over-Rigid

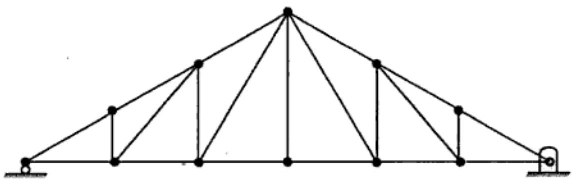


(d) $m = (2j-3)$, Non-Just-Rigid Trusses

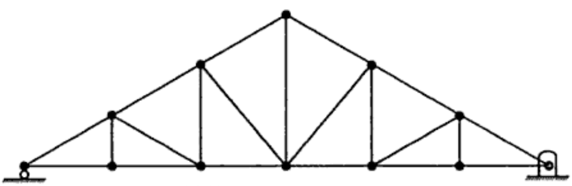
Fig. S1.2 ***Conditions of Rigidity of a Truss***



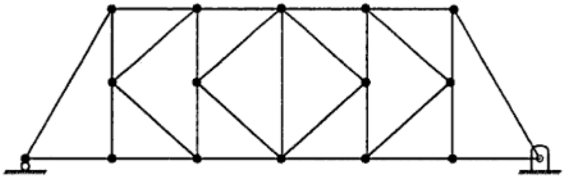
(a) Warren Truss



(b) Pratt Truss



(c) Howe Truss



(d) K-Truss

Fig. S1.3 ***Some Standard Types of Trusses***

This is because the basic space truss is a tetrahedron consisting of 6 members and 4 joints, i.e., $m = 6, j = 4$. It can be extended to a bigger space truss by adding 3 non-coplanar members from any 3 previously existing joints and creating one additional joint, i.e., $m = 9, j = 5$ and $m = 12, j = 6$, etc. which leads to the equation $m = 3j - 6$. Like plane truss relationship, this relationship is necessary but not a sufficient condition for a space truss to be just-rigid. For example, a truss may be partly over-rigid and partly non-rigid even if $m = 3j - 6$.

S1.2 INTERNAL FORCES: TENSION AND COMPRESSION

In a frame, members are interconnected at joints. If a member is in tension as shown in Fig. S1.4(a), the member is pulled towards its ends on either side by external forces. Consequently, the internal forces in the member tend to resist it and hence act in a sense away from the end.

Likewise, if a member is in compression as shown in Fig. S1.4(b) the external forces compress it at its ends and the internal forces tend to resist it.

Fig. S1.4 **(a)** ***A Member in Tension*** **(b)** ***A Member in Compression***

External and internal forces on a member are indeed in accordance with the action and reaction principle.

Members in a frame may be subjected only to forces along them. A member cannot be subjected to a transverse force. That is why members are said to be 2-force (or two-force) members, i.e., subjected to a pair of tensile forces or a pair of compressive forces.

S1.3 ANALYSIS BY THE METHOD OF JOINTS

A plane truss or frame can be subjected only to a coplanar force system. Any joint of a plane truss or frame may be subjected only to a coplanar and concurrent force system. A space truss or frame can be subjected to a spatial force system. Any joint of a space truss may be subjected to a spatial concurrent force system. The condition of concurrency of a force system at a joint in a truss follows from the equilibrium of the forces at that joint which is also the point of concurrency.

The method of joints consists of taking up one joint at a time and analysing it for equilibrium. At every joint in a truss the forces must be along the members at that joint. The forces acting at every joint must satisfy the necessary condition of equilibrium:

$$\Sigma \mathbf{F} = 0$$

which implies that

$$\Sigma F_x = 0, \qquad \Sigma F_y = 0, \qquad \Sigma F_z = 0$$

for spatial forces at a joint
and

$$\Sigma F_x = 0, \qquad \Sigma F_y = 0$$

for plane forces at a joint.

In addition to the equations of equilibrium at each joint, the overall equilibrium of a truss provides additional equations which can be used to determine the reactions of the supports. In fact, one of the first tasks in the method of joints is to evaluate the reactions from the supports. The equations of over-all equilibrium are:

$$\Sigma \mathbf{F} = 0$$

$$\Sigma \mathbf{M} = 0$$

which, for a plane truss reduce to only three equations:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0$$

and $$\Sigma M = 0$$

or an alternative set of equations

$$\Sigma F_y = 0$$

and $$\Sigma M_1 = 0, \qquad \Sigma M_2 = 0$$

as has been shown earlier.

The points about which moments are taken may be the points of application of the support reactions for convenience and for simplicity in analysis.

Once the reactions are known, there must be a pin-joint in a simple just-rigid plane truss where there are only two or less unknown forces in magnitude. These values are determined by analysing the joint for equilibrium. The force in any member at a joint being known, the force at the joint at the other extremity of that member is known by the *action-reaction principle*. We then look for another joint where there are a maximum of two unknown forces in magnitudes and analyse it for equilibrium. In this manner, a chain process is set up to proceed from one joint to another and analyse the forces in the members. When the objective is to determine the force in a particular member, it is necessary to search a joint nearest to the member where to start with and proceed towards a joint where the member is connected.

It is a useful convention in the method of joints to indicate the forces at a joint such that the known forces are taken in the correct directions and the unknown forces are assumed positive, away from the joint. On evaluation, if a force turns out to be positive, the member must be in tension and if a force comes out to be negative, the member must be in compression. It is also a universal practice to designate the force at a joint due to a member by the name of the member itself. For example, the force exertd by a member *BC* at *B* or *C* is refered to as **BC.**

Remarks on the loading conditions of some joints are also in order:

1. If there are only two members and no external force at a joint, the two members must be collinear in order that any force is taken by them. The reason for this is that the two forces maintaining a point in equilibrium must act along the same line of action and their magnitudes must be equal. The two members at the joint should, therefore, not only be collinear but also have equal forces and their nature must be the same, i.e., either both tensile or both compressive.

   It follows from this that, in the special case when the two members do not have any forces, they may not be collinear. Conversely, if in a truss, there exists a joint of two members which are non-collinear and there is no external force, the forces in the members must be zero. This is shown in Fig. S1.5(a).

   It also follows that if an external force acts on a two-member joint then the members cannot be collinear.

2. If there are only three members and no external force at a joint, the members will carry forces in accordance with the condition of equilibrium. If two of the three members are collinear, these two members should have equal forces and the third must be a zero-force member, otherwise equilibrium will not be maintained.

   If an external force is applied at a joint of three members, the forces in the members can again be determined in the light of equilibrium of the joint. If two of the three members are collinear and the force acts in line with the third member, then the force in the third member must be equal and opposite to the external force. These facts are illustrated in Fig. S1.5(b).

3. If there are four members at a joint, the members will carry forces in accordance with equilibrium. If, however, there are two pairs of collinear members at a joint and there is no external force then the forces in the collinear members must be equal. One such example is shown in Fig. S1.5(c) where the joint B is in equilibrium under the action of two pairs of collinear forces.

***Example* S1.1** A pin-jointed frame *ABCO* is supported and loaded as shown in Fig. Ex. S1.1. The members *AB* and *BC* are each 3 m long. Find the magnitude and nature of force in each of the members due to a load of 10 kN at the apex.

Solution Considering the equilibrium of the structure as a whole

$$\Sigma F_y = 0; \quad R_{Ay} + R_c - 10 = 0$$

$$\Sigma F_x = 0; \quad R_{Ax} = 0$$

and

$$\Sigma M_A = 0; \ 3\,R_c - 10 \times 3/2 = 0$$

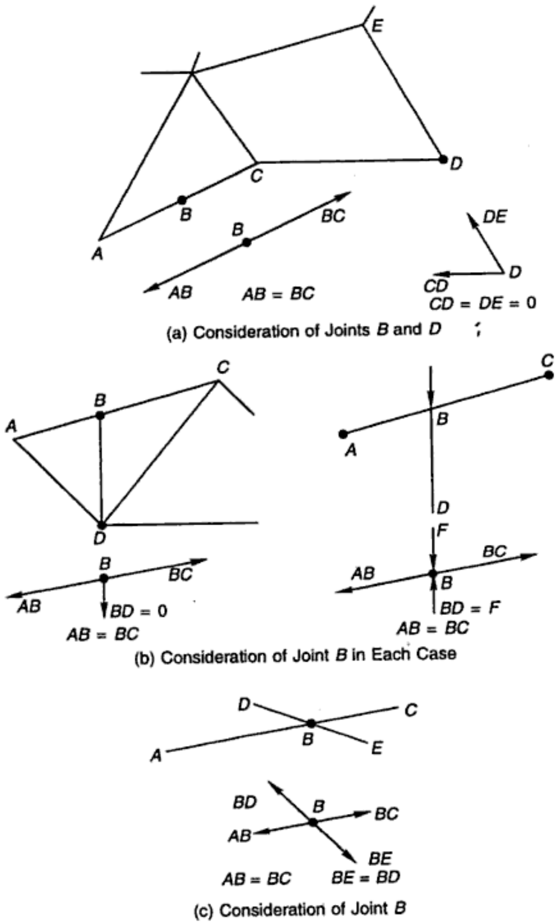


Fig. S1.5 ***Consideration of Joints***

whence $$R_c = 5 \text{ kN}, R_{Ay} = 5 \text{ kN}, R_{Ax} = 0$$

For equilibrium of the joint A (Fig. Ex. S1.1 (Solution)),

$$\Sigma F_y = 0; 5 + OA \sin 30° + AB \sin 60° = 0$$

$$\Sigma F_x = 0; AB \cos 60° + OA \cos 30° = 0$$

$$OA = -0.577\, AB$$

and $$5 + 0.866\, AB - 0.577/2\, AB = 0$$

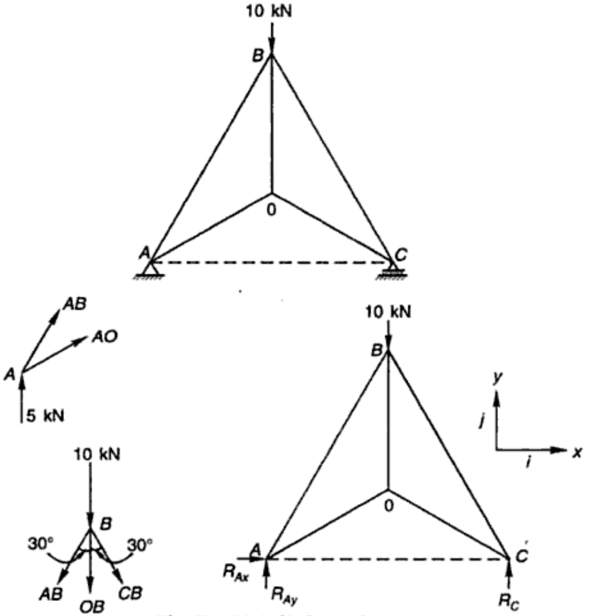


Fig. Ex. S1.1 ***(Solution)***

whence $AB = -8.66$ kN; compression

and $OA = 5.00$ kN; tension

For equilibrium of the joint *B*,

$$\Sigma F_x = 0;\ CB \sin 30° - AB \sin 30° = 0$$

$$\Sigma F_y = 0;\ -10 - OB - AB \cos 30° - CB \cos 30° = 0$$

whence, $CB = AB = -8.66$ kN; compression

and $OB + 2 \times 0.866\, AB = -10$

or $OB = -10 + 2 \times 8.66 \times 0.866$

or $OB = 5$ kN; tension

and $OC = OA = 5$ kN; tension

***Example* S1.2** A frame *PQRSTU* is hinged to a rigid support at *P* and is simply supported at *T*. It is loaded as indicated in the Fig. Ex. S1.2. Estimate the magnitude and nature of the force in the members *PQ* and *UT*.

Solution The free-body diagram of the structure is shown in Fig. Ex. S1.2. For equilibrium,

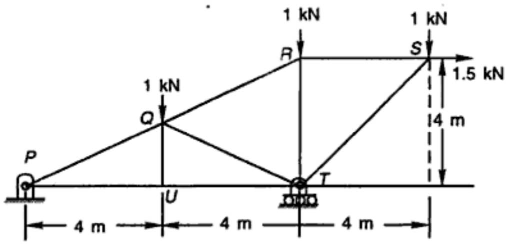


Fig. Ex. S1.2

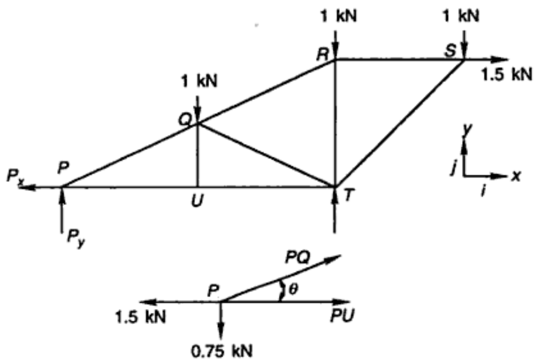


Fig. Ex. S1.2 *(Solution)*

$$\Sigma \mathbf{F} = 0$$

$$\Sigma F_y = 0; P_y + T - 1 - 1 - 1 = 0$$

$$P_y + T = 3 \text{ kN}$$

$$\Sigma F_x = 0; 1.5 - P_x = 0$$

$$P_x = 1.5 \text{ kN}$$

$$\Sigma \mathbf{M}_p = 0$$

$$8T - 1 \times 4 - 1 \times 8 - 1 \times 12 - 1.5 \times 4 = 0$$

whence $T = 3.75$ kN

and hence $P_y = 3 - 3.75 = -0.75$ kN

Consider the equilibrium at joint P:

$$\Sigma F_x = 0; -1.5 + PU + PQ \cos \theta = 0$$

$$\Sigma F_y = 0; PQ \sin \theta - 0.75 = 0$$

Taking $\sin \theta = \dfrac{2}{\sqrt{20}} = 0.447$

and $$\cos\theta = \frac{4}{\sqrt{20}} = 0.895$$

$$\mathbf{PQ} = 0.75/0.447 = 1.68 \text{ kN; tension}$$

$$\mathbf{PU} = 1.5 - 1.68 \times 0.895 = 0$$

Since UQ is perpendicular to PU and UT, UQ can transmit no force and the force in UT must be the same as in PU which is zero.

Hence $$UT = 0$$

***Example* S1.3** A simple structure *ABCDE* is supported on a hinge at *A* and on rollers at *B* while it carries a horizontal force of 1000 kN at *E* as shown in Fig. Ex. S1.3. Determine the force in member *AC*, using the method of joints.

Solution

Method of Joints Starting with E

For joint *E*,

$$\Sigma F_x = 0;\ 1000 + EC \sin 30° - ED \sin 30° = 0$$

$$\Sigma F_y = 0;\ -EC \cos 30° - ED \cos 30° = 0$$

whence $-EC = ED = 1000$ kN

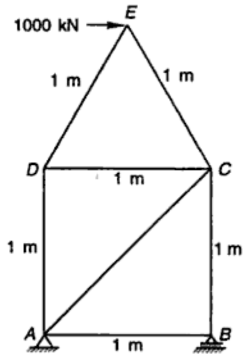


Fig. Ex. S1.3

Next, for joint *D*,

$$\Sigma F_x = 0;\ DE \cos 60° + CD = 0$$

$$\Sigma F_y = 0;\ DE \sin 60° - DA = 0$$

whence, $CD = -500$ kN

$DA = 866$ kN

Finally, for joint *C*,

$$\Sigma F_x = 0;\ -DC\ - EC \cos 60° - AC \cos 45° = 0$$

$$\Sigma F_y = 0;\ -BC\ - AC \sin 45° + EC \sin 60° = 0$$

and from the former

$$500 + 1000/2 - AC\ \times 0.707 = 0$$

whence $$AC = \frac{1000}{0.707} = 1414 \text{ kN}$$

which implies that the force in AC is 1414 kN tensile. The internal forces in AC are shown in Fig. Ex. S1.3 (Solution) and it is clear that the external force on AC must be tensile.

Method of Joints Starting with B

For this method, we should first determine the reactions at *A* and *B*.

Let the reaction at hinge *A* be

$$A_x\,\mathbf{i} + A_y\,\mathbf{j}$$

and the reaction at the roller support *B* be vertical, i.e., $B\mathbf{j}$.

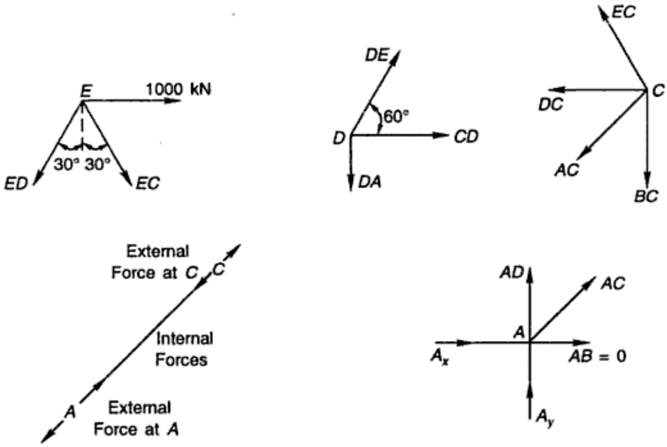


Fig. Ex. S1.3 ***(Solution)***

For the equilibrium of the structure,

$$\Sigma F_x = 0; \qquad A_x + 1000 = 0$$

$$\Sigma F_y = 0; \qquad A_y + B = 0$$

$$\Sigma M_A = 0; \; B \times 1 - 1000 \times 1.866 = 0$$

whence $$B = 1866, \; A_x = -1000 \text{ and } A_y = -1866 \text{ kN}$$

At *B*, the reaction is vertical and there are two members *BA* and *BC*; one horizontal and the other vertical. The horizontal member can carry no force because if it did it would not be balanced.

For joint *A*,

$$\Sigma F_x = 0; \; A_x + AC \cos 45° = 0$$

$$\Sigma F_y = 0; \; A_y + AD + AC \sin 45° = 0$$

whence $$AC = \frac{1000}{0.707} = 1414 \text{ kN}$$

The internal or resistive forces in *AC* are outwards which, as before, imply that the member *AC* carries a tensile force of 1414 kN.

Experiment E3

Forces in a Plane Truss

Objective

To determine the forces in the members of a statically determinate plane truss.

Apparatus

A pin-joined, simply supported plane determinate truss, some members of which have spring balances installed in them. Standard weights and a metre rod.

Background Information

The forces in the members of a truss may be computed analytically by the method of joints and by the method of sections and also graphically based upon the concept of equilibrium of the whole or of a part of the truss. Experimentally, the spring balances installed within the members are read off without loading and with loading in order to estimate the forces developed due to the loading of the truss. A typical truss *ABCDEFGHIJ* hinge-supported at *A* and roller-supported at *G* as shown in Fig. E3.1, may be subjected to loads at *C*, *D* and *E*, for example. If it is required to determine the force in member *AC*, the procedure would be to determine the reactions R_A and R_G at *A* and *G* respectively in the first instance. This may be done by considering the equilibrium of the entire truss. For the vertical loading as given, R_A and R_G are directed upwards.

$$\Sigma \mathbf{F} = 0; \qquad R_A + R_G + W_C + W_D + W_E = 0 \tag{i}$$

$$\Sigma \mathbf{M}_G = 0; \qquad W_C\, 3l + W_D\, 2l + W_E l - R_A 4l = 0 \tag{ii}$$

whence R_A and R_G are determined.

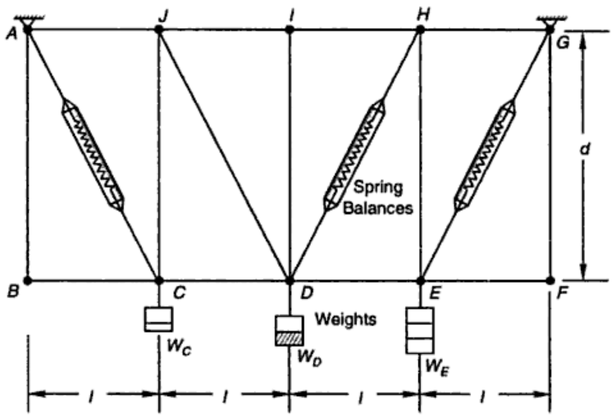


Fig. E3.1 ***Free-Body Diagram of a Loaded Plane Truss***

By the method of joints, one would first consider the simplest joint *B* where only two members *AB* and *BC* are pinned at right angles. From the free-body diagram of joint *B*, it is seen that it can be in equilibrium only if each of *BA* and *BC* is zero, because if either is non-zero it will leave an unbalanced force at *B*. Next, consider the joint *A*. The free-body diagram of joint *A* shows that there are two forces *AJ* and *AC* unknown in magnitude as shown in Fig. E3.2. From the equilibrium of *A*,

$$\Sigma F_x = 0; \qquad AJ + AC \sin \theta = 0 \tag{iii}$$

$$\Sigma F_y = 0; \qquad R_A - AC \cos \theta = 0 \tag{iv}$$

The solution of Eqs (iii) and (iv) provide the desired force in the member AC. Whether the member is in tension or compression may be sorted out by observing the sign of AC. It would come out to be positive in this case which may be interpreted as tension, following the text; otherwise, a free-body diagram of the member AC may be drawn as also shown in Fig. E3.3. The internal forces being inward, the external forces at A and C must be outward which suggest that the member AC must bear a tensile force.

By the method of sections, one would cut a section through members AJ, AC and BC. Consequently external forces AJ, AC and BC are shown acted upon the left section in its free-body diagram drawn in Fig. E3.3. Consider the section for equilibrium:

$$\Sigma M_A = 0; \qquad BC\,d = 0$$

whence the force BC is seen to be zero.

$$\Sigma F_x = 0; \qquad AJ + AC \sin \theta + BC = 0$$

$$\Sigma F_y = 0; \qquad R_A - AC \cos \theta = 0$$

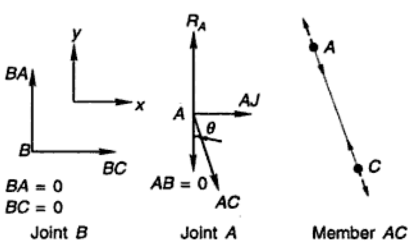


Fig. E3.2 ***Free-body Diagram***

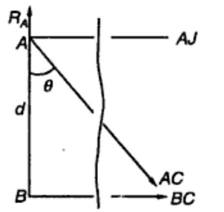


Fig. E3.3 ***Free-body Diagram of the Section***

The desired force AC is thus obtained from these equations. The sign of AC would decide whether it is tensile or compressive. It would come out to be positive in this case which may be interpreted as tension; otherwise, as noted from the fact that the external force AC on the member AC is outward, the member must be in tension.

Experimentally, the spring balance installed in the member AC would show a tensile force acting on the member when the truss is loaded as shown.

Table of Results

Forces in the members with spring balances for the prescribed loading should be reported.

Forces in members	*AC*	*DH*	*EG*
Experimentally			
Analytically			
Difference %			

Results on the linearity of response of the system should be reported in the form of a curve plotted between the load at C and the force in AC.

Points for Discussion

1. Recognise the sources of error in the experiment. In particular, observe the play in the spring balances and joints. The use of a spring balance which operates on the principle of elongation of the member is inherently prone to error.
2. Suggest some means of eliminating the sources of error. One of the ways of minimising the error in the experiment would be to use larger loads for the first part of the experiment and to change the load in larger steps in the second part of the experiment.
3. Explain the truth in the statement: "The method of joints is a special case of the method of sections".
4. Do the forces in members of a truss respond linearly to loading? How?

S1.4 ANALYSIS BY THE METHOD OF SECTIONS

The method of sections consists of hypothetically cutting a section of the given truss and analysing it for equilibrium. Equilibrium of the entire truss guarantees the equilibrium of every part of the truss. In the method of joints, equilibrium of every joint was considered. In the method of sections, equilibrium of any selected section of the truss is considered. The section of the truss is selected in such a way as to 'cut' the desired member only once. The free-body diagram of the section will thus include the unknown force in the member. The analysis of the section for equilibrium requires the application of

$$\Sigma \mathbf{F} = 0 \quad \text{and} \quad \Sigma \mathbf{M} = 0$$

For a plane section, the necessary conditions of equilibrium reduce to

$$\Sigma F_x = 0, \Sigma F_y = 0 \quad \text{and} \quad \Sigma M = 0$$

or an equivalent set of three equations.

The force system acting on a section of a plane truss can only be a coplaner force system. The three necessary equations as above are also sufficient conditions of equilibrium.

The free-body diagram of the desired section of a truss may have one, two, three or more unknown forces. The equations of equilibrium written for a section may or may not be adequate to determine all the unknowns. In case a selected section of the truss is not amenable to solution, an attempt must be made to locate a section which includes some of the unknown forces of the desired section that can be determined. The knowledge gained from the analysis of simpler sections must enable us to determine all the unknowns. It may be appreciated that the method of joints is a special case of the method of sections. When a section is chosen in the vicinity of a joint so as to enclose the joint, the section in question reduces to the joint only. Equilibrium of the section implies equilibrium at the joint. The method of joints is thus the method of sections applied to enclose one joint at a time.

It is a useful convention in the method of sections to indicate the forces on a section such that the known forces are taken in the correct directions and the unknown forces are assumed acting away from the section under consideration. On evaluation, if a force turns out to be positive, the member must be in tension and if a force comes out to be negative, the member must be in compression. The forces are named after the names of the members for convenience in recognition.

***Example* S1.4** A simple structure *ABCDE* is supported on a hinge at *A* and on rollers at *B* while it carries a horizontal force of 1000 kN at *E* as shown in Fig. Ex. S1.4. Determine the force in member *AC*, using the method of sections.

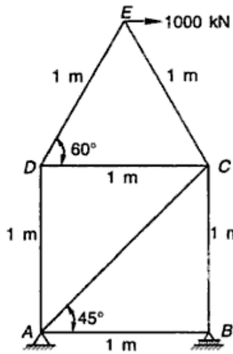


Fig. Ex. S1.4

Solution

Method of Sections (a)

A section *SS* can be cut through members *AD*, *AC* and *BC* as shown in Fig. Ex. S1.4 (Solution), and a consideration of equilibrium of either side of the section should lead to the force in *AC*. Consider the upper part of the structure. The external forces on the part are

(a) 1000 kN at *E*, acting horizontally

(b) Forces along *AD*, *AC* and *BC* due to the lower part of the structure; tensile if away from the section under consideration.

For equilibrium,

$$\Sigma F_x = 0;\ \ 1000 - AC\cos 45° = 0$$

$$\Sigma F_y = 0; - AD - BC - AC\sin 45° = 0$$

and from the former,

$$AC = \frac{1000}{0.707}$$

$$= 1414 \text{ kN}$$

which shows that the member *AC* is subjected to 1414 kN tensile force.

Method of Sections (b)

It is instructive to consider the free-body diagram and the equilibrium of the lower part of the section *SS* for the desired purpose and to show that it involves more work. The additional work is required to calculate the reactions by considering the equilibrium of the entire structure first. After obtaining that

$$B_y = 1866 \text{ kN}$$

$$A_x = -1000 \text{ kN}$$

and $$A_y = -1866 \text{ kN}$$

the equilibrium of the lower part requires that

$$\Sigma F_x = 0; A_x + AC\cos 45° = 0$$

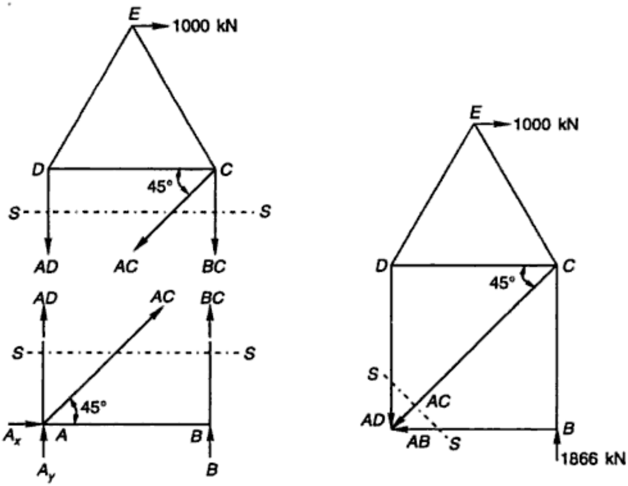


Fig. Ex. S1.4 *(Solution)*

$$\Sigma F_y = 0;\ A_y + B + AD + BC + AC \sin 45° = 0$$

whence

$$AC = \frac{1000}{0.707} = 1414 \text{ kN}$$

which shows that the member *AC* is subjected to 1414 kN tensile force.

Method of Sections (c)

The section *SS* may instead be cut through members *AD*, *AC* and *AB* and the part of the structure on its right can be considered for equilibrium

$$\Sigma F_x = 0;\ 1000 - AB - AC \cos 45° = 0 \qquad \text{(i)}$$

$$\Sigma F_y = 0;\ 1866 - AD - AC \sin 45° = 0 \qquad \text{(ii)}$$

$$\Sigma M_B = 0;\ -1000 \times 1.866 + 1 \times AD + AC \times 0.707 = 0 \qquad \text{(iii)}$$

whence,

$$AB + 0.707\ AC = 1000 \qquad \text{(i)}$$

$$AD + 0.707\ AC = 1866 \qquad \text{(ii)}$$

$$AD + 0.707\ AC = 1866 \qquad \text{(iii)}$$

and it can be noticed that the equations (ii) and (iii) are identical. The three unknown forces *AD*, *AC* and *AB* cannot be evaluated by only two independent equations.

Another look on the section reveals that the three forces under question are concurrent. Such sections which render three concurrent forces as unknowns are not plausible.

The way out of this difficulty can be found by evaluating one of thè three unknowns by considering some other section or joint for equilibrium. In this case, we notice from the consideration of equilibrium at B, that the force in AB is zero.

Hence, from Eq. (i),

$$0.707\, AC = 1000$$

$$AC = 1414 \text{ kN}$$

which shows that AC must carry a tensile force of 1414 kN.

***Example* S1.5** A derrik has ball-and-socket joints at A, B, C and D as shown in Fig. Ex. S1.5. Determine the forces in links AD, BD and CD when it is supporting a dead load of 1000 kN.

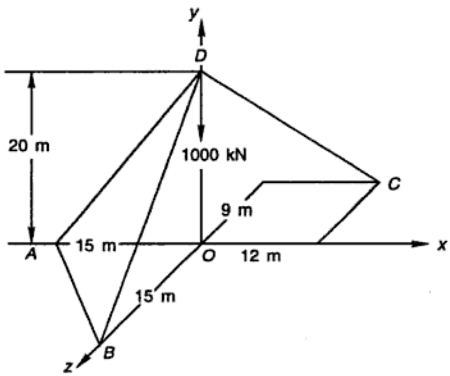


Fig. Ex. S1.5

Solution With respect to the given origin O, position vectors of the salient points are as follows:

$$\mathbf{OA} = -15\,\mathbf{i}$$

$$\mathbf{OB} = 15\,\mathbf{k}$$

$$\mathbf{OC} = 12\,\mathbf{i} - 9\,\mathbf{k}$$

$$\mathbf{OD} = 20\,\mathbf{j}$$

It follows that the unit vectors along the three members all pointing to D are:

$$\mathbf{e}_{AD} = \frac{\mathbf{OD} - \mathbf{OA}}{\left|\sqrt{\mathbf{OD}^2 + \mathbf{OA}^2}\right|} = \frac{20\,\mathbf{j} + 15\,\mathbf{i}}{25} = 0.8\,\mathbf{j} + 0.6\,\mathbf{i}$$

$$\mathbf{e}_{BD} = \frac{20\,\mathbf{j} - 15\,\mathbf{k}}{25} = 0.8\,\mathbf{j} - 0.6\,\mathbf{k}$$

$$\mathbf{e}_{CD} = \frac{20\,\mathbf{j} - 12\,\mathbf{i} + 9\,\mathbf{k}}{25} = -0.48\,\mathbf{i} + 0.8\,\mathbf{j} + 0.36\,\mathbf{k}$$

Let the magnitudes of forces in members be denoted by AD, BD and CD, then for equilibrium,

$$AD\,(0.8\,\mathbf{j} + 0.6\,\mathbf{i}) + BD\,(0.8\,\mathbf{j} - 0.6\,\mathbf{k}) + CD\,(-0.48\,\mathbf{i} + 0.8\,\mathbf{j} + 0.36\,\mathbf{k})$$
$$- 1000\,\mathbf{j} = 0$$

This vector equation is written as three component scalar equations:

$$0.6\,AD - 0.48\,CD = 0, \qquad \text{for } x\text{-direction}$$

$$0.8\,AD + 0.8\,BD + 0.8\,CD - 1000 = 0, \qquad \text{for } y\text{-direction}$$

and
$$-0.6\,BD + 0.36\,CD = 0, \qquad \text{for } z\text{-direction}$$

Solving these equations,

$$AD = 416.7 \text{ kN},\ BD = 312.5 \text{ kN},\ CD = 520.83 \text{ kN}.$$

The unit vectors were taken pointing to the common point D. Now that AD, BD and CD all turn out positive the members exert forces pointing to D as shown in Fig. Ex. S1.5 (Solution) which means the members must be under compression.

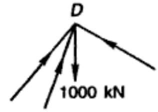


Fig. Ex. S1.5 *(Solution)*

***Example* S1.6** A simple crane rests on a ball-and-socket joint O and it is supported by two strings AB and AC as shown in Fig. Ex. S1.6. Determine the forces in the cables and the reaction at O when it holds a load of 1280 kN.

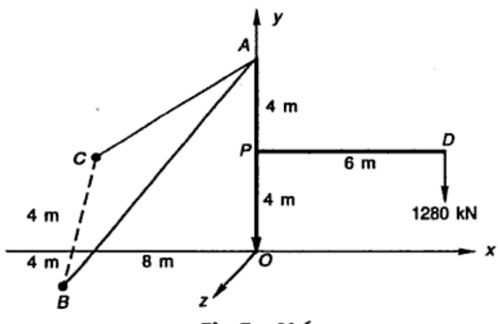


Fig. Ex. S1.6

Solution Position vectors of A, B, C and D are as follows:

$$\mathbf{OA} = 8\,\mathbf{j}$$

$$\mathbf{OB} = -8\,\mathbf{i} + 4\,\mathbf{k}$$

$$\mathbf{OC} = -8\,\mathbf{i} - 4\,\mathbf{k}$$

$$\mathbf{OD} = 4\,\mathbf{j} + 6\,\mathbf{i}$$

and the load applied at D is $-1280\,\mathbf{j}$ kN.

The unit vectors along the cables, pointing towards A are

$$\mathbf{e}_{CA} = \frac{8\,\mathbf{j} + 8\,\mathbf{i} + 4\,\mathbf{k}}{12} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{e}_{BA} = \frac{8\,\mathbf{j} + 8\,\mathbf{i} - 4\,\mathbf{k}}{12} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

The negative sign implies that the forces exerted by the strings are not pointing towards A but away from A, which means that they must be in tension. This is indeed true for strings and cables.

The ball-and-socket joint at O cannot resist any moment. It can provide only force reactions. Therefore, using

$$\Sigma\mathbf{F} = 0$$

for equilibrium of the crane structure,

$$720\left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) + 720\left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right)$$

$$-1280\,\mathbf{j} + R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_z\,\mathbf{k} = 0$$

whence

$$R_x = 960 \text{ kN}$$

$$R_y = 2240 \text{ kN}$$

$$R_z = 0$$

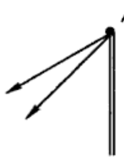


Fig. Ex. S1.5(a) ***(Solution)***

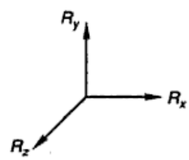


Fig. Ex. S1.5(b) ***(Solution)***

Taking moments about the joint O,

$$(-8\,\mathbf{i} + 4\,\mathbf{k}) \times AB\,\mathbf{e}_{BA} + (-8\,\mathbf{i} - 4\,\mathbf{k}) \times AC\,\mathbf{e}_{CA} + (4\,\mathbf{j} + 6\,\mathbf{i}) \times (-1280\,\mathbf{j})$$

$$= (-8\mathbf{i} + 4\mathbf{k}) \times \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right) AB + (-8\mathbf{i} - 4\mathbf{k}) \times \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) AC$$

$$+ (6\,\mathbf{i} + 4\,\mathbf{j}) \times (-1280\,\mathbf{j})$$

$$= \left(-\frac{16}{3}\mathbf{k} - \frac{8}{3}\mathbf{j} + \frac{8}{3}\mathbf{j} - \frac{8}{3}\mathbf{i}\right) AB - \left(\frac{16}{3}\mathbf{k} + \frac{8}{3}\mathbf{j} - \frac{8}{3}\mathbf{j} + \frac{8}{3}\mathbf{i}\right) AC - 7680\,\mathbf{k}$$

$$= -3000\,\mathbf{k} - \frac{8}{3}(AB - AC)\mathbf{i} - \frac{16}{3}(AB + AC)\mathbf{k}$$

Now $$\mathbf{e}_{OB} = \frac{-8\mathbf{i}+4\mathbf{k}}{\sqrt{80}} = \frac{-2\mathbf{i}+\mathbf{k}}{\sqrt{5}}$$

and $$\mathbf{e}_{OC} = \frac{-2\mathbf{i}-\mathbf{k}}{\sqrt{5}}$$

The moment about axis *OB*, i.e.,

$$M_{OB} = \mathbf{M}_O \cdot \mathbf{e}_{OB} \quad \text{must be zero for equilibrium.}$$

$$\left[-\frac{8}{3}(AB-AC)\mathbf{i} - \frac{16}{3}(AB+AC+1440)\,\mathbf{k}\right]\cdot\left[\frac{-2\mathbf{i}+\mathbf{k}}{\sqrt{5}}\right] = 0$$

$$\frac{16}{3\sqrt{5}}(AB-AC) - \frac{16}{3\sqrt{5}}(AB+AC+1440) = 0$$

or $$\frac{-32}{3\sqrt{5}}AC = \frac{16}{3\sqrt{5}}\times 1440$$

or $$AC = -720 \text{ kN}.$$

By symmetry (or by considering the moment about axis *OC*), *AB* = –720 kN.

***Example* S1.7** A vertical pole *CD* is supported on a hinge *D* at the base and two guy wires tied at its top as shown in Fig. Ex. S1.7. The pole is supporting a horizontal pull of 1 kN along *x*-axis. Determine the reactions at the support *A*, *B* and *C*.

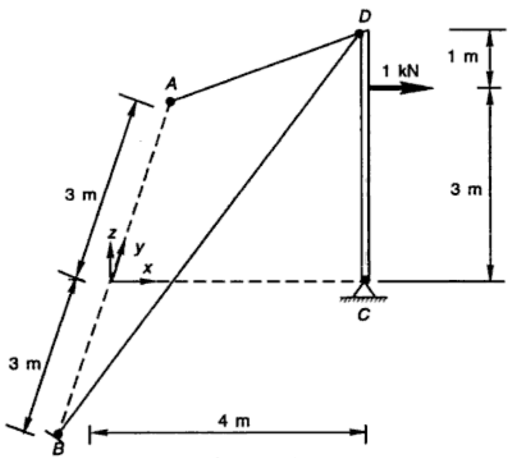


Fig. Ex. S1.7

Solution Reactions at A and B should be along the guy wires; Reaction at C consists of R_x, R_y and R_z.

From the coordinates of the points $A(0, 3, 1)$, $B(0, -3, 1)$, $C(4, 0, 0)$ and $D(4, 0, 4)$, unit vectors along the wires are

$$\mathbf{e}_{DA} = (-4\,\mathbf{i} + 3\,\mathbf{j} - 3\,\mathbf{k})/\sqrt{34} \qquad \mathbf{F}_{DA} = F\,\mathbf{e}_{DA}$$

$$\mathbf{e}_{DB} = (-4\,\mathbf{i} - 3\,\mathbf{j} - 3\,\mathbf{k})/\sqrt{34} \qquad \mathbf{F}_{DB} = F\,\mathbf{e}_{DB}$$

By symmetry forces in guy wires must be equal. As shown in Fig. Ex. S1.7 (Solution) the force exerted by the wires on the pole at D is

$$\mathbf{F}_{DA} + \mathbf{F}_{DB} = (-8\,\mathbf{i} - 6\,\mathbf{k})/\sqrt{34}\ \mathbf{F} = (-8\,\mathbf{i} - 6\,\mathbf{k})F'$$

For equilibrium of the pole,

$$(-8\,\mathbf{i} - 6\,\mathbf{k})\ F' + 1000\,\mathbf{i} + R_x\mathbf{i} + R_y\mathbf{j} + R_z\,\mathbf{k} = 0$$

The isolated component along y; $R_y = 0$.

$$-8F' + 1000 + R_x = 0$$

$$-6F' + R_z = 0$$

and $\quad \Sigma\mathbf{M}_c = 0;\ 4\,\mathbf{k} \times (-8\,\mathbf{i} - 6\,\mathbf{k})\ F' + 3\,\mathbf{k} \times 1000\,\mathbf{i} = 0$

whence $\quad (-32F' + 3000) = 0;\ F' = 93.75$

and $\quad F = \sqrt{34} \times 93.75 = 547$ N

Also, $\quad R_x = 8 \times 93.75 - 1000 = -250$ N

$\quad R_z = 6 \times 93.75 = 562.5$ N

Let us check the solution. Taking moments about D, for example,

$$-1000 \times 1 - 4\,R_x = 0;\ R_x = -250 \text{ N}$$

which is the same as before.

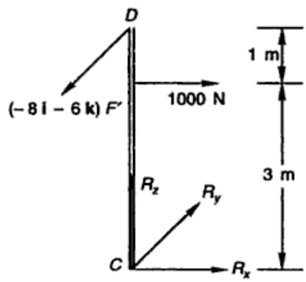


Fig. Ex. S1.7 *(Solution)*

Experiment E4

Forces in a Space Frame

OBJECTIVE

To determine the forces in the members of a loaded shear-legs space frame experimentally, vectorially and graphically.

APPARATUS

Shear-legs apparatus consisting of two rigid bars AB and AC and a tie-bar AD

together with a provision for loading at A as shown in Fig. E4.1. Metre rod, spring balances and standard weights.

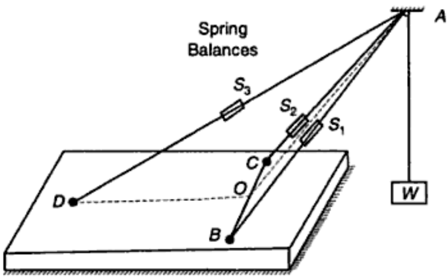


Fig. E4.1 ***Shear Legs Apparatus***

BACKGROUND INFORMATION

Since the frame is in equilibrium, every part and sub-part of the frame must also be in equilibrium. Consider, for example, the joint A. The forces that act at A to keep it in equilibrium are the known loads acting vertically downward and the forces along AB, AC and AD. The directions of AB, AC and AD are determined from coordinate geometry; only the magnitudes of forces are unknown which may be obtained experimentally by reading the spring balances installed in the members. Theoretically, the method of vector analysis or a graphical construction may be employed to estimate the forces.

Vectorially, let the unit vectors along AB, AC and AD be **ab**, **ac** and **ad** respectively. These are obtained by measuring the coordinates of the end points. For example, if the origin is chosen at O,

$$\mathbf{r}_A = x_A\,\mathbf{i} + z_A\,\mathbf{k}$$

$$\mathbf{r}_B = -y_B\,\mathbf{j}$$

$$\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A = -(x_A\,\mathbf{i} + y_B\,\mathbf{j} + z_A\,\mathbf{k})$$

whence

$$AB = \sqrt{(x_A^2 + y_B^2 + z_A^2)}$$

and

$$\mathbf{ab} = -\frac{x_A}{AB}\mathbf{i} - \frac{y_B}{AB}\mathbf{j} - \frac{z_A}{AB}\mathbf{k}$$

Let the unknown forces be F_1, F_2 and F_3 in the members AB, AC and AD respectively. Then, for equilibrium of the joint A

$$\Sigma\mathbf{F} = 0$$

$$F_1\,\mathbf{ab} + F_2\,\mathbf{ac} + F_3\,\mathbf{ad} - W\,\mathbf{k} = 0$$

This equation may be written as three equations, one along each coordinate direction to obtain the three unknowns.

Graphically, the forces in the members may be determined in two steps. First, a fictitious member AO is assumed to replace the bars AB and AC. The point A may then be considered for equilibrium under the action of the load and forces in AO and AD. A triangle of forces is drawn whence, by measurement to appropriate scale, the forces in AO and AD are determined, as shown in Fig. E4.2.

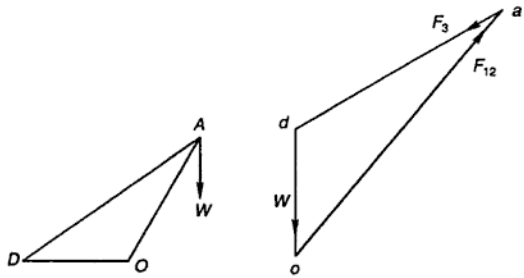


Fig. E4.2 ***Construction of Triangle of Forces***

It is advisable to draw a scale diagram ADO of the frame with the fictitious member AO before drawing the triangle of forces. It may be seen that the force in AD must be F_3 and the force in AO in F_{12}. The nature of forces in AD and AO is observed by drawing the arrows of the internal forces and of the external forces on the members.

Since the force exerted by the member AD on the joint is F_3, directed A to D, the external force acting at A on AD must be equal and opposite to it as shown by the dotted lines in Fig. E4.3. Outward external forces on AD imply that it must be in tension. Similarly, the fictitious member AO is observed to be in compression.

The second step is to resolve the force F_{12} along AO into forces F_1 and F_2 along AB and AC respectively. After drawing the force F_{12} along AO, the directions of AB and AC are drawn and the force F_{12} is resolved into F_1 along AB and F_2 along AC by completing the parallelogram of forces with AO as the resultant diagonal as shown in Fig. E 4.4.

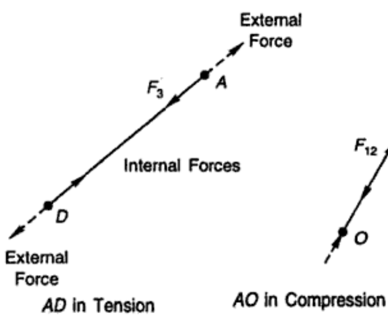


Fig. E4.3 ***Internal Forces in Members***

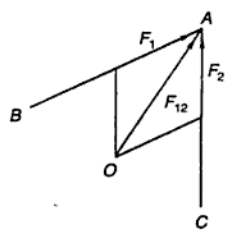


Fig. E4.4 ***Construction of Parallelogram of Forces***

RESULTS

Force in member	Experimental values			Vectorial analysis	Graphical procedure
	Initial reading	*When loaded*	*Force (by difference)*		
AB					
AC					
AD					

POINTS FOR DISCUSSION

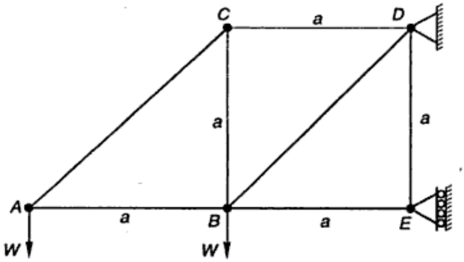
1. What are the assumptions made in the vectorial and graphical analyses in respect of the rigidity and mass of the members?
2. Examine the joints and supports of the frame and comment on the validity of your analysis in view of the same.
3. Is it possible to calibrate one of the spring-balances to determine the weight of a given body?
4. Comment on the validity of the assumption of a fictitious member *AO* lying in the plane of *AB* and *AC*.

Concept Review Questions

1. (a) What is meant by a simple structure?
   (b) Differentiate between a structure and mechanism.
   (c) Can a simple structure be a space structure?
2. (a) Draw a just-rigid structure with five members and another just-rigid structure with five joints. Draw one additional member in each of them to render them over-rigid.
   (b) Is it possible to have a just-rigid structure with an even number of members? Why or why not?
3. In the analysis by the method of joints, should one proceed from a joint on the extreme left to the joint on the extreme right or the reverse or are there some other important considerations?
4. Draw the free-body diagram of a member subjected to tension. What would be the internal forces at the two joints?
5. Recalling the comments on loading conditions, fill in the blanks:
   (a) If there are only two members and no external force at a joint,.........
   (b) If there are three members, two of which are collinear and there is no external force at a joint,......
6. Illustrate the implication of the Bow's notation. What are the space diagram and rays diagram?
7. Comment on the graphical method of analysing a simple structure. Under what conditions is the graphical method preferred and for what conditions does it fail?

Tutorial Problems

S1.1 Determine the forces in the members of the pin-jointed truss shown in Fig. Prob. S1.1.



(Ans. $D_x = 3W$
$D_y = 2W$, $E_x = -3W$,
AB = $-W$ (Comp),
BC = $-W$ (Comp),
AC = $\sqrt{2}\,W$ (tens),
CD = W (tens.),
BD = $2\sqrt{2}\,W$ (tens),
BE = $-3W$ (comp),
DE = 0)

Fig. Prob. S1.1

S1.2 A coplanar simple truss *ABCDE* is loaded with a force of 50 kN at *A* as shown in Fig. Prob. S1.2. Determine

(a) the forces exerted in the members of the truss

(b) the force exerted on the pins at the joints.

(**Ans. AB** = –86.6 kN (comp) = **BC**,
AE = 50 kN = **ED**,
EB = 0 = **EC**)

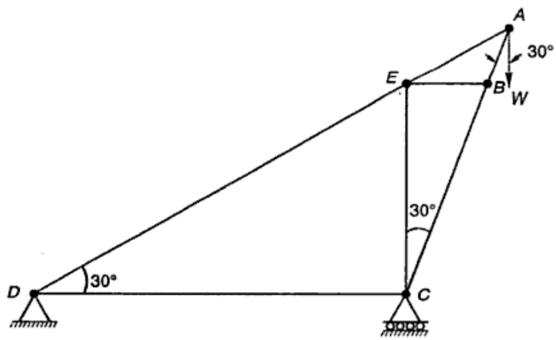


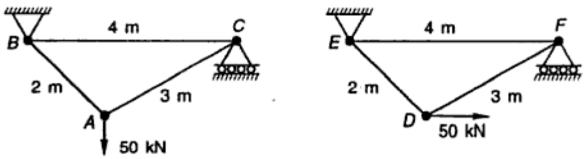
Fig. Prob. S1.2

S1.3 Using the method of joints or the method of sections, calculate the force in each member of the trusses shown in Fig. Prob. S1.3. State whether the members are in tension or in compression

(**Ans.** **AB** = 40.6, **AC** = 34.6, **BC** = 27.95 kN | **PN** = $2\sqrt{2}$, **MN** = 2
DE = 29.8, **DF** = –36.6, **EF** = 29.5 kN | **MP** = 2, **LM** = 2
ON = 5, **PO** = 0 = **KP** | **LP** = $2\sqrt{2}$, **KL** = 5)

S1.4 Determine the forces in the members of the given simple truss. (Fig. Prob. S1.4)

AB = 100 $\sqrt{2}/3$
(**Ans. CD** = –100 $\sqrt{2}/3$, **DE** = 100/3 = **CE**
CB = –100/3 = **FE** = **AF** etc.)



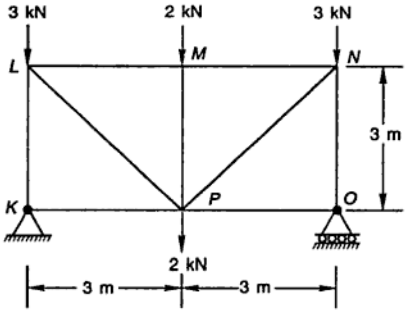


Fig. Prob. S1.3

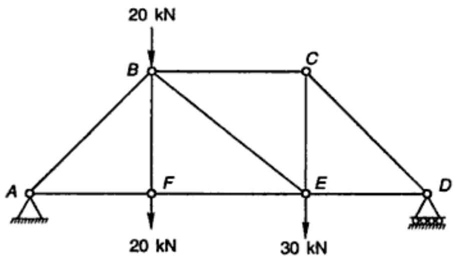


Fig. Prob. S1.4

S1.5 A simple truss is loaded as shown in Fig. Prob. S1.5. Calculate the forces in the members.

(**Ans. AB = BC = 0 = CD = FE, AC = –24.0, AH = 13.3 = HG HC = –10.0, CG = 24.0, CD = 0.0 = DE, GF = 106.7, GE = 96.1 kN**)

S1.6 Compute the forces in the members of the given pin-jointed truss. (Fig. Prob. S1.6).

(**Ans. DH = 9.1, BH = 12.93, EH = 20.0 AB = 28.28, AG = 40.0, AF = 28.28, GE = 27.87 kN**)

S1.7 A triangular simple truss is loaded as shown in Fig. S1.7. Determine the forces in the members.

(**Ans. CD = 0.46, AD = 10.15, BD = 10.60, BC = 0.55 = AB**)

S1.8 A frame is subjected to a horizontal load to 2 kN at *C*, a vertical load of 1 kN and a moment of 1.5 kNm at the mid-point of *DE* as shown in Fig. Prob. S1.8. Determine the reactions at the support *A* and *B* and the force on the pin at *D*.

(**Ans.** $R_{Ax} = -2$ **kN,** $R_{Ay} = -1.21$ **kN,** $R_B = 2.21$ **kN; Zero**)

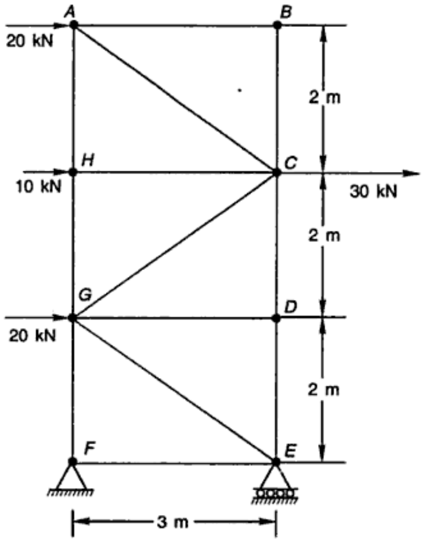


Fig. Prob. S1.5

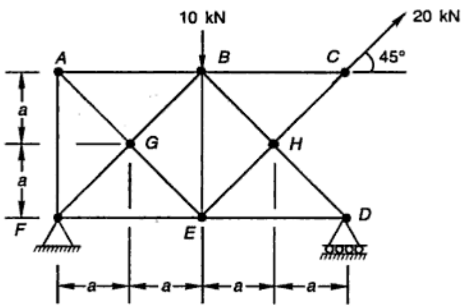


Fig. Prob. S1.6

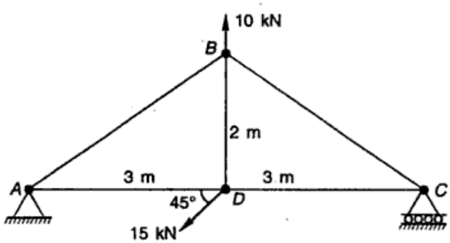


Fig. Prob. S1.7

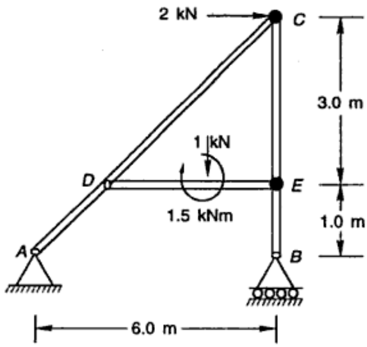


Fig. Prob. S1.8

S1.9 A ball-and-socket jointed space frame is loaded with a single load of 10 kN at C in the *y*-*z* plane as shown in Fig. Prob. S1.9. All dimensions are in metres. Determine the force in the member *CG*.

(**Ans.** **CG** = 7.07 kN (comp); **AC** = **BC** = –1.6; **DG** = **FG** = 1.6; **EG** = 3.35 (comp))

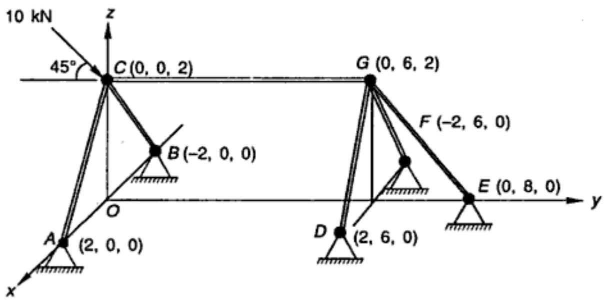


Fig. Prob. S1.9

S1.10 Determine the forces produced in the bars of the system due to the horizontal force *P* applied at the hinge *B* (Fig. Prob. S1.10)

[**Ans.** **AB** = **AD** = *P* (tension); **BC** = 0 **AC** = 1.414 *P* (comp)]

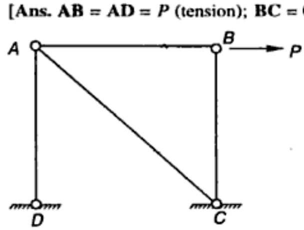


Fig. Prob. S1.10

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. If, for a plane pin-jointed truss, $m = 2j - 3$ with the usual nation,
   (a) it must be a just-rigid truss
   (b) it cannot be over-rigid over any part of the truss
   (c) it may or may not be a just-rigid truss
   (d) it cannot be non-rigid over any part of the truss
2. The method of joints for the analysis of forces in the members of a pin-jointed truss
   (a) is a special case of method of sections
   (b) does not need the determination of reactions at the supports
   (c) works equally well, irrespective of starting point for the analysis
   (d) fails when there are only two members at a joint and no external load is applied there
3. In the method of sections for the analysis of forces in the members of a pin-jointed truss,
   (a) the section can be cut through any set of members for equal ease of analysis
   (b) the sections must be cut so that the number of unknowns is limited and determined by employing the conditions of equilibrium.
   (c) care must be taken to ensure that the section being cut is in equilibrium
   (d) the sections to be cut are as small as possible

Answers to the Multiple-Choice Questions

1 (c), 2 (a), 3 (b)

S2 THIN RIGID BEAMS

S2.1 BEAMS AND LOADING

Beams are structural members primarily subjected to transverse forces. Forces in the longitudinal direction and twisting moments about the longitudinal axis may act in addition to the transverse loading. One basic feature of a beam is that internal forces called shear forces and the internal moments called bending moments are developed in a beam so as to resist the applied-force system. Beams dealt with under this section are assumed to be rigid; they do not deform under the application of loads. Further, beams are qualified as thin to imply that their transverse dimensions are negligible in comparison with their length. These assumptions are universally accepted in studying the shear-force and bending moment variation over the span of a beam.

A beam may be *simply supported* if one of its ends is hinged and other is supported on a roller as in Fig. S2.1(a). A beam may be confined within the space between the supports or it may extend on one or both ends with forces acting on it; it is then said to be an *overhanging beam* (Fig. S2.1(b)). If a beam is fixed on one end and is free on the other end, it is said to be a *cantilever beam* or a *cantilever* as shown in Fig. S2.1(c).

A beam may be fixed on one or both ends as shown in Figs. S2.1(d) and (f). It may as well be propped by a support within the span. A beam may also be supported on hinges at a number of points over the span; it is then said to be a *continuous beam* as in Fig. S2.1(e). Beams may be combined by means of ball-and-socket joints or otherwise to constitute a longer combined beam as shown in Fig. S2.1(g).

A beam may be loaded in a variety of ways. A *concentrated load* is that which acts over so small a length that it is assumed to act at a point. On the other hand, a *distributed load* acts over a finite length of the beam. A distributed load may be uniform over the length or it may vary uniformly or non-uniformly. A distributed load is specified by the *intensity of loading* per unit length, say w N/m in SI units. A *uniformly-distributed load*, therefore, implies a constant intensity of loading w, whereas a *uniformly-varying load* implies the increase or decrease of loading intensity at a constant rate along the length

$$w = w_1 + kx \tag{S2.1}$$

where k is the rate of change of the loading intensity, w_1 being the loading at the reference point. Similarly, distributed load may be represented by a parabolic, cubic or a higher order curve for non-uniformly varying load

$$w = w_1 + k_1x + k_2x^2 \qquad \text{(parabolic)}$$

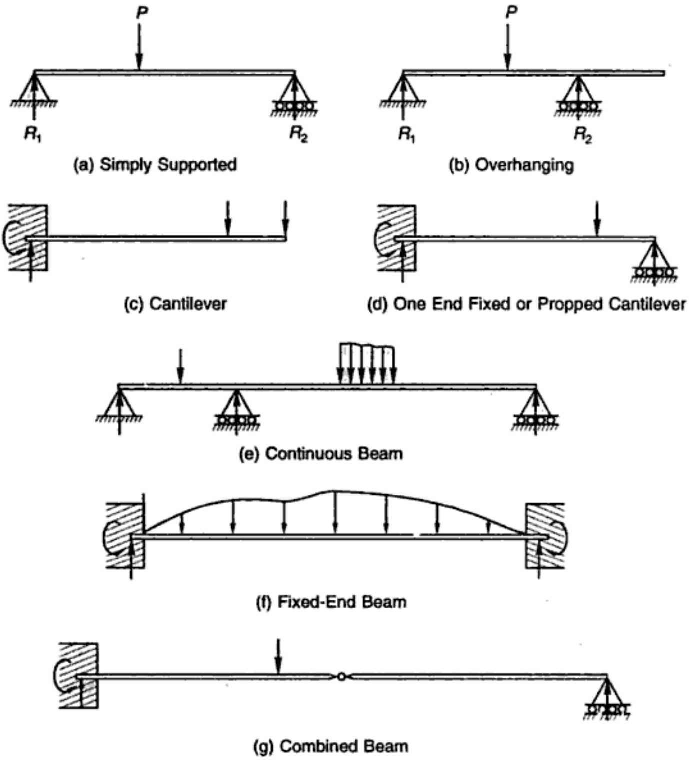


Fig. S2.1 ***Types of Beams and Reactions at Supports***

$$w = w_1 + k_1x + k_2x^2 + k_3x^3 \qquad \text{(cubic)}$$

$$w = w_1 + k_1x + k_2x^2 + k_3x^3 + k_4x^4 \qquad \text{(quartic)}$$

A load may also be a combination of a uniform load and a uniformly-varying load: a uniform load is shown by a rectangular distribution, a uniformly-varying load by a triangle and the combination by a trapezium drawn on the beam.

S2.2 SHEAR FORCE AND SF DIAGRAM

The shear force in a beam at any section is the transverse force tending to cause shear across the section.

Consider a simple beam supported at its ends and subjected to an external concentrated load P at a given point p as shown in Fig. S2.2. The shear force SF at a cross-section SS is the transverse force tending to shear the beam at this cross-

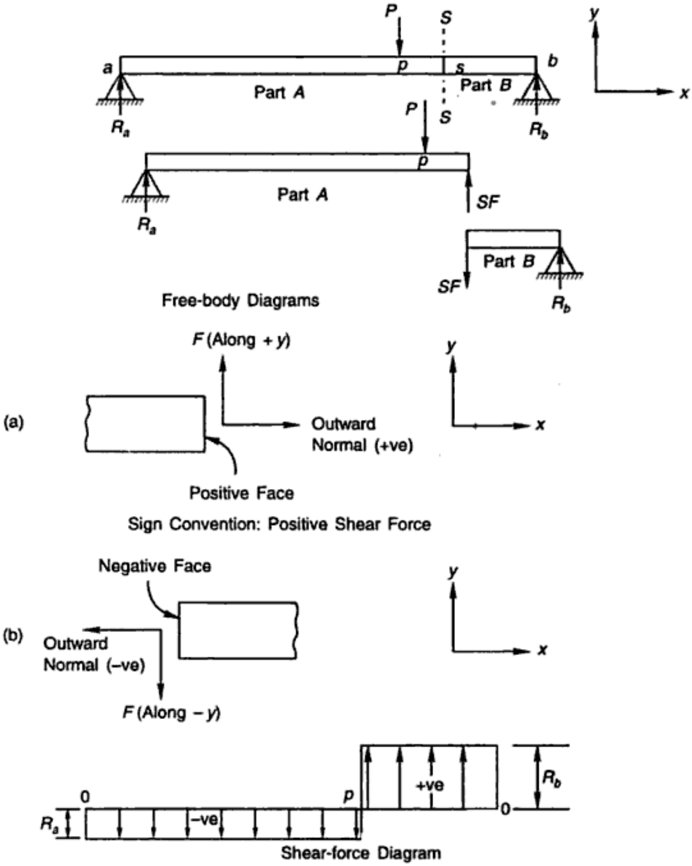


Fig. S2.2 ***Shear Force: Definition, Sign and Diagram***

section. In order to determine the shear force, consider the free-body diagram of a part of the beam on either side of the section *SS*. The free-body diagram of parts *A* and *B* of the beam are shown in Fig. S2.2. The transverse force *F* acting on the cross-section *SS* is the shear force at that section. The sign of the shear force should be determined with reference to the sign convention:

Sign Convention

The sign convention for shear force is *A positive shear force is that which acts in a positive direction on a positive face or in a negative direction on a negative face.*

It may be understood that a face is said to be positive if the outward normal to it acts along the positive coordinate direction and negative if otherwise. The concept of positive and negative faces as well as the sign convention for shear force is illustrated in Fig. S2.2.

Shear-force Diagram

The shear-force diagram generally referred to as *SF* diagram is a graph showing the variation of shear force along a beam. If the positive shear force is shown above an arbitrary reference line *OO* and the negative shear force below it, the line joining the extremities of the shear forces at different points is the shear-force diagram. The *SF* curve over a given length may remain constant, vary linearly or non-linearly depending upon the loading condition. In general, for no load between two points, *SF* remains constant, for uniformly distributed load *SF* varies linearly and for non-uniformly varying load *SF* varies non-linearly. The implication of this statement will be clear when we relate the loading diagram to the shear-force diagram.

Drawing of Shear-force Diagram

The drawing of a shear-force diagram is an important task because the *SF* diagram provides a picture of the shear force at all points along the length of the beam. The procedure to plot an *SF* diagram is as follows:

1. Draw the symbolic loading diagram of the given beam to scale along the length of the beam.
2. Find the reactions at the supports by using the fact that the entire beam is in equilibrium and that

$$\Sigma \mathbf{F} = 0$$

and $\Sigma \mathbf{M} = 0$ about any point; say about the points of the supports

3. Start from the right-hand end of the beam for convenience. The shear force at the end equals the loading or the reaction, as the case may be, in magnitude and in direction.
4. The shear force at a section to the left of the right-hand end is obtained by considering a free-body diagram of the length of the beam up to that section. In general, the *SF* equals the sum of the loads and reactions starting from the right up to that section; positive, if the sum is positive and negative, if the sum is negative. Loads and reactions are referred positive upwards, i.e., along the positive y direction.
5. Determine the *SF* at a number of salient points, i.e., where the *SF* changes in magnitude or sign and where the *SF* is an extremum, i.e., maximum or minimum.
6. Plot the *SF* diagram to a suitable scale, preferably under the loading diagram with the same scale along its length.

Consider, for example, the beam shown in Fig. S2.2. From the loading diagram, the reactions, R_a and R_b are determined from the relations governing the equilibrium of the beam

$$R_a + R_b = P$$

and $$R_b \times ab = P \times ap \qquad \text{(for moments about } a\text{)}$$

or $$R_a \times ba = P \times bp \qquad \text{(for moments about } b\text{)}$$

Starting from the right-hand end, the shear force equals the reaction itself; it is positive because the reaction is positive, i.e., upwards. Since there is no other force between b and p, the shear force remains same up to p. At p, the load p acts downwards bringing the shear force to

$$F = R_b - P = -R_a$$

which is now equal to the reaction at a in magnitude but negative in sign. From p to a the shear force remains the same, i.e., $-R_a$. The *SF* diagram has consequently been drawn under the beam.

S2.3 BENDING MOMENT AND BM DIAGRAM

The bending moment in a beam at any section is the transverse moment tending to cause bending of the beam in the plane of loading.

Consider a simple beam supported at its ends and subjected to an external concentrated load P at a given point p as shown in Fig. S2.2. The bending moment at a cross-section SS is the transverse moment tending to bend the beam at this cross-section. In order to determine the bending moment M, consider the free-body diagram of a part of the beam on either side of the section SS as shown in Fig. S2.3.

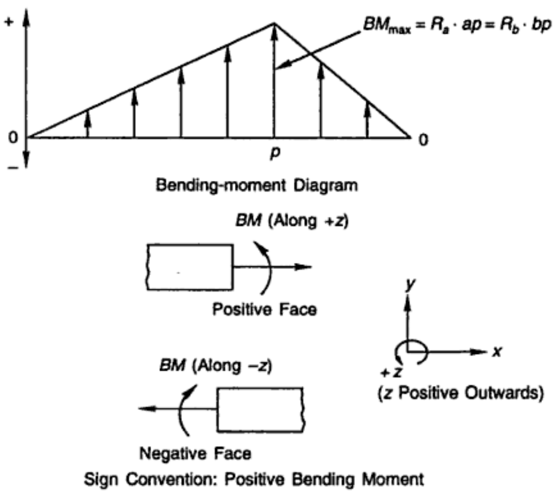


Fig. S2.3 ***Bending Moment: Definition, Sign and Diagram***

The moment of the forces of either part of the beam about the section results in the bending moment. The magnitude of the bending moment is therefore, given by

$$(P \times ps - R_a \times as) \qquad \text{or} \qquad R_b \times sb$$

considering the moments acting on part A or part B due to the other part of the beam. These two expressions imply the same value because the algebraic sum of the moments about any point must be zero for equilibrium of the whole beam. The sign of the bending moment should be determined with reference to the sign convention:

Sign Convention

The sign convention for bending moment is *A positive bending moment is that which acts in a positive direction (shown counterclockwise) on a positive face or in a negative direction on a negative face.*

The Bending Moment Diagram

The bending moment diagram generally referred to as *BM* diagram is a graph showing the variation of bending moment along a beam. If the positive bending moment is shown above an arbitrary reference line *OO* and the negative bending moment below it, the line joining the extremeties of the bending moments at different points is the *BM* diagram. The *BM* curve over a given length may remain constant, vary linearly or non-linearly depending upon the loading condition. In general, for no shear force between two sections, the *BM* remains constant; for constant shear force, the *BM* varies linearly; for varying shear force, the *BM* varies non-linearly. Further implication of this statement will be clear when we relate the *BM*, *SF* and the loading diagrams.

Drawing of a *BM* diagram is an important task because the *BM* diagram provides a picture of the moments tending to bend the beam at all points along the length of the beam. The procedure to plot the *BM* diagram is as follows:

1. Draw the symbolic loading diagram.
2. Determine the reactions at the supports.
3. Start from the right-hand end of the beam for convenience. The bending moment at the end equals the applied moment or the reaction moment, as the case may be, in magnitude and direction.
4. The bending moment at a section to the left of the right-hand end is obtained by the summations of the moments due to the reactions and other forces acting on the right-hand side only. This bending moment acts on the other part of the beam with a positive face and hence provides the bending moment with proper sign in accordance with the sign convention.
5. Determine the *BM* at a number of salient points, i.e., where the *BM* changes in magnitude or sign and where the *BM* is an extremum, i.e., maximum or minimum.
6. Plot the *BM* diagram to a suitable scale preferably under the loading diagram and the *SF* diagram with the same scale along the length.

For the simple beam shown in Fig. S2.2, the bending moment at the right-hand end is zero. At a section *SS*, *sb* from the right end,

$$M_s = R_b \times sb$$

At a section at point *p*,

$$M_p = R_b \times pb$$

At a section to the left of point *p* the bending moment can be obtained correctly in magnitude and direction by the summation of the moments of the forces to the right of it as outlined. However, it may sometimes be more convenient to determine the summation of the moments from the left-hand side. In that case a negative sign is necessary because that moment would act on a negative face of the right-hand part of the beam. It is therefore possible to draw the *SF* and *BM* diagram starting from either end. It may also be added that the *SF* and *BM* diagrams need not necessarily be drawn to scale. The salient values should be mentioned in the diagrams.

S2.4 RELATIONSHIP BETWEEN BENDING MOMENT, SHEAR FORCE AND LOADING

The bending moment, shear force and loading at any section of a thin beam subjected to distributed loading are mutually related.

Consider a continuously-loaded thin beam with variable loading w per unit length.

Clearly, $$w = w(x) \tag{S2.2}$$

The loading has been shown positive upwards, i.e., along the positive y-direction in Fig. S2.4.

In order to derive the desired relationship, consider the equilibrium of a differential element Δx of the beam as shown in an enlarged sketch. The differential

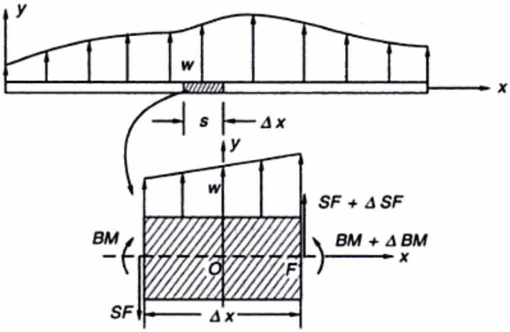


Fig. S2.4 ***A Section of a Beam***

element of the beam has a positive face on the right and a negative face on the left. Consequently, the directions of positive shear force and bending moment on these faces are mutually opposed as also shown in the same sketch. It is also conventional to assume increments in the shear force and in the bending moment in the positive x-direction over the element of the beam. The shear force changes from *SF* to $SF + \Delta SF$ and the bending moment from *BM* to $BM + \Delta BM$ over the length dx. For equilibrium,

1. The sum of the forces in the y-direction must be zero, i.e.,

$$SF + \Delta(SF) - SF + w\Delta x = 0$$

whence $$\frac{\Delta(SF)}{\Delta x} = -w$$

which, in the limit $\Delta x \to 0$, becomes

$$\boxed{\frac{d(SF)}{dx} = -w} \tag{S2.3}$$

2. The sum of the moments in the plane of the bending, i.e., about the z-axis must be zero. Taking the moments about the point F on the positive face,

$$BM + \Delta BM - BM + SF\Delta x - w\Delta x\Delta x/2 = 0$$

whence
$$\frac{\Delta(BM)}{\Delta x} = -SF + \frac{w\Delta x}{2}$$

which, in the limit $\Delta x \to 0$ becomes

$$\boxed{\frac{d(BM)}{dx} = -SF} \tag{S2.4}$$

From the two conditions of equilibrium, it follows that for a distributed-loaded thin beam,

1. The rate of change of shear force along the length of the beam equals the loading with a negative sign.
2. The rate of change of bending moment along the length of the beam equals the shear force with a negative sign.
3. By combining Eqs. (S2.3) and (S2.4),

$$\frac{d^2(BM)}{dx^2} = w \tag{S2.5}$$

which shows that the second longitudinal derivative of the bending moment equals the loading at that cross-section.

It is interesting to observe the implications of Eqs. (S2.3) and (S2.4) in sketching the *SF* and *BM* diagrams.

From Eq. (S2.3), by integration,

$$SF_1 - SF_2 = \int_{x_1}^{x_2} w\,dx \tag{S2.6}$$

It shows that the difference of shear force between two points along the length of a beam equals the integral or the summation of the vertical forces over the length which is also equal to the area under the loading diagram. If there is no loading between two points, there can be no change in the shear force.

In order to start the construction of an *SF* diagram, it is more convenient to write the integral equation (S2.6) as

$$SF = -\int_0^x w\,dx + SF_0 \tag{S2.7}$$

where SF_0 is the shear force at the origin O selected at the end of a beam. SF_0

equals the loading at the end or the reaction force on the beam if it is supported at the end. It can also be observed from Eq. (S2.7) that

$$\frac{d(SF)}{dx} = -w$$

and

$$\frac{d}{dx}\left(\frac{d(SF)}{dx}\right) = -\frac{dw}{dx}$$

which imply that

1. The slope of the *SF* diagram equals the negative of the loading at that section. If the loading is constant, the slope of the *SF* diagram must be constant, i.e., the shear force varies linearly. If the loading is zero, the shear force remains constant. If the loading is variable, the shear force varies non-linearly.
2. The rate of change of the slope of the *SF* diagram equals the negative of the rate of change of the loading. It shows that the curve of the *SF* diagram is one degree higher than the curve of the loading diagram.

Table S2.1 Relationship between Loading and SF Diagram

Type of Loading	*Shape of the Loading Diagram*	*Shape of the SF Diagram*
1. Zero	Zero line	Rectangular
2. Uniform	Rectangular	Triangular
3. Uniformly varying	Triangular	Parabolic
4. Linearly varying	Parabolic	Cubic

3. Although the case of concentrated loads is not covered by these relationships, it is possible to use these relations in the presence of concentrated loads and reactions provided these are applied *between* the positions of the concentrated loads and reactions and *not at* the positions of the concentrated loads and reactions.

From Eq. (S2.4), by integration,

$$BM_1 - BM_2 = \int_{x_1}^{x_2} SF\, dx \tag{S2.8}$$

It shows that the difference of the bending moments between two points along the length of a beam equals the integral of the shear forces over the length which is also equal to the area under the *SF* diagram. If there is no shear force over a length, there can be no change in the bending moment.

It is perhaps more convenient to write the integral equation as

$$BM = -\int_0^x SF\, dx + BM_0 \tag{S2.9}$$

where BM_0 is the bending moment at the origin *O* selected at the end of the beam.

BM_0 equals the moment at the end or the reaction moment on the beam if it is supported at the end.

It can also be observed from Eq. (S2.9) that

$$\frac{d(BM)}{dx} = -SF$$

$$\boxed{\frac{d}{dx}\left(\frac{d(BM)}{dx}\right) = -\frac{d(SF)}{dx} = w}$$

which imply that

1. The slope of the *BM* diagram equals the negative of the shear force at that section. If the shear force is constant, the slope of the *BM* diagram must be constant, i.e., the bending moment varies linearly. If the shear force is zero, the bending moment remains constant. If the shear force is variable, the bending moment varies non-linearly.
2. The rate of change of the slope of the *BM* diagram equals the negative of the rate of change of the shear force. It shows that the curve of the *BM* diagram is one degree higher than the curve of the *SF* diagram.

Table S2.2 Relationship between SF and BM Diagrams

Type of Loading	*Shape of the SF Diagram*	*Shape of the BM Diagram*
Zero	Rectangular	Triangular
Uniform	Triangular	Parabolic
Uniformly varying	Parabolic	Cubic
Parabolic	Cubic	Quartic

3. Although the step variations in the shear force due to concentrated loads or otherwise are not covered by these relations, it is possible to use these relations in the presence of such step variations provided these are applied *between* such points and *not at* such points.

***Example* S2.1** Draw the *SF* and *BM* diagrams for the beam loaded as shown in Fig. Ex. S2.1. Also locate the points of contraflexure.

Solution From the conditions of equilibrium of the beam

$$\Sigma F = 0; -10 + R_1 - 4 \times 2 - 10 + R_2 - 2 \times 2 = 0$$

$$R_1 + R_2 = 32$$

$$\Sigma \mathbf{M}_B = 0; \; -10 \times 1 - 8 \times 1 - 10 \times 3 + R_2 \times 4 - 4 \times 5 = 0$$

$$R_2 = 12 \text{ kN} \qquad \text{and} \qquad R_1 = 32 - 12 = 20 \text{ kN}$$

The shear-force distribution in kN is calculated as follows:

$$0 < x < 1 \quad SF - 10 = 0; SF = 10$$

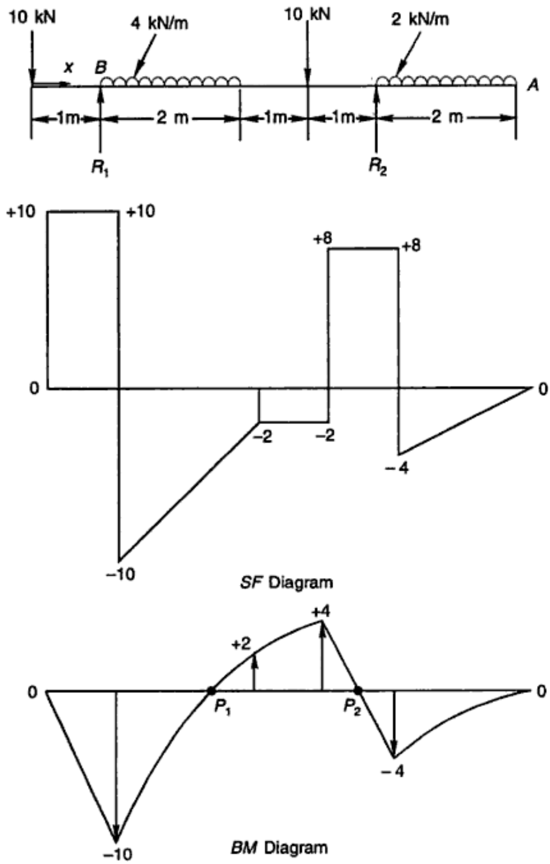


Fig. Ex. S2.1

At $x = 1$ SF changes from +10 to – 10

$1 < x < 3$ $SF - 10 + 20 - 4\,(x - 1) = 0$; $SF = -14 + 4x$

At $x = 3$ $SF = -2$

$3 < x < 4$ $SF = -2$

At $x = 4$ SF changes from –2 to + 8.

$4 < x < 5$ $SF - 10 + 20 - 8 - 10 = 0$; $SF = 8$

At $x = 5$ SF changes from +8 to – 4

$$5 < x < 7 \quad SF - 10 + 20 - 8 - 10 + 12 - 2(x - 5) = 0; SF = 2x - 14$$

$$\text{At } x = 7 \quad SF = 0, \text{ as is expected at the end } A.$$

The bending-moment distribution in kN m is estimated as follows:

$$0 < x < 1 \quad BM + 10x = 0; BM = -10x$$

$$\text{At } x = 1 \quad BM = -10$$

$$1 < x < 3 \quad BM + 10x - 20(x - 1) + 4(x - 1)^2/2 = 0$$

$$BM = -2(x - 1)^2 + 10x - 20$$

$$\text{At } x = 1 \quad BM = -10; \text{ at } x = 2 \; BM = -2$$

$$\text{At } x = 3 \quad BM = 2$$

$$3 < x < 4 \quad BM + 10x - 20(x - 1) + 8(x - 2) = 0$$

$$BM = 2x - 4$$

$$\text{At } x = 3 \quad BM = 2; \text{ at } x = 4 \; BM = 4$$

$$4 < x < 5 \quad BM + 10x - 20(x - 1) + 8(x - 2) + 10(x - 4) = 0$$

$$BM = -8x + 36$$

$$\text{At } x = 4 \quad BM = 4; \text{ at } x = 5 \; BM = -4$$

$$5 < x < 7 \quad BM + 10x - 20(x - 1) + 8(x - 2) + 10(x - 4) - 12(x - 5) + 2(x - 5)^2/2 = 0$$

$$BM = -(x - 5)^2 + 4x - 24$$

$$\text{At } x = 5 \quad BM = -4; \text{ at } x = 6 \; BM = -1$$

$$\text{At } x = 7 \quad BM = 0, \text{ as expected at the end } F.$$

The points of contraflexure can be located by observing the change of sign of *BM*

For P_1, $-2(x - 1)^2 + 10x - 20 = 0$ $\quad x = 2.38$ m

For P_2, $-8x + 36 = 0$ $\quad x = 4.50$ m

***Example* S2.2** A beam carrying a uniformly-distributed load rests on two supports b m apart with an equal overhang of a m at each end. Determine the ratio b/a for the maximum bending moment to be as small as possible. Use this result to determine the most economical length for a railway sleeper if the rail centres are 1.6 m apart. and also for the metre-gauge rails.

Solution Consider the variation of the bending moment along the length of the beam

For $\quad 0 < x < a \quad BM + wx \cdot x/2 = 0; BM = -wx^2/2$

For $\quad x = a \quad BM = -wa^2/2$

The reactions are symmetrically located; these are each

$$w(2a + b)/2$$

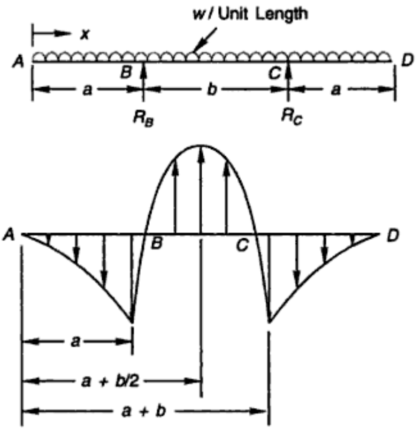


Fig. Ex. S2.2 *(Solution)*

For $a < x < (a + b)$ $BM + wx \cdot x/2 - w(2a + b)/2 \cdot (x - a) = 0$

$$BM = -wx^2/2 + wax - wa^2 + wbx/2 - wab/2$$

As is clear from the nature of the *BM* diagram (Fig. Ex. S2.2 (Solution)) the bending moment can be maximum either at $x = a$ and $x = a + b$ or at $x = a + b/2$. The maximum bending is the least when the numerical values at these places are equal:

$$wa^2/2 = -w(a + b/2)^2/2 + wa(a + b/2) - wa^2$$
$$+ wb(a + b/2)/2 - w\,ab/2$$

whence, cancelling w and simplifying,

$$a = b/\sqrt{8}$$

and $$a/b = 1/\sqrt{8}$$

It is clear that, for $a > b/\sqrt{8}$ the *BM* at $x = a$, i.e., at the supports is maximum and for $a < b/\sqrt{8}$ the *BM* at $x = a + b/2$, i.e., at the centre is maximum.

The railway sleepers rest uniformly on the ground and are subjected to two equal point loads as a first approximation. For 1.6 m gauge rails,

$$b = 1.60;\ a = 1.6/\sqrt{8} = 0.57 \text{ m}$$

The most economical length of the railway sleeper is, therefore,

$$l = 0.57 + 1.60 + 0.57 = 2.74 \text{ m}$$

For the metre gauge rails,

$$b = 1.00;\ a = 1/\sqrt{8} = 0.36 \text{ m}$$

and the most economical length of the sleeper is

$$l = 0.36 + 1 + 0.36 = 1.72 \text{ m}$$

***Example* S2.3** Figure Ex. S2.3 shows the *SF* diagram for a beam supported at two points and loaded in some manner. Determine the position of the supports and details of loading over the entire length. Also, determine the positions of maximum positive and negative bending moments and the points of contraflexure.

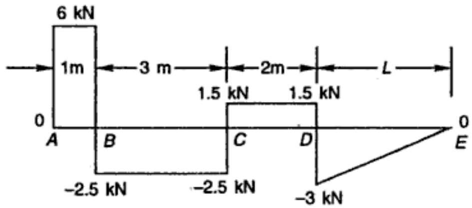


Fig. Ex. S2.3

Solution Conclusions in respect of the relationship between the loading, shear force and *BM* diagrams can be drawn from the fact that

$$\frac{d(SF)}{dx} = -w \tag{i}$$

and

$$\frac{d(BM)}{dx} = -SF \tag{ii}$$

Between *A* and *B*, the shear force is constant; the loading must be zero. At *A* and at *B*, the shear force changes abruptly indicating the presence of concentrated loads at *A* and *B*. Positive shear force at *A* and from *A* to *B* indicates negative, i.e., downward load of 6 kN at *A*. A change of sign of *SF* at *B* requires an upward concentrated force of 8.5 kN at *B*. Between *B* and *C*, there is no loading. Again, at *C*, there must be a downward load of 4 kN raising the *SF* to 1.5 kN. Again, no loading between *C* and *D* and an upward force of 4.5 kN at *D* is observed from the *SF* diagram. From *D* to *E* the *SF* increases linearly showing that the loading must be uniform and downward. This load is, by difference, 3 kN.

The upward forces at *B* and *D* are the reactions from the supports as shown in the loading diagram Fig. Ex. S2.3 (Solution).

The bending moment distribution in kN m is calculated as follows:

For $0 < x < 1$, $BM + 6x = 0$

$$BM = -6x$$

At $x = 0$, $BM = 0$,

At $x = 1$, $BM = -6$

For $1 < x < 4$, $BM + 6x - 8.5(x - 1) = 0$

$$BM = 2.5x - 8.5$$

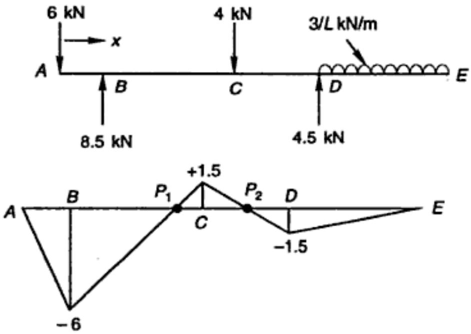


Fig. Ex. S2.3 ***(Solution)***

At	$x = 1,$	$BM = -6,$
At	$x = 4,$	$BM = 1.5$
For	$4 < x < 6,$	$BM + 6x - 8.5(x - 1) + 4(x - 4) = 0$
		$BM = -1.5x + 7.5$
At	$x = 4,$	$BM = +1.5$
At	$x = 6,$	$BM = -1.5$
For	$6 < x < (6 + L),$	$BM + 6x - 8.5(x - 1) + 4(x - 4) - 4.5(x - 6) + 3(x - 6)^2/2L = 0$
		$BM = +3x - 19.5 + 3(x - 6)^2/2L$
At	$x = 6,$	$BM = -1.5$
At	$x = 6 + L$	$BM = 4.5L - 1.5$

Since the *BM* at *E* should be zero,

$$4.5L - 1.5 = 0;\ L = 0.33 \text{ m}$$

The total length of the beam is, therefore, 6.33 m.

The maximum positive bending moment occurs at *C* and its value is 1.5 kN m.

The maximum negative bending moment occurs at *B* and its value is – 6 kN m.

Points of contraflexure, P_1 and P_2, where the bending moment changes sign, can be located as follows:

$$BP_1/6 = (3 - BP_1)/1.5;\ BP_1 = 2.4$$

$$CP_2/1.5 = 2 - CP_2/1.5;\ CP_2 = 1$$

The points of contraflexure are, therefore, at distances of 3.4 m and 5 m from the left end *A* of the beam.

***Example* S2.4** A log of wood, specific gravity 0.78, 3 m long and 25 cm × 25 cm in cross-section floats in water. Determine the load that should be placed centrally on the log so that the log is just completely immersed in water. Draw the *SF* and *BM* diagrams of the log.

Solution The log of wood is subjected to an upward hydrostatic force from below, as shown in Fig. Ex. S2.4 (Solution) the hydrostatic pressure being

$$p = \rho gh = 1000 \times 9.81 \times 0.25 = 2452.5 \text{ N/m}^2$$

which comes to an upward uniform loading

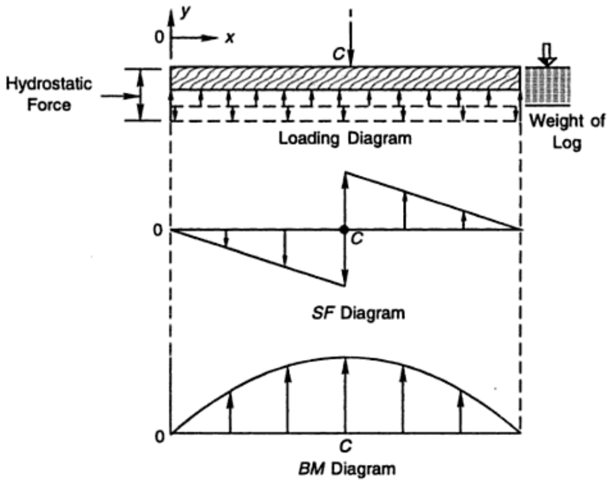


Fig. Ex. S2.4 ***(Solution)***

$$w_1 = 2452.5 \times 0.25 = 613 \text{ N/m}$$

The weight of the log appears as downward uniform loading

$$w_2 = 0.78 \times 1000 \times 9.81 \times 0.25 \times 0.25 = 478 \text{ N/m}$$

The effective uniform loading is

$$w = w_1 - w_2 = 613 - 478 = 135 \text{ N/m upward}$$

The centrally placed weight should, therefore, be

$$W = 135 \times 3 = 405 \text{ N}$$

in order to keep the log in equilibrium.

The shear-force distribution is calculated thus:

For $0 < x < 1.5$, $SF + 135x = 0$; $SF = -135x$

At $x = 0$, $SF = 0$

At	$x = 1.5$,	*SF* changes from – 202.5 N to + 202.5 N
For	$1.5 < x < 3$,	$SF + 135x - 405 = 0$; $SF = 405 - 135x$
At	$x = 1.5$,	$SF = 202.5$ N
At	$x = 3$,	$SF = 0$, as expected at the free end

The *BM* diagram can be plotted from the following:

For	$0 < x < 1.5$,	$BM - 135x.\, x/2 = 0$; $BM = 67.5x^2$
At	$x = 0$,	$BM = 0$; at $x = 1.5$ $BM = 152$ N m
For	$1.5 < x < 3$,	$BM - 135x.\, x/2 + 405(x - 1.5) = 0$
		$BM = 67.5x^2 - 405(x - 1.5)$
At	$x = 1.5$,	$BM = 152$ N m
At	$x = 3$,	$BM = 0$, as expected at the free end

Example **S2.5** Draw the *SF* and *BM* diagrams of the beam shown in Fig. Ex. S2.5.

Solution From the equilibrium of the beam,

$$\Sigma F = 0;\ R_y - (1/2 + 1)/2 \times 5 - 1 = 0;\ R_y = 4.75 \text{ kN}$$

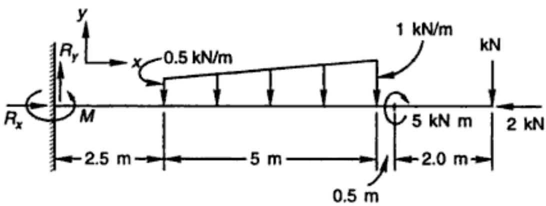


Fig. Ex. S2.5

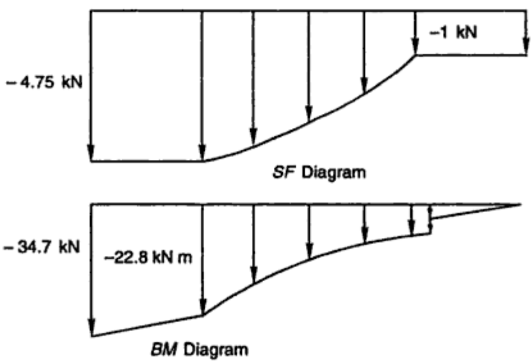


Fig. Ex. S2.5 ***(Solution)***

$$R_x - 2 = 0; R_x = 2 \text{ kN}$$

$$\Sigma \mathbf{M}_0 = 0; M - \int_{2.9}^{7.5} wx\,dx - 5 - 1 \times 10 = 0$$

where the loading w is determined as a function of x as follows:

$$w = ax + b$$

$$\left.\begin{array}{l} 0.5 = 2.5a + b \\ 1.0 = 7.5a + b \end{array}\right] a = 0.1, b = 0.25$$

or $$w = 0.1x + 0.25$$

and $$M = 34.7 \text{ kN m}$$

Consideration of the shear force at different sections shows that

For $0 < x < 2.5$, $SF + 4.75 = 0; SF = -4.75$ kN

For $2.5 < x < 7.5$, $SF + 4.75 - \int_{2.5}^{x} (0.25 + 0.1x)\,dx = 0$

$$SF = 0.05x^2 + 0.25x - 5.67$$

At $x = 2.5$, $SF = -4.75$ kN

At $x = 5.0$, $SF = -3.17$ kN

At $x = 7.5$, $SF = -1.00$ kN

For $7.5 < x < 10$, $SF + 4.75 - \int_{2.5}^{7.5} (0.25 + 0.1x)\,dx = 0$

$$SF = -1.00 \text{ kN}$$

The bending moment at different sections is determined as follows:

For $0 < x < 2.5$, $BM - 4.75x + 34.7 = 0; BM = 4.75x - 34.7$

At $x = 0$, $BM = -34.7$ kN m

At $x = 2.5$, $BM = -22.8$ kN m

For $2.5 < x < 7.5$, $BM = 4.75x + 34.7 + (0.25 + 0.1)(x - 2.5)$

$$BM = 4.75x - 34.7 - 0.125\,(x - 2.5)^2$$
$$0.05x(x - 2.5)^2 + 0.033\,(x - 2.5)^3$$

At $x = 2.5$, $BM = -22.8$ kN m

At $x = 5.0$, $BM = -11.77$ kN m

At $x = 7.5$, $BM = -7.5$ kN m

For	$7.5 < x < 8$,	*BM* varies linearly from –7.5 kN m to –7 kN m
At	$x = 8$,	*BM* changes abruptly from –7 kM m to –2 kN m
For	$8 < x < 10$,	*BM* varies linearly from – 2 kN m to 0.

***Example* S2.6** Compute the shear force, axial force and bending moment over the length of the bent beam shown in Fig. Ex. S2.6.

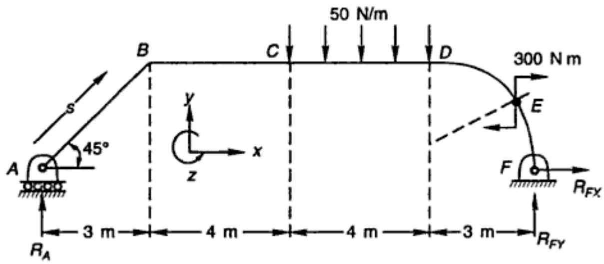


Fig. Ex. S2.6

Solution From the equilibrium of the beam,

$$\Sigma \mathbf{M}_F = 0; -300 - 14\, R_A + 50 \times 4 \times 5 = 0;\ R_A = 50 \text{ N}$$

$$\Sigma \mathbf{F} = 0; -50 \times 4 + 50 + R_{FY} = 0;\ R_{FY} = 150 \text{ N};\ R_{FX} = 0$$

Consider a section at a distance *s* from *A* along *AB*. From the free-body diagram of the beam segment (Fig. Ex. S2.6 (Solution)).

$$AF \sin 45° + SF \cos 45° + 50 = 0$$

$$AF \cos 45° - SF \sin 45° = 0$$

whence,

$$AF = SF = -70.7 \text{ N}$$

$$BM = 50\, s \cos 45°$$

$$= 35.35\, s$$

At $s = 4.24$, $BM = 150$ N m

Consider now a section between *B* and *C*. From the free-body diagram,

$$SF + 50 = 0;\ SF = -50 \text{ N and } AF = 0$$

$$BM - 50(s - 4.24 + 3) = 0$$

or

$$BM = 50s - 62.0$$

At $s = 4.24$, $BM = 150$ N m; at $s = 8.24$ $BM = 350$ N m

From the free body of a section between *C* and *D* of the beam,

$$SF + 50 - 50(s - 4.24 - 4) = 0;\ SF = 50s - 462$$

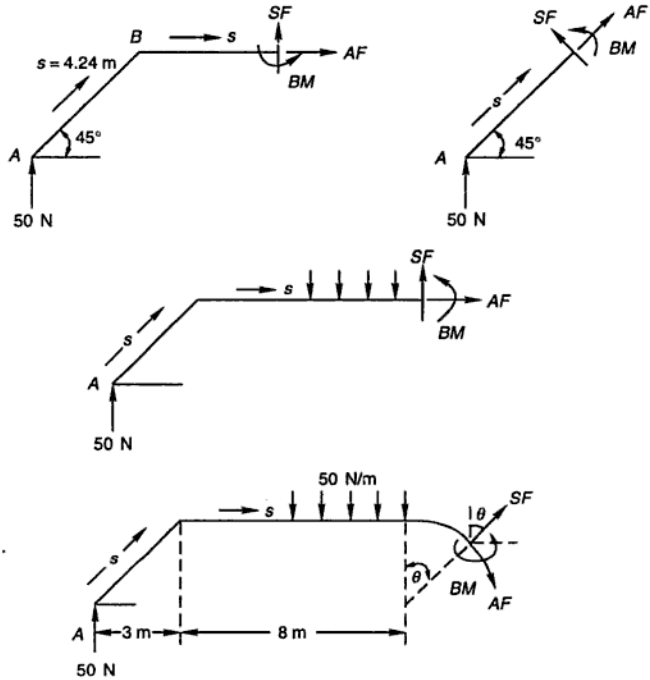


Fig. Ex. S2.6 ***(Solution)***

$$AF = 0$$

At $s = 12.24$, $SF = 150$ N

$$BM - 50\ (s - 4.24 + 3) + 50(s - 4.24 - 4)^2/2 = 0$$

$$BM = 50s - 62 - 25(s - 8.24)^2$$

At $s = 8.24$, $BM = 350$ N m

At $s = 12.24$, $BM = 150$ N m

For a section between *D* and *E*, from the free-body diagram of the part on the left,

$$SF \cos\theta - AF \sin\theta + 50 - 200 = 0$$

$$SF \sin\theta + AF \cos\theta = 0$$

whence $SF = 150 \cos\theta$ and $AF = -150 \sin\theta$

$$BM - 50(s - 4.24 + 3 + 3 \sin\theta) + 200(2 + 3 \sin\theta) = 0$$

$$BM = 50s - 450 \sin\theta - 462$$

At $s = 12.24$, $BM = 150$ N m and $\theta = 0$

At $s = 12.24$, $BM = -168$ N m and $\theta = \pi/4$

At E, the BM changes abruptly from –168 N m to 132 N m and varies linearly down to zero between E and F. The shear force and the axial force are given by

$$SF = 150 \cos \theta;\ AF = -150 \sin \theta$$

until F where these values drop down to zero.

***Example* S2.7** An overhanging beam AB 20 m long simply supported at A and D carries a uniformly-distributed load and two concentrated loads as shown in Fig. Ex. S2.7. Determine the location of the supports placed 12 m apart sharing the load equally. Draw the BM diagram and locate the points of zero shear force in the beam.

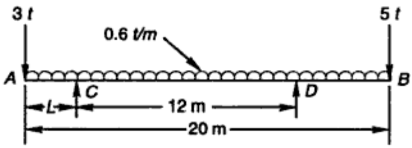


Fig. Ex. S2.7

Solution From the equilibrium of the beam,

$$\Sigma \mathbf{F} = 0;\ R_C + R_D - 3 - 0.6 \times 20 - 5 = 0$$

$$R_C + R_D = 20$$

and using the condition $R_C = R_D$,

$$R_C = 10 = R_D$$

$$\Sigma \mathbf{M}_A = 0;\ 10L + 10(L + 12) - 5 \times 20 - 20 \times 0.6 \times 10 = 0$$

whence $L = 5$ m, the distance of C from A.

For evaluating the bending moment in tonne metres along the beam, taking the origin at A and proceeding for the equilibrium of different sections:

$0 < x < 5$ $\quad BM + 3x + 0.6x^2/2 = 0;\quad BM = -3x - 0.3x^2$

At $x = 0$, $BM = 0$, as expected at the end

At $x = 2.5$, $BM = -9.375$

At $x = 5$, $BM = -22.5$

$5 < x < 17$ $\quad BM + 3x + 0.6x^2/2 - 10\,(x - 5) = 0$

$$BM = 7x - 0.3x^2 - 50$$

At $x = 5$, $BM = -22.5$, providing a check

At $x = 11$, $BM = -9.3$

At $x = 17$, $BM = -17.7$

$$17 < x < 20 \quad BM + 3x + 0.6x^2/2 - 10\,(x - 5) - 10(x - 17) = 0$$

$$BM = +17x - 0.3x^2 - 220$$

At $x = 17$, $BM = -17.7$, providing a check

At $x = 20$, $BM = 0$, as expected at the end

The *BM* diagram is shown in Fig. Ex. S2.7 (Solution). There is no point of contraflexure in the beam because the bending moment does not change sign at any point.

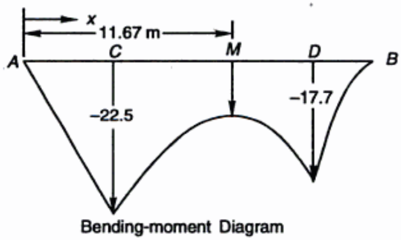


Fig. Ex. S2.7 ***(Solution)***

The point of zero shear in the beam corresponds to the location of the minimum bending moment:

$$\frac{d}{dx}\,(7x - 0.3x^2 - 50) = 0$$

$$7 - 0.6x = 0$$

or at $x = 11.67$ m; at point M as shown.

It may also be noted that the shear force changes sign, i.e., passes through zero at points C and D but the value at these points is not said to be zero.

***Example* S2.8** A flat plate (4 m × 1 m) (Fig. Ex. S2.8) hinged at the top 1 m edge serves as a wide wall of a tank containing oil of specific gravity 0.9 to a depth of 3 m above the lower edge of the plate. Determine the horizontal force required at the lower edge of the plate to keep it in equilibrium. Draw the *SF* and *BM* diagrams of the plate.

Solution The pressure at a depth h in oil is given by

$$p = \rho\, gh$$

$$= 900 \times 9.81 \times h = 8829h \text{ N/m}^2$$

$$= 8.83h \text{ kN/m}^2$$

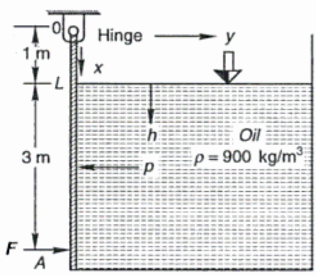


Fig. Ex. S2.8

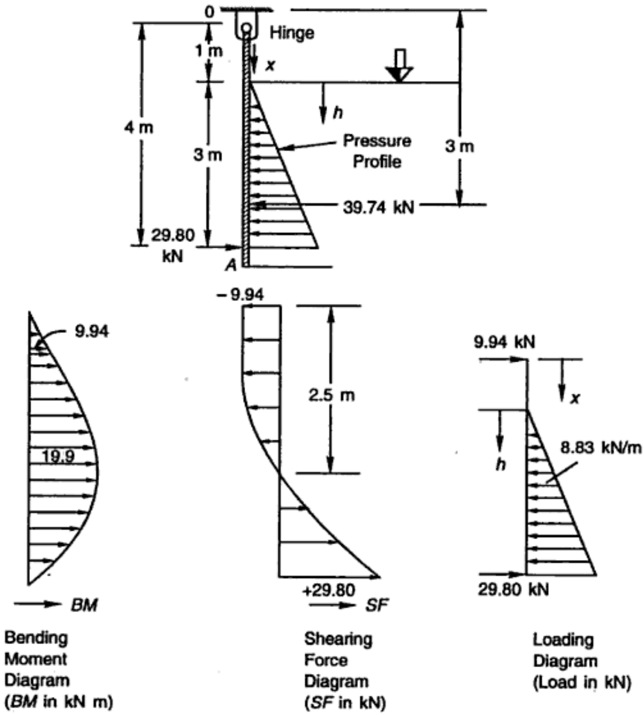


Fig. Ex. S2.8 ***(Solution)***

Total hydrostatic force acting on the plate is

$$P = \int p\,dA = \int_0^3 8.83h \times (1 \times dh)$$

$$= \left[8.83\,h^2/2\right]_0^3 = 8.83 \times 9/2$$

$$= 39.74 \text{ kN}$$

which acts at 1 m from the base of the plate, i.e., at 3 m below the hinge.

For equilibrium of the plate

$$\Sigma \mathbf{M}_0 = 0;$$

$$4F = 39.74 \times 3$$

whence $F = 29.80$ kN

The reaction at the hinge 0 must be (39.74 − 29.80) = 9.94 kN towards right in the horizontal direction.

The shearing force is estimated as follows:

For $0 < x < 1$, $SF = -9.94$ kN

For $1 < x < 4$,

or $0 < h < 3$, $SF = -9.94 + \int_0^h 8.83h. (1 \times dh)$

$= -9.94 + 4.41h^2$ kN

which is zero at $h = \sqrt{\frac{9.94}{4.41}} = 1.50$ m

or at $x = 1.50 + 1 = 2.5$ m

and maximum being $-9.94 + 4.41 \times 3^2$ or 29.80 kN

At $h = 3$ m or $x = 4$ m

which was expected because the reaction of 29.80 kN would bring it to zero at the end.

The bending moment is calculated as follows:

For $0 < x < 1$, $BM = 9.94x$ kN m

At $x = 1$, $BM = 9.94$ kN m

For $1 < x < 4$,

or $1 < h < 3$, $BM = 9.94x - 4.41\ h^2 \cdot h/3$

$= 9.94(h + 1) - 1.47h^3$

$= 9.94 + 9 + 94h - 1.47h^3$ kN m

It may be seen that *BM* is maximum at $h = 1.5$ m.

Maximum $BM = 9.94 + 9.94 \times 1.5 - 1.47 \times 1.5^3 = 19.9$ kN m

It should also be appreciated that, at $h = 3$ m, the *BM* reduces to

$$9.94 + 9.94 \times 3 - 1.47 \times 3^3 = 0$$

as is expected at the lower end of the plate.

The *SF* and *BM* diagrams are consequently as shown in Fig. Ex. S2.8 (Solution).

Experiment E5

Shear Force and Bending Moment in a Beam

Objective

To determine the shear force and bending moment at a cross section of a beam and to compare the same with the corresponding theoretical values.

APPARATUS

A simply supported level beam with provision for loading at the desired points and spring balances provided to enable the measurement of shear force and bending moment at a cross section.

BACKGROUND INFORMATION

The shear force at a cross section X of a beam is defined as the transverse force tending to cause shear across the section. The bending moment in a beam at any cross-section X is the transverse moment tending to cause bending of the beam in the plane of the loading.

Experimental determination of the shear force and bending moment at a cross-section is made by improvising a beam in two parts and by measuring the reaction of one part on the other.

Initially, the unloaded beam is positioned to be horizontal by adjusting the wing nuts provided on the spring and the initial readings on the spring are noted. The beam is then loaded as desired. This may be achieved by suspending masses m_1, m_2 and m_3 at three places at distances of x_1, x_2 and x_3 from the left end A as shown in Fig. E5.1. The loads acting at these locations are, therefore, m_1g, m_2g and m_3g. The two parts of the beam tend to get disturbed from the horizontal position on the application of the loads. The wing nuts are turned suitably to bring them in the horizontal position.

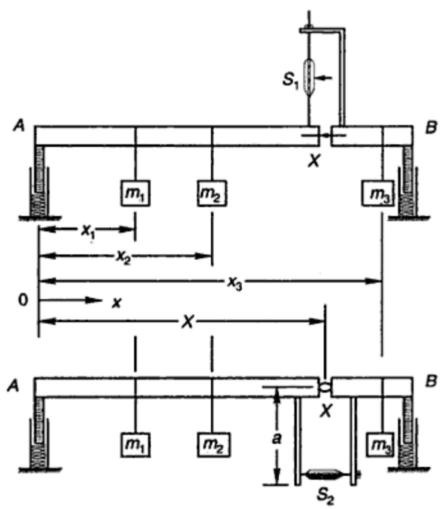


Fig. E5.1 ***Measurement of Shear Force and Bending Moment***

The magnitude of the shear force at the cross section equals the corrected reading on the spring balance S_1

$$SF = S_1 \tag{E5.1}$$

The theoretical value may be obtained by computing the reactions R_A and R_B and then by noting that, for the equilibrium of the left-hand part,

$$R_A - m_1 g - m_2 g + SF = 0$$

whence
$$SF = m_1 g + m_2 g - R_A \tag{E5.2}$$

The magnitude of the bending moment at the cross-section equals the moment exerted by the force observed as the corrected reading S_2 on the spring balance placed at an arm of length a, i.e.,

$$BM = S_2 \cdot a \tag{E5.3}$$

The theoretical value may be obtained by observing that, for the equilibrium of the left-hand part,

$$-R_A \cdot X + m_1 g \cdot (X - x_1) + m_2 g \cdot (X - x_2) + BM = 0$$

whence
$$BM = R_A \cdot X - m_1 g\,(X - x_1) - m_2 g\,(X - x_2) \tag{E5.4}$$

Observations and Calculations

The initial and final readings of the springs for the unloaded and loaded beams respectively are recorded:

A set of observations and calculations for a prescribed loading may be arranged as follows:

Length of the Beam:
Loading on the Beam:
Distances of Loads:
Distance 'a':

	Initial	*Final*	*Difference*
Spring balance A			
Spring balance B			
Spring balance S_1			
Spring balance S_2			

$SF = S_1 =$
$BM = S_2 a =$

Results

The measured values of the shear force and bending moment at the section X may be recorded and compared with the theoretical values.

Points for Discussion

1. Comment on the difference between the experimentally measured and the theoretical values of shear force and bending moment.
2. Compute the percentage error for each system of loading and observe whether the percentage error increases as the value of theoretical shear force and bending moment decrease. If so, why? If not, why not?
3. Have you taken the weight of the beam into account? Explain how?

4. Can the method of measuring the shear force and the bending moment be employed on a cantilever? Explain how with the help of a sketch. Can the method be used on a beam with the fixed end supported and with other end conditions?
5. Is it possible to state from the observations S_1 and S_2 taken from the spring balances whether the shear force and bending moment are positive or negative?

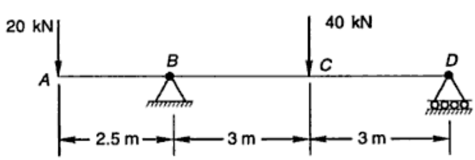
Concept Review Questions

1. (a) What is the difference between a beam and a member of a simple truss?
   (b) What are the implications of a beam referred to as thin and rigid?
2. State the different means of supporting a beam and differentiate between an overhanging beam, a cantilever beam and a continuous beam.
3. (a) Define the terms shear force and bending moment at a cross-section in a beam. How are the *SF* and *BM* diagrams drawn and what useful purpose is achieved by drawing them?
   (b) Comment on the sign conventions for the shear force and bending moment at a section.
   (c) Can a structural member have shear force and bending moment at a cross-section?
4. (a) How are the distributions of the loading, the shear force and bending moment related to each other? Are there any pre-conditions for the relationship?
   (b) Sketch a simply supported beam with some transverse and inclined loading. Draw the *SF* and *BM* diagrams for the same alongside the sketch.
5. What are the implications of sudden changes of
   (a) loading
   (b) shear force
   (c) bending moment
   for a simply supported beam?
6. Is it possible to predict the location and mode of failure of a beam subjected to a given loading? Is it necessary to draw the *SF* and *BM* diagrams before predicting the same?

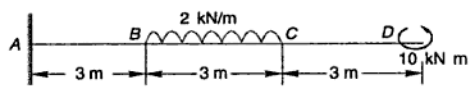
Tutorial Problems

S2.1 Draw the *SF* and *BM* diagrams for the beams and loading shown in Fig. Prob. S2.1(a), (b), (c).

(a)



(b)



(c)

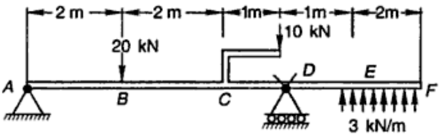


Fig. Prob. S2.1 (a), (b), (c)

S2.2 A beam of length L is loaded by an external moment M as shown in Fig. Prob. S2.2. Draw the *SF* and *BM* diagrams.

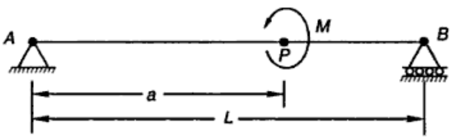


Fig. Prob. S2.2

S2.3 A simply supported overhanging beam 4 m long is uniformly loaded at 2 kN/m over the entire span and carries a triangularly distributed load over the left half of the span as shown in Fig. Prob. S2.3. Draw the *SF* and *BM* diagrams for the beam.

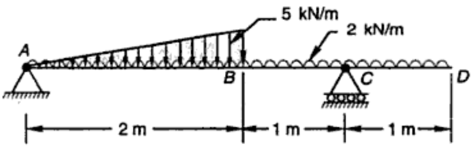


Fig. Prob. S2.3

S2.4 A uniform beam AB of weight W rests horizontally on two supports C and D at one third of the span from either and respectively. There are two loads, each $2W$ at one third the distance between C and D measured from C and D respectively. Draw the *SF* and *BM* diagrams for the beam.

S2.5 A train of weight W and length L is in the centre of a bridge whose span is twice the length of the train as shown in Fig. Prob. S2.5. Assuming that the weight of the train is uniformly distributed throughout its length, calculate the bending moment at the centre of the bridge. Compare it with the value when one end of the train just reaches the pier-end of the bridge.

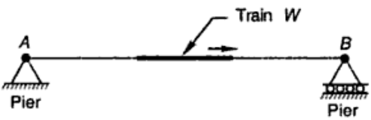


Fig. Prob. S2.5

S2.6 Draw the *SF* and *BM* diagrams for a light horizontal cantilever 4 m long carrying concentrated loads each of 2 kN at its centre and its free end.

S.27 The *SF* diagram for a part of a loaded beam is shown in Fig. Prob. S2.7. All the *SF* values are in kN. Draw the loading diagram and the *BM* diagram for the beam.

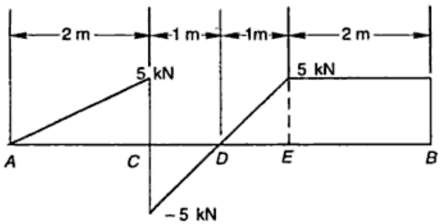


Fig. Prob. S2.7

S2.8 The *BM* diagram for a beam 1 m long is parabolic with a maximum of 20 kN m at the mid-span of a beam as shown in Fig. Prob. S2.8. Sketch the corresponding *SF* and loading diagrams.

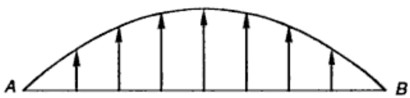


Fig. Prob. S2.8

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions.

1. A thin rigid beam hinged at one end and roller-supported at its mid-points is said to be
   (a) a symmetrical simply supported beam
   (b) an overhanging simply supported beam
   (c) a cantilever beam
   (d) a fixed beam
2. The shear force at a section in a beam is given by
   (a) the external force at that section
   (b) the transverse component of the external force at that section
   (c) the transverse force from the part of the beam on one side of the section to that on the other side of the section
   (d) the addition of the forces at (b) and (c) above
3. The bending moment at a section in a beam is given by
   (a) the external moment at that section
   (b) the summation of all the moments about that section
   (c) the summation of moments of all the forces about the section
   (d) the net moment exerted by the part of the beam on one side of the section to that on the other side of the section
4. The point of contraflexure in a loaded beam is one where
   (a) the bending moment is maximum

(b) the shear force is maximum
(c) the bending moment changes sign
(d) the shear force changes sign

5. In a symmetrically overhung simply supported beam, the maximum bending moment will be the least possible when
   (a) the supports are near the ends
   (b) the supports coincide to become a single support at the centre
   (c) the distance between the supports becomes one-third of the length of the beam.
   (d) the numerical values of the bending moment at either supports and at the centre of the beam are equal
6. The maximum bending moment in a simply supported beam length L loaded by a concentrated load W at the mid point is given by
   (a) WL (b) $WL/2$ (c) $WL/4$ (d) $WL/8$

Answers to the Multiple-Choice Questions

1 (b), 2 (c), 3 (d), 4 (c), 5 (d), 6 (c)

S3 FRICTION

S3.1 FRICTION AND IMPENDING MOTION

Forces of friction come into play when two surfaces in contact with each other exert force normal to each other and one surface slides or tends to slide with respect to the other. The force of friction is also called *frictional resistance* or simply *friction*. The mechanism of friction can be explained by the interlocking of the roughnesses of the surfaces or by the development of adhesive forces as the molecules of the surfaces come close together or by some other hypothesis; these explanations are of little consequence if the net effect is expressed in terms of a macroscopic parameter, *coefficient of friction*, defined as follows:

Consider a body of mass m resting on a surface as shown in Fig. S3.1. If a normal force $\mathbf{F}_n$ acts normal to the lower surface, the body continues to be at rest because an equal and opposite force **R** acts upon it by the lower surface. The normal force $\mathbf{F}_n$ may be due to the weight of the body or by some other action. If now a small tangential force $\mathbf{F}_t$ is also acted upon the body, the body may still continue to be at rest. The applied tangential force $\mathbf{F}_t$, is balanced by the friction force **f** due to the lower surface. Until a certain limiting value of $\mathbf{F}_t$

$$\mathbf{f} = \mathbf{F}_t$$

the body stays at rest. Clearly, the body would slide if the magnitude of $\mathbf{F}_t$ is increased beyond this limiting value. This state of the body is called *state of impending motion*; a critical border line condition between the static and dynamic conditions of the body.

The state of impending motion can be modelled in terms of **Coulomb's laws* of dry friction:**

1. *The maximum force of friction is independent of the magnitude of area in contact between the surfaces.*
2. *The maximum force of friction is proportional to the normal force on the area of contact.*
3. *The maximum force of friction is less and practically constant at low velocities of sliding than that at the state of impending motion.*

The first law allows us to ignore the extent of the area in contact. The area does not enter the picture, perhaps because the extent of mechanical interlocking or cohesive forces adjusts with respect to the normal forces along which acts the total force of friction. The second law provides a proportionality

* These laws are stated without proofs at this stage. Premature statements are given merely to introduce the pressure intensity and force concepts in the context of force fields.

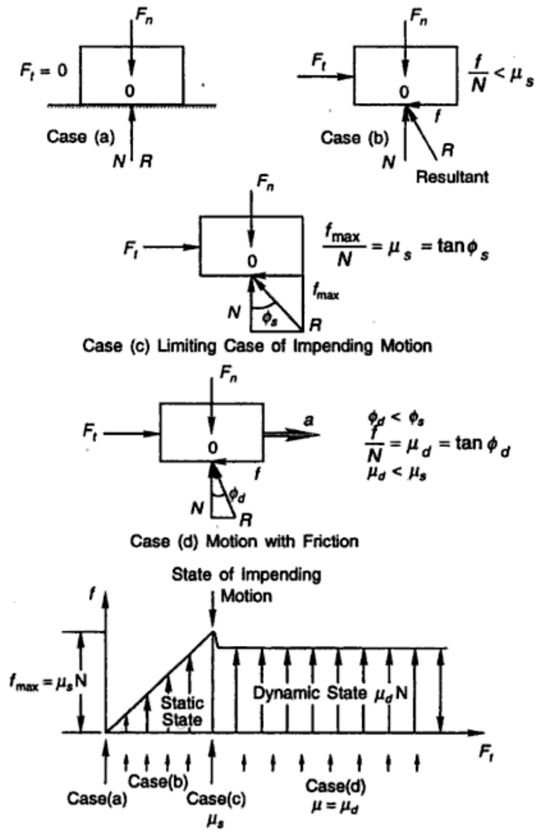


Fig. S3.1 ***Concept and Variation of Frictional Force***

$$f_{max} \propto N$$

or $$f_{max} = \mu_s N \qquad \text{(S3.1)}$$

where f_{max} is the limiting force of friction, **N** the normal reaction force and μ_s the constant of proportionality.

Coefficient of friction between two surfaces is the constant of proportionality μ_s between the limiting for f_{max} and the normal reaction *R*.

$$\mu_s = \frac{f_{max}}{N}$$

If two surfaces have the coefficient of friction $\mu_s = 0$, the surfaces are said to be *smooth*. Surfaces with a non-zero coefficient of friction are called *rough surfaces*.

Obviously, a smooth surface cannot provide a force of friction and hence the reaction of a smooth surface must always be normal to it. If, however, a body with a smooth surface rests on another smooth surface and a tangential force is applied to it, the body must slide and accelerate under the action of the applied forces. The friction force

$$\boxed{f_{max} = \mu_s N}$$

given by the coefficient of friction is the *maximum frictional force* which can be developed between the two surfaces for a given normal reaction force **N**. In actual practice,

(i) if a body is at rest on a surface, the friction force at an instant may be less than the limiting value, it being only equal to the applied tangential component of the force such as that shown in case (b) in Fig. S3.1 where

$$f < \mu_s N$$

(ii) if the body is in motion, the friction force is given by the coefficient of dynamic friction or the coefficient of kinetic friction. This coefficient is indeed less than the coefficient of static friction; the difference being a function of the velocity and the nature of surfaces in contact. The coefficient of dynamic friction is written as μ_d to differentiate it from the coefficient of static friction which is often simply denoted as μ. In the dynamic state, as shown in case (d), Fig. S3.1.

$$f = \mu_d N \qquad \text{(S3.2)}$$

where $$\mu_d \approx 75\% \ \mu_s$$

Table S3.1 Coefficients of Static Friction μ_s

Pair of Surfaces	*Range of μ_s**
Wood and wood	0.2 – 0.6
Wood and leather	0.2 – 0.5
Rope and wood	0.6 – 0.7
Steel and cast iron	0.4 – 0.5
Steel and leather	0.4 – 0.6
Mild steel and mild steel	0.5 – 0.6

* The coefficient of dynamic friction is about 25% less than that for static friction.

(iii) if the body is in an unsteady state, e.g., in intermittent or reversed sliding motion, the coefficient of friction may be quite different from the steady state value.

It is usual to define the term *angle of friction*. Angle of friction is the angle between the line of action of the total reaction by one body on the other and the normal to the common tangent to the surfaces in contact in the state of impending motion.

Angle of friction $$\phi_s = \tan^{-1}\left(\frac{f_{max}}{N}\right) = \tan^{-1} \mu_s$$

or $$\boxed{\mu_s = \tan \phi_s} \tag{S3.3}$$

The angle of friction ϕ_s is, therefore, the maximum angle between the normal reaction **R** and the resultant reaction at the instant of impending motion.

It can also be shown that the angle to which an inclined plane may be raised before the object resting on it slides under the action of its weight and the reaction of the plane, also called as the *angle of repose*, equals the angle of friction. This is shown in Fig. S3.2. Also shown in the figure are the cases of α less than ϕ_s and greater than ϕ_s

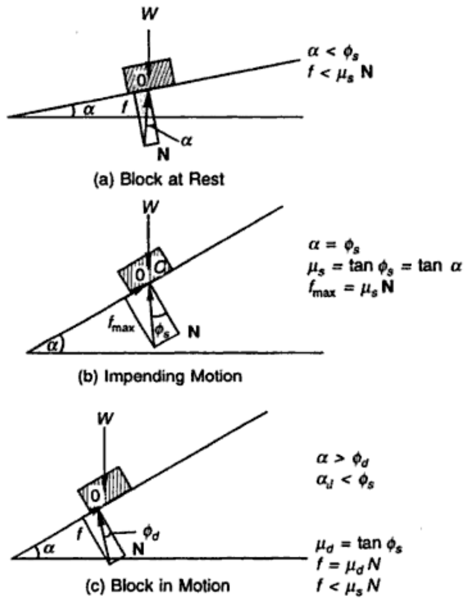


Fig. S3.2 ***Frictional Force on a Body on an Incline***

Fluid Resistance

Surface contact between two material bodies which results in friction forces is permitted only when the frictional forces are usefully employed. This is, however, not the case in most circumstances when the relative velocity of sliding is required and the resistive forces must be minimised. In such case the interspaces must be filled with suitable fluid lubricants. Drag forces resulting from the fluid resistance appear on the moving components. In general, a drag force appears when a solid body moves in a mass of fluid or when a fluid flows around a solid body.

Two different modes of drag formation resulting from the relative velocity between a solid and a fluid must be recognised. First, the *laminar flow drag*, when the relative velocity is low and the fluid flow is characterised by thin sliding laminas sliding over each other; second, the *turbulent flow drag*, when the relative velocity is higher and the fluid flow is characterised by random turbulence and eddy formation.

While the mechanics of drag formation in laminar and turbulent flow is not discussed at this stage, it is worthwhile stating that the variation of drag with the basic parameters is different in the two cases as shown in Table S3.2.

Table S3.2 Variation of Drag

	Laminar Flow	*Turbulent Flow*
Drag Force	$\propto$ speed $\propto$ viscosity $\propto$ size	$\propto$ speed^2 $\propto$ density $\propto$ size^2

***Example* S3.1** A wooden block of mass 1 kg rests on a rough incline at an angle θ as shown in Fig. Ex. S3.1. If the coefficient of friction between the contact surfaces is 0.5, determine the force required to be applied to the string passing through a frictionless pulley to initiate motion of the block down the plane. At what angle θ would the force required be zero? What would happen if the angle is increased beyond that value?

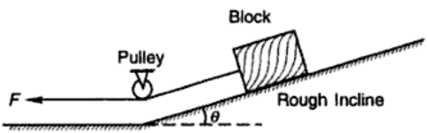


Fig. Ex. S3.1

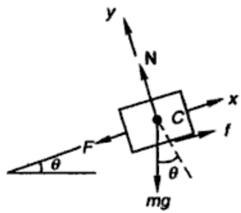


Fig. Ex. S3.1 *(Solution)*

Solution Consider the free-body diagram of the block shown in Fig. Ex. S3.1 (Solution). It is subjected to a weight force mg acting downward, a string force F down the plane and a frictional force f up the plane because the impending motion is down the plane and the normal reaction R perpendicular to the plane. At the state of impending motion, the frictional force must be such that

$$f = \mu R \quad \text{(i)}$$

where R is the normal reaction by the plane on the block.

For equilibrium of the block,

$$\Sigma F_x = 0; f - F - mg \sin \theta = 0 \quad \text{(ii)}$$

$$\Sigma F_y = 0; N - mg \cos \theta = 0 \quad \text{(iii)}$$

From Eq. (iii), $N = mg \cos \theta$

and from Eq. (i), $f = \mu mg \cos \theta$

which when substituted in Eq. (ii) provides

$$F = \mu mg \cos\theta - mg \sin\theta$$

or $$F = (\mu \cos\theta - \sin\theta)\, mg \tag{iv}$$

Substituting $\mu = 0.5$ and $m = 1$ kg

$$F = (0.5 \cos\theta - \sin\theta)\, 1 \times 9.81$$

$$= (4.905 \cos\theta - 9.81 \sin\theta)\, N$$

The force required would be zero when

$$\mu \cos\theta - \sin\theta = 0$$

or $$\tan\theta = \mu$$

For the present case,

$$\theta = \tan^{-1} 0.5 = 26.56°$$

If the angle of incline is increased beyond 26.56°, the force required to initiate the motion is negative which means that an upward force is required to hold the block in equilibrium. If the angle of incline is decreased below 26.56°, the force required to initiate the motion increases.

***Example* S3.2** Two boxes of weights W_1 and W_2 are stacked on the floor as shown in Fig. Ex. S3.2. The coefficient of friction between the boxes is μ_1 and between the box and the floor is μ_2. A horizontal force F is applied to the top of the boxes and gradually increased. What is the maximum force before the equilibrium is destroyed? How will the equilibrium be destroyed?

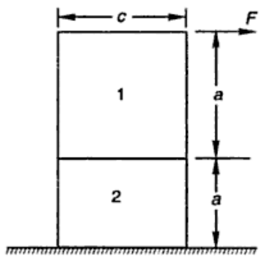


Fig. Ex. S3.2

Solution The free-body diagram of each box is drawn in Fig. Ex. S3.2 (Solution). The upper box 1 is subjected to the applied force F, weight W_1, normal reaction R_1 from box 2 and frictional force f_1 acting by virtue of contact with box 2. In turn, box 2 is acted upon by a frictional force equal and opposite to f_1 as well as a normal reaction equal and opposite to R_1 at its upper surface. It is, in addition, subjected to a normal reaction R_2 and a friction force f_2 at its base surface due to contact with the floor and its own weight W_2.

Let us examine the different possibilities of upsetting the equilibrium.

1. Box 1 may slide to the right, box 2 remaining intact.
2. Boxes 1 and 2 may remain stuck and slide together over the ground.
3. Box 1 may tip over its right edge A, box 2 remaining intact.
4. Boxes 1 and 2 may remain stuck and tip together over the bottom right edge B.

Since the force F is increased from a zero value, the mode of upsetting the equilibrium depends upon which of the four possibilities materialises first. We,

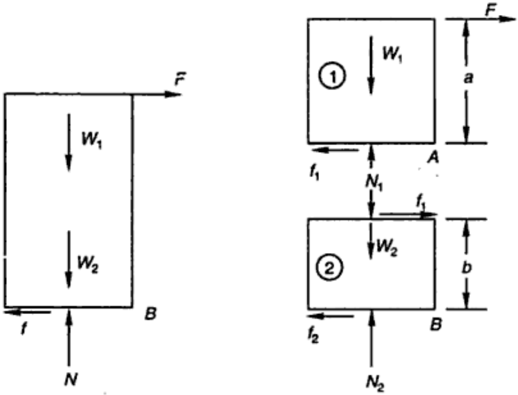


Fig. Ex. S3.2 *(Solution)*

therefore, evaluate the minimum forces required to materialise each possibility and the least of them would be the decisive force.

For the first possibility, consider the equilibrium of box 1 at the state of impending motion.

$$\Sigma F_x = 0;\ F_1 = f_1 = \mu_1 N_1$$

$$\Sigma F_y = 0;\ N_1 = W_1$$

whence $$F_1 = \mu_1 W_1 \qquad \text{(i)}$$

Similarly for the second possibility,

$$N = W_1 + W_2$$

$$F_2 = f = \mu_2 N = \mu_2 (W_1 + W_2) \qquad \text{(ii)}$$

For the tipping of the upper box alone,

$$F_3 \times a = W_1 \times c/2$$

whence $$F_3 = \frac{W_1 c}{2a} \qquad \text{(iii)}$$

For the tipping of the two blocks together,

$$F_4 \times (a + b) = (W_1 + W_2) \times c/2$$

whence $$F_4 = \frac{(W_1 + W_2)c}{2(a+b)} \qquad \text{(iv)}$$

An observation of the results (i) to (iv) reveals that the magnitudes of F_1, F_2, F_3 and F_4 depend upon the values of weights, coefficients of friction and the dimensions of the blocks. In particular, if

$$\mu_1 = \mu_2 = 0.3$$

$$W_1 = W_2 = 100 \text{ N}$$

$$a = b = c = d, \text{ say}$$

$$F_1 = 0.3 \times 100 = 30 \text{ N}$$

$$F_2 = 0.3 \times (100 + 100) = 60 \text{ N}$$

$$F_3 = \frac{100 \times d}{2 \times d} = 50 \text{ N}$$

$$F_4 = \frac{(100 + 100) \times d}{2(d + d)} = 50 \text{ N}$$

Since F_1 is the smallest, the boxes will no longer be in equilibrium if the applied force F exceeds 30 N. The equilibrium will be destroyed by the slippage of the upper box, the lower remaining intact.

Experiment E6

Coefficient of Static Friction

OBJECTIVE

To determine the coefficient of static friction between two given material surfaces with the help of an inclined plane.

APPARATUS

An adjustable inclined plane with a frictionless pulley, a block, inextensible string and standard weights.

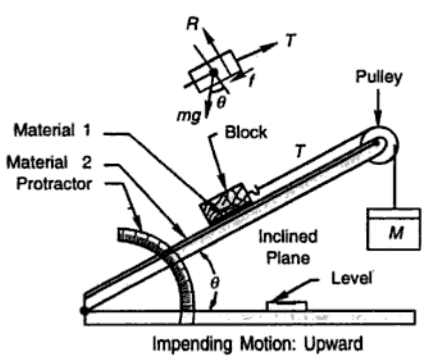


Fig. E6.1 ***Determination of Coefficient of Static Friction*** μ

Background Information

The mating surfaces of the block and the incline are faced with sheets of materials between which the coefficient of friction is desired. At a fixed angle of inclination θ, the suspended mass is increased until the block is at the verge of upward slippage, i.e., in the state of impending motion. Referring to the free-body diagram of the block at such a state, as shown for equilibrium,

$$T = Mg = f + mg \sin \theta$$

$$R = mg \cos \theta$$

whence

$$\mu = f/R = \frac{Mg - mg \sin \theta}{mg \cos \theta} = \frac{M - m \sin \theta}{m \cos \theta}$$

Observations and Calculations

The mass of the block as well as the mass of the suspended weight together with the chosen angle of inclination should be recorded:

Material 1				*Material 2*		
S.No.	θ	m	M	$\sin \theta$	$\cos \theta$	μ
1						
2						

The observations and calculations are repeated with different selections of the two masses or two angles of inclination or both.

Result

The average value of the coefficient of static friction μ may be obtained and the range of variation of μ may be noted.

Points for Discussion

1. Compare the value of μ between the two surfaces with the value given in a standard handbook. Account for the difference, if any.
2. Would the value of μ be the same if the materials on the incline and block are interchanged?
3. In the first method, supposing a weight W is placed over the block, would the angle of repose remain the same?
4. What is the corresponding value of the coefficient of dynamic friction μ_d between the same pair of surfaces? Explain, giving examples, as to where μ is used and where μ_d is to be used?
5. Supposing a block is placed on an incline and there is no pulley and string, etc. Can't we find the coefficient of static friction by creating a condition of downward impending motion? By increasing the inclination θ; by adding more weight on the block? Can we? How would it compare with our method?
6. The assumptions made in the analysis of observations are frictionlessness of the pulley, inextensibility of the string, correctness of masses and angles measured. How far are the assumptions justified and what are the possible sources of error?

7. If the rectangular block were faced with the same material on all faces, would the observations and calculations alter if the block were placed one face or the other resting on the incline?

***Example* S3.3** A ladder of length 5 m and weight 120 N is placed on a flat floor against a vertical wall as shown in Fig. Ex. S3.3. If the coefficients of friction are 0.3 and 0.2 and the ladder is considered homogeneous, determine the smallest angle θ the ladder can be placed at the floor for equilibrium.

Solution At the position of the smallest angle θ, the ladder would be in the state of impending skid at *A* and at *B*.

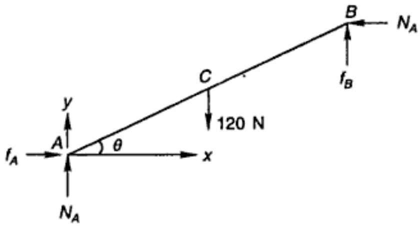


Fig. Ex. S3.3

$$f_A = 0.3\ N_A \text{ and } f_B = 0.2\ N_B$$

For equilibrium,

$$\Sigma F_x = 0 \qquad f_A - N_B = 0;\ f_A = N_B$$

$$\Sigma F_y = 0 \qquad N_A - 120 + f_B = 0;\ f_B = 120 - N_A$$

$$\Sigma M_A = 0 \qquad N_B \times 5 \sin\theta + f_B \times 5 \cos\theta - 120 \times 2.5 \cos\theta = 0$$

From these equations,

$$N_B = 0.3\ N_A \quad \text{and} \quad 120 - N_A = 0.2\ N_B$$

whence, $N_A = 113.2$ N and $N_B = 34$ N

and $f_A = 34$ N and $f_B = 6.8$ N

Finally, $170 \sin\theta + (34 - 300)\cos\theta = 0$

$$\tan\theta = 1.565,\ \theta = 57.4°$$

***Example* S3.4** A man wishes to climb a 5 m long ladder placed at 60° on a horizontal surface ($\mu = 0.3$) against a vertical wall ($\mu = 0.2$) as shown in Fig. Ex. S3.4. How far can he climb without the ladder slipping? The man and the ladder weigh 800 N and 150 N respectively.

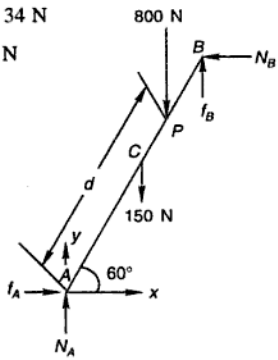


Fig. Ex. S3.4

Solution The free body diagram of the ladder shows that it is subjected to 6 forces of which 4 are unknown and we wish to find the maximum value of d, i.e., at the state of impending slip at A and B.

Then, $f_A = 0.3\ N_A$ and $f_B = 0.2\ N_B$

For equilibrium,

$$\Sigma F_x = 0; \quad f_A - N_B = 0; \quad f_A = N_A$$

$$\Sigma F_y = 0; \quad N_A + f_B - 150 - 800 = 0; \quad f_B = 950 - N_A$$

$$\Sigma M_A = 0; \quad 5\ N_B \sin 60° + 5\ f_B \cos 60° - 800\ d \cos 60°$$

$$- 150 \times 2.5 \cos 60° = 0; \quad 4.33\ N_B + 2.5\ f_B - 400\ d = 187.5$$

From these equations,

$$N_B = 0.3\ N_A, \quad 950 - N_A = 0.2\ N_B$$

whence $N_A = 896.3$ and $N_B = 268.9$

and $f_A = 268.9$ and $f_B = 53.7$ in N units.

Finally, $-400\ d = 187.5 - 1164.3 - 134.3 = -1111.1$ and $d = 2.77$ m

The ladder will skid as the man reaches a distance of 2.77 m up the ladder.

Let us verify this result by considering the summation of moments about point B.

$$\Sigma M_B = 800\ (5 - 2.77) \cos 60° + 150 \times 2.5 \cos 60° + 268.9 \times 5 \sin 60° - 896.3 \times 5 \cos 60°$$

$$= 889 + 187.5 + 1164.4 - 2240.7 \approx 0.0$$

which shows that the distance of 2.77 m up the ladder is the right answer for the equations. Any mistake in the setting up of the equations is not checked by the verification. That can be ensured by checking the results vis-a-vis the data in the problem. From the fact that $\mu = 0.3$ at the floor, $f_A = 0.3\ N_A$. The value of f_A as 268.9 N is compatible with that of N_A as 896.3 N. Similarly, the value of f_B as 53.7 N is compatible with that of N_B as 268.9 N for a μ of 0.2 at the wall. The result is, therefore, correct.

***Example* S3.5** A homogeneous ladder is placed on a flat horizontal surface to rest against a vertical wall as shown in Fig. Ex. S3.5. Assuming that the coefficient of friction at each surface is μ, determine the minimum possible inclination of the ladder with the horizontal. Can the angle be less than 45°?

Solution From the free body diagram of the ladder, the position of minimum inclination corresponds to that of impending skidding.

$$f_A = \mu N_A \quad \text{and} \quad f_B = \mu N_B$$

For equilibrium,

$$\Sigma F_x = 0 \quad f_A - N_B = 0; \quad \mu N_A = N_B$$

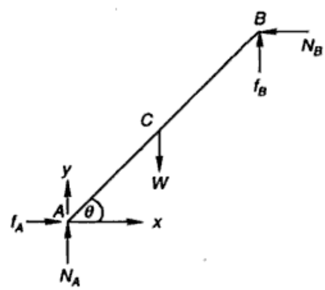


Fig. Ex. S3.5

$$\Sigma F_y = 0 \qquad N_A + f_B - W = 0; \qquad N_A + \mu N_B = W$$

$$\Sigma F_A = 0 \qquad N_B \times L \sin\theta + f_B \times L\cos\theta - W \times L/2 \cos\theta = 0$$

From these equations,

$$N_A + \mu^2 N_A = W; \qquad N_A = \frac{W}{1+\mu^2}$$

$$N_B = \mu N_A = \frac{\mu W}{1+\mu^2}$$

and $$\frac{\mu W}{1+\mu^2}\sin\theta = \left(\frac{W}{2} - \frac{\mu^2 W}{1+\mu^2}\right)\cos\theta = \frac{1-2\mu^2}{1+\mu^2} = W\cos\theta$$

whence $$\tan\theta = \frac{1-2\mu^2}{\mu}$$

For smooth surfaces, $\mu = 0$, $\tan\theta \to \infty$, $\theta \to 90°$

For θ to be 45°, $\tan\theta = 1$, $1 - 2\mu^2 = \mu$, or $\mu = 0.5$

Hence it is possible for the inclination to be less than 45° if the coefficient of friction is greater than 0.5.

For example a value of $\mu = 0.6$ implies that $\tan\theta = (1 - 0.72)/0.6 = 0.467$. $\theta = 25°$. Interestingly, $\theta = 0$ corresponds with $1 - 2\mu^2 = 0$; $\mu = 0.707$.

***Example* S3.6** A small block of mass 100 kg is placed on a 30° incline with a coefficient of friction of 0.25 as shown in Fig. Ex. S3.6. Determine the horizontal force to be applied on it in order to keep it in equilibrium at rest.

Solution The block would stay in equilibrium at rest for a range of horizontal forces between F_1 and F_2 where F_1 is the minimum force required to prevent it from sliding down and F_2 is the maximum force applied without sliding it up. For the case of impending sliding down the incline,

$$F_1 \cos 30° - 981 \sin 30° + f_1 = 0$$

where $$f_1 = 0.25\, N_1 = 0.25 \times (F_1 \sin 30° + 981 \cos 30°).$$

$$0.866\, F_1 - 490.5 + 0.125\, F_1 + 212.4 = 0$$

whence $$F_1 = 280.6 \text{ N}$$

For the case of impending sliding up the incline,

$$F_2 \cos 30° - 981 \sin 30° - f_2 = 0$$

where $$f_2 = 0.25\, N_2 = 0.25 \times (F_2 \sin 30° + 981 \cos 30°)$$

$$0.866\, F_2 - 490.5 - 0.125\, F_2 - 212.4 = 0$$

whence $$F_2 = 948.6 \text{ N}$$

The block would, therefore, be in equilibrium if the horizontal force applied is between 280.6 N and 948.6 N as shown.

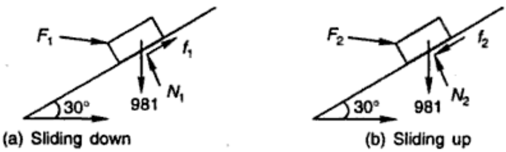


Fig. Ex. S3.6

***Example* S3.7** A horizontal force F acts on a block at a height h above the surface of the table where it lies as shown in Fig. Ex. S3.7. Explain the condition when the body (a) slides (b) overturns.

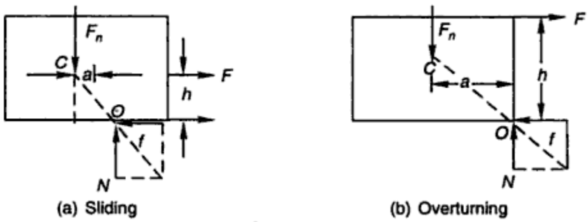


Fig. Ex. S3.7

Solution When a horizontal force F acts at some distance from the base, the point of application of R and f, i.e., point O is not vertically below C: The point O locates itself such that the moment due to couple F and f is balanced by the moment due to couple F_n and N. As long as O lies within the base, the body has a tendency to slide.

Now, if the point O moves to a corner because the frictional force f keeps increasing due to a high value of μ_s, the coefficient of static friction then, the body has a tendency to overturn or tip over that corner. In that case,

$$f\,h \geq F_n\,a$$

or

$$\mu_s\,F_n\,h \geq F_n\,a$$

or

$$h \geq a/\mu$$

The tendency to overturn is more if the base width is small, coefficient of friction is large or the height, h is large.

***Example* S3.8** A 'levelling plank' is often used in a field to level off the earth by sliding the plank over it. One such plank requires a horizontal force of 600 N to slide it on a horizontal field at a constant velocity when the frictional resistance is 1.5 N per kg mass of the plank. It is employed to level a field inclined at 5° to the horizontal. The span of the plank is 2 m and the force is applied along the direction of movement. It is used for levelling, both going up and coming down the incline end to end, alternately. Estimate the force required each way and compare the average force required with that required for a horizontal field.

Solution The horizontal force of 600 N on a levelling plank must equal the frictional resistance for levelling at a constant velocity.

$$\text{Mass of the plank } = 600/1.5 = 400 \text{ kg}$$
$$\text{Weight of the plank } = 400 \times 9.81 = 3920 \text{ N}$$

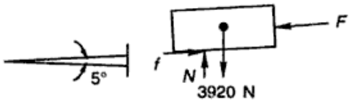


Fig. Ex. S3.8 (a) ***(Solution)***

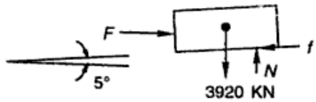


Fig. Ex. S3.8 (b) ***(Solution)***

The normal reaction on the horizontal surface is 3920 N. The coefficient of friction μ is, therefore, 600/3920 = 0.153.

For going up the incline, from the free-body diagram of the plank shown in Fig. Ex. S3.8(a) (Solution),

$$N = 3920 \cos 5° = 3900 \text{ N}$$
$$f = \mu R = 0.153 \times 3900 = 597 \text{ N}$$
$$F = 3920 \sin 5° + 597 = 939 \text{ N}$$

For going down the incline,
R = 3900 N and f = 597 N as shown in the free-body diagram (Fig. Ex. S3.8(b) Solution).

$$F = 597 - 3920 \sin 5° = 255 \text{ N}$$

The average force required for the inclined field is

$$(939 + 255)/2 = 597 \text{ N}$$

The force required for a horizontal lawn is 600 N which is slightly more than 597. In fact, it can be observed that the difference is due to the decrease of the normal component of reaction for the inclined surface.

Example **S3.9** A block of mass 150 kg is to be raised by means of inserting a 10° wedge weighing 50 kg under it and by applying a horizontal force at it as shown in Fig. Ex. S3.9. Assuming the coefficient of friction between all surfaces of contact as 0.3, determine what minimum horizontal force should be applied to raise the block. What would happen if the horizontal force is removed?

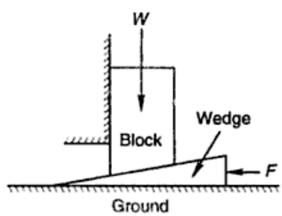


Fig. Ex. S3.9

Solution It is necessary to visualise the forces and to draw the free-body diagrams of the block and of the wedge for the impending motion of the block upwards.

For equilibrium of the block (Fig. Ex. S3.4 (Solution)),

$$W = R_2 \cos \alpha - f_1 - f_2 \sin \alpha$$
$$N_1 = f_2 \cos \alpha + R_2 \sin \alpha$$

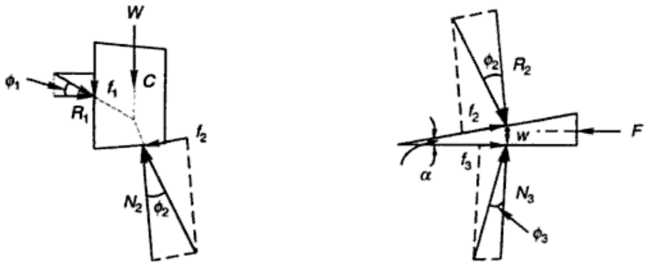


Fig. Ex. 3.9 ***(Solution)***

where $\qquad f_1 = \mu_1 R_1, f_2 = \mu_2 R_2$, and W is the weight of the block.

For equilibrium of the wedge,

$$F = f_2 \cos \alpha + f_3 + N_2 \sin \alpha$$

$$N_3 = R_2 \cos \alpha + w - f_2 \sin \alpha$$

where $\qquad f_2 = \mu_2 N_2 \qquad f_3 = \mu_3 N_3$

Putting $\qquad W = 150 \text{ kg} = 1470 \text{ N}$

$$w = 50 \text{ kg} = 490 \text{ N}$$

$\mu_1 = \mu_2 = \mu_3 = 0.3$, and solving.

$$N_1 = 869.8 \text{ N} \qquad f_1 = 261.0 \text{ N}$$

$$N_2 = 1854.6 \text{ N} \qquad f_2 = 556.4 \text{ N}$$

$$N_3 = 2219.8 \text{ N} \qquad f_3 = 665.9 \text{ N}$$

$$F = 1535.94 \text{ N}$$

S3.2 ROLLING RESISTANCE

Rolling resistance occurs as a result of the small deformation of the surface upon which a rolling object rolls. The surface of the rolling object may also be deformed in the process. A cylindrical or spherical object is shown rolling in Fig. S3.3. The force F required parallel to the surface must be adequate to lift the object out of the depression in the surface. The process of deformation of the surfaces is continuous during the rolling movement and hence a constant force F must be applied to overcome the resistance offered by the deformation.

With reference to Fig. S3.3, the sum of the moments about the point O must be zero for equilibrium.

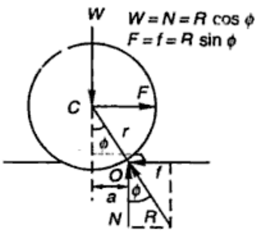


Fig. S3.3 ***Mechanism of Rolling Resistance***

$$W \cdot r \sin \phi - F \cdot r \cos \phi = 0$$

or

$$F = W \tan \phi$$

Since the angle ϕ subtended by the length deformation must be very small,

$$\tan \phi \approx \sin \phi \approx \frac{a}{r}$$

Hence

$$\boxed{F = a\frac{W}{r}}$$

The distance a reckoned as the 'forward length of deformation' is defined as the *coefficient of rolling resistance*. The coefficient of the rolling resistance, unlike the coefficient of friction, has the units of length.

Table S3.3 Coefficient of Rolling Resistance

Pair of Surfaces	*Range of a (cm)*
Steel on steel	0.02-0.04
Steel on wood	0.15-0.25
Tyre on road	0.05-0.15
Hardened steel surfaces	0.0005-0.0015

It is interesting to observe that the coefficient of rolling resistance is defined as the forward length of deformation rather than a non-dimensional quantity. The rolling resistance is, therefore, a function of (a) the weight or load W normal to the surface, (b) the radius of the cylinder and (c) the coefficient of rolling resistance.

It follows that less force is required to roll a wheel of bigger radius than a wheel of smaller radius for the same load and the same pair of materials.

S3.3 SLIDING AND ROLLING OF CYLINDERS

If a circular cylinder is placed in contact with a surface and subjected to a force, it may either slip or roll about the point of contact. The same is likely to happen when it is placed on an incline. Whether it will actually slip or roll depends upon the friction characteristics, force and angle of incline. In many problems, it is necessary to ascertain whether a cylinder will tend to slip or roll, particularly when it is in contact with two surfaces. Consider, for example a cylinder and a flat-based block on an incline and the angle of incline is gradually raised until the equilibrium is disturbed as shown in Fig. S3.4.

In the limiting condition, the block must slide down the incline at its base whereas the cylinder may slip at B and roll at A or it may slip at A and remain stuck at B. It may be seen that a cylinder cannot roll about two points on it because the meaning of rolling about a point is that the velocity at that point is zero and at all other points non-zero. However, slipping of a cylinder about two points on it cannot be ruled out in general and it is decided by the constraints. In the present case, slipping at A and B would mean rotation about O, i.e., the cylinder rotates about its axis which is obviously not possible without an external moment. Hence, the cylin-

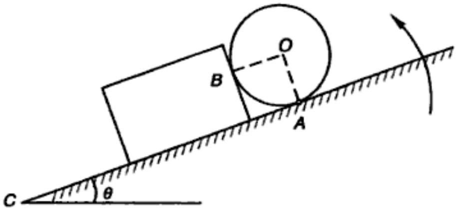


Fig. S3.4 *Cylinder and Block on an Incline*

der may slip at A or B and roll about the other point. There is no simpler way to decide as to which is the actual mode except by considering each mode for the state of impending motion and analysing the situation.

At each contact point, a cylinder experiences a normal reaction directed towards the centre of the cylinder. It also experiences the frictional force at each contact point. At a point where it may slip, the frictional force is related to the normal reaction,

$$f = \mu N$$

but at the point where it may roll, the frictional force is left as an unknown. It should be between zero and the maximum possible value μR at that point. For rolling, therefore,

$$0 < f < \mu N \tag{S3.2}$$

where μ is the coefficient of static friction between the contacting surfaces.

***Example* S3.10** A circular cylinder of radius 0.5 m and mass 200 kg is placed in contact with a rectangular block of mass 150 kg on an incline at 30° as shown in Fig. Ex. S3.10. If the coefficient of static friction is 0.6, determine the minimum force F to be applied up the plane at the block to prevent the bodies from sliding down.

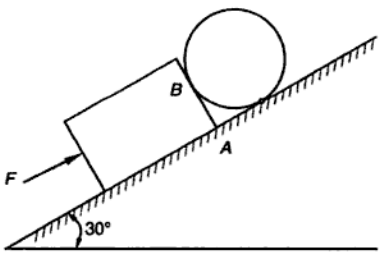


Fig. Ex. S3.10

Solution We should first consider the equilibrium of the cylinder at the state of impending downwards motion. The free-body diagram is constructed by including the normal and frictional reactions by the contact surfaces as well as the weight of the cylinder itself as shown in Fig. Ex. S3.10 (Solution). There are two possible

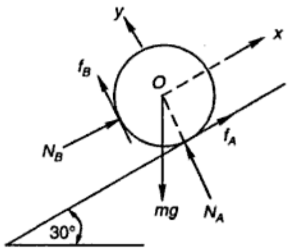


Fig. Ex. S3.10 (a) ***(Solution)***

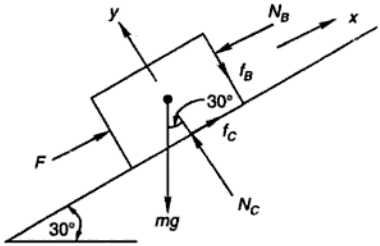


Fig. Ex. S3.10 (b) ***(Solution)***

modes of the cylinder coming down the incline. It may roll at A and slide at B or it may slide at A. Let us consider the first possibility, i.e., sliding at B,

$$f_B = \mu N_B \tag{i}$$

and $\quad f_A$ = any thing between zero and μR_A

For equilibrium of the cylinder

$$\Sigma F_x = 0; \qquad N_B + f_A - mg \sin 30° = 0 \tag{ii}$$

$$\Sigma F_y = 0; \qquad f_B + N_A - mg \cos 30° = 0 \tag{iii}$$

$$\Sigma M_0 = 0; \qquad f_A \times 0.5 - f_B \times 0.5 = 0 \tag{iv}$$

From these relations,

$$N_B = \frac{mg}{2(1+\mu)} = \frac{200 \times 9.81}{2(1+0.6)} = 613 \text{ N}$$

$$f_B = 0.6 \times 613 = 368 \text{ N}; f_A = 368 \text{ N}$$

$$N_A = 1331 \text{ N}$$

A check on the value of f_A shows that it lies between 0 and 0.6 × 1331, i.e., 799 N. The possibility that the cylinder rolls at A and slips at B is, therefore, feasible.

Now, consider the equilibrium of the block at the state of impending downward motion,

$$\Sigma F_x = 0; \; F - N_B + f_C - mg \sin 30° = 0$$

$$\Sigma F_y = 0; \; N_C - f_B - mg \cos 30° = 0$$

Since $\quad N_B = 613 \text{ N}, f_B = 368 \text{ N}$

$$f_C = \mu R_C = 0.6 \, R_C$$

these relations provide

$$N_C = 1615 \text{ N} \qquad \text{and} \qquad F = 380 \text{ N}$$

A force of 380 N is, therefore, required to be applied up the plane at the block to check the state of impending downward motion; this is the minimum force required to keep the assembly of the block and cylinder in equilibrium on the incline.

The problem could have been formulated in a different way. The block and cylinder could have been shown on the 30° incline in the absence of any applied force F and one could be asked to determine whether the bodies are in a state of equilibrium or not. It can now be suggested to attempt the problem in exactly the same way as done here and determine the force F required. If F comes out to be positive, as it did come in the given case, the bodies would not be in equilibrium without holding them up by any up-the-incline force. If F comes out to be negative, it means that the bodies were in equilibrium at rest.

***Example* S3.11** A circular cylinder of radius 0.5 m and mass 200 kg is placed in contact with a rectangular block of mass 150 kg on an incline at 30° as shown in Fig. Ex. S3.11. If the coefficient of static friction is 0.6, determine the minimum force F to be applied up the plane at the block to initiate an upward motion of the bodies.

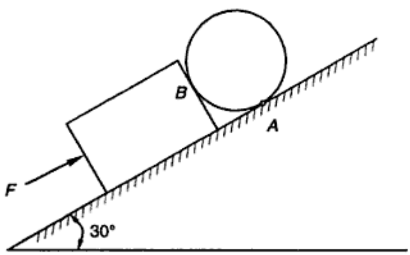


Fig. Ex. S3.11

Solution The cylinder, while tending to move up, may either tend to roll at A and slide at B or slide at A. Taking the former possibility first, consider the free-body diagram drawn in Fig. Ex. S3.11(a) (Solution), for equilibrium.

$$\Sigma F_x = 0; \qquad R_B - f_A - mg \sin 30° = 0$$

$$\Sigma F_y = 0; \qquad -f_B + N_A - mg \cos 30° = 0$$

$$\Sigma M_0 = 0; \qquad f_B \times 0.5 - f_A \times 0.5 = 0$$

and $\quad f_B = \mu N_B = 0.6\, N_B$

From these relations,

$$N_B = 2452 \text{ N}; \qquad N_A = 3170 \text{ N}$$

$$f_B = 0.6 \times 2452 = 1471 \text{ N}; \qquad f_A = 1471 \text{ N}$$

A check on the value of f_A shows that it lies between 0 and 0.6×3170, i.e., 1902 N It may, therefore, be taken that the cylinder would roll at A and slip at B.

Next, consider the equilibrium of the block with reference to the free-body diagram shown in Fig. Ex. S3.11(b) (Solution).

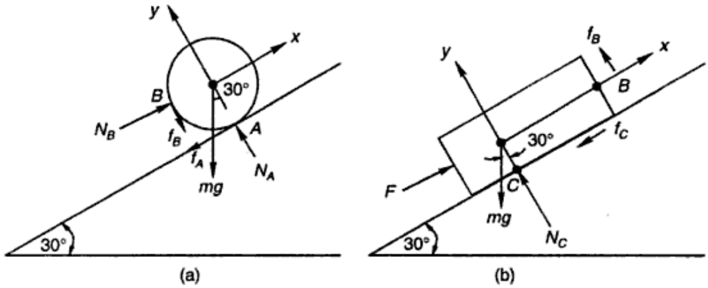


Fig. Ex. S3.11 ***(Solution)***

$$\Sigma F_x = 0; \qquad F - N_B - f_C - mg \sin 30° = 0$$
$$\Sigma F_y = 0; \qquad N_C + f_B - mg \cos 30° = 0$$

and $f_C = \mu N_C = 0.6\ R_C$

Since $N_B = 2452$ N and $f_B = 1471$ N, these relations provide,

$$F = 3070 \text{ N}$$

A force of 3070 N is, therefore, required to be applied up the incline at the block in order to initiate an upward motion of the bodies.

It was observed in the previous problem that a force of 380 N is required to hold the bodies in equilibrium against coming down the plane. It has now been seen that a 3070 N force is required to move the bodies up the plane. Obviously, if a force in between 380 N and 3070 N is applied, the bodies would stay in equilibrium.

The problem could have been formulated in a slightly different manner. The block and cylinder would have been shown on the 30° incline and a force of some magnitude shown applied at the block and one could be asked to ascertain whether the bodies are moving down, in equilibrium, or moving up. If the applied force was less than 380 N, the bodies would be moving down; if more than 3070, the bodies would be moving up and if in between these two values, the body must stay at rest in equilibrium.

***Example* S3.12** A cylinder of radius a and weight W is wedged between a vertical wall and a light-hinged bar as shown in Fig. Ex. S3.12. The coefficient of friction between the cylinder and wall is 0.2 and that between the bar and cylinder is 0.4. What is the force F which, when applied at the end of the bar at right angles to it, is just sufficient to cause the cylinder to move upward?

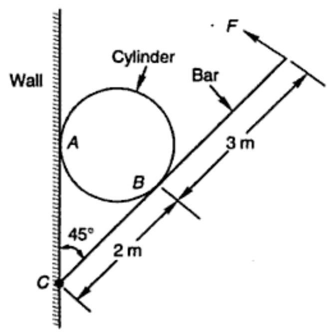


Fig. Ex. S3.12

Solution The cylinder can slip upward at either point; not necessarily at both points simultaneously. In other words, the cylinder may slip at the bar point B and roll about A or it may slip at the wall point A and roll about B. We should, therefore, examine both possibilities.

When there is impending motion at B, the free-body diagram drawn at Fig. Ex. S3.7(a) (Solution) shows the forces acting on the cylinder.
By considering the equilibrium,

$$\Sigma F_x = 0; \qquad N_A - f_B \cos 45° - 2.5F \cos 45° = 0$$

$$\Sigma F_y = 0; \qquad -f_A + 2.5F \sin 45° - f_B \sin 45° - W = 0$$

$$\Sigma M_0 = 0; \qquad f_B \times \text{radius} = f_A \times \text{radius}$$

and $\quad f_B = 0.4 \times 2.5F = F$

From these relations,

$$f_A = F = 16.5\ W \qquad \text{and} \qquad N_A = 40.8\ W$$

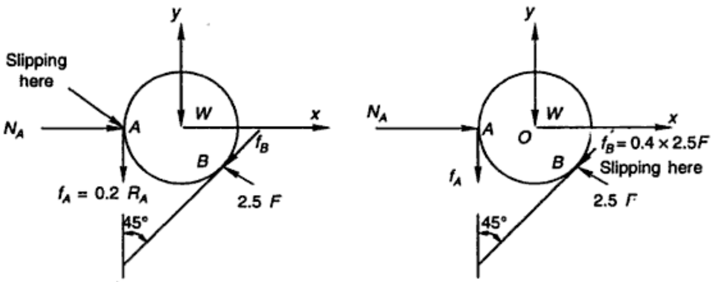


Fig. Ex. S3.12 (a) ***(Solution)*** **Fig. Ex. S3.12 (b)** ***(Solution)***

It may be seen that the ratio of the friction to the normal reaction at A is

$$\frac{f_A}{N_A} = \frac{16.5}{40.8} = 0.4$$

but the maximum possible ratio is 0.2. Since the slippage at B leads to this impasse, it is impossible for slippage to occur at B before it occurs at A.

The free-body diagram for the second possibility, i.e., slippage at A is shown in Fig. Ex. S3.12(b) (Solution).

By equilibrium consideration,

$$\Sigma F_x = 0; \qquad N_A - 2.5F \cos 45° - f_B \cos 45° = 0$$

$$\Sigma F_y = 0; \qquad -0.2N_A \times W - f_B \sin 45° + 2.5F \sin 45° = 0$$

$$\Sigma M_0 = 0; \qquad f_B \times \text{radius} = 0.2\ N_A \times \text{radius.}$$

From these relations,

$$f_B = 0.39\ W, \qquad N_A = 1.94\ W$$

and $\quad F = 0.94\ W$

We may again like to check the ratio of friction to the normal reaction at B.

$$\frac{f_B}{N_B} = \frac{0.39}{2.5 \times 0.94} = 0.17$$

which is considerably less than the limiting value of 0.4.

It may, therefore, be concluded that the force required at the end of the rod is 0.94 W and that the cylinder will slip at A, the point of contact with the wall.

S3.4 BAND-BRAKE AND BELT FRICTION

A flexible member, such as a band, belt or rope passing over or wrapped around a cylinder can be used as band-break for power absorption or as a belt rope drive for power transmission.

Consider a flexible member passing over a cylinder as shown in Fig. S3.5. The total angle subtended at the centre by the flexible member is θ. Consider the equilibrium of a small segment over an angle $d\theta$. The tensions on either side are T and $T + dT$ which may be considered collinear over the small segment. The difference in the tensions is due to the force of friction which would be a maximum at the state of impending motion between the flexible member and the cylinder.

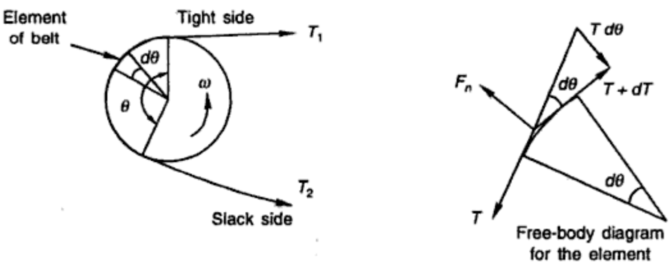


Fig. S3.5 ***Analysis of Belt Drive***

$$dT = f = \mu F_n$$

where μ is the static coefficient of friction between the two surfaces. The normal force F_n must be given by

$$F_n = T\,d\theta$$

Thus
$$dT = \mu T\,d\theta$$

or
$$\frac{dT}{T} = \mu\,d\theta$$

Integrating T from T_2 to T_1 over an angle θ

$$\int_{T_2}^{T_1} \frac{dT}{T} = \int_0^{\theta} \mu\,d\theta = \mu\theta$$

or $$\log_e \frac{T_1}{T_2} = \mu\theta$$

and $$\frac{T_1}{T_2} = e^{\mu\theta} \tag{S3.3}$$

It may be noticed that the tension T_1 and T_2 can be widely different depending only upon the coefficient of friction μ and the angle of contact θ. The side of the belt with greater tension, T_1 is called the *tight side* and the other with less tension, T_2 is called the *slack side* of the drive. The difference between T_1 and T_2 is responsible for transmitting the power. The difference can be increased by placing the pulleys closer and by employing a cross-belt drive instead of an open belt drive as far as possible. The *power transmitted* is given by

$$\text{Power} = (T_1 - T_2)\, v$$

where v is the linear speed of the belt.

If T_2 is desired to be much smaller than T_1, as is the case of drum pulley blocks and capstans, the rope can be wound round the drum a few turns. For n turns round the drum the angle of contact θ becomes 2π times n.

***Example* S3.13** A rope is placed round a fixed circular post over an angle θ as shown in Fig. Ex. S3.13. Determine the ratio of tensions T_1 and T_2 for impending slip in the anticlockwise direction.

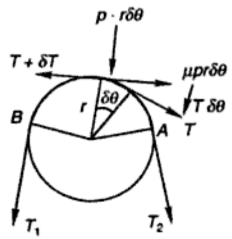


Fig. Ex. S3.13

Solution Let p be the pressure per unit length of the rope on the post. For an elementary length $r\delta\theta$ of the rope, the normal force is $pr\delta\theta$ and the frictional force is $\mu pr\delta\theta$. It is subjected to tensions T and $T + \delta T$ on either side. For equilibrium,

$$T + \delta T - T = \mu p\, r\, \delta\theta$$

or $$\delta T = \mu p\, r\, \delta\theta \tag{i}$$

and $$T\delta\theta = p\, r\, \delta\theta$$

or $$T = \mu r \tag{ii}$$

From these two equations

$$\frac{\delta T}{T} = \mu\delta\theta$$

Integrating over the length of contact

$$\log \frac{T_1}{T_2} = \mu\theta$$

or $$\frac{T_1}{T_2} = e^{\mu\theta}$$

where θ is the angle of lap over the post.

If the post was non-circular, even then the T_1/T_2 ratio would be the same as $e^{\mu\theta}$ where θ is the angle of lap determined w.r.t. a centre located by drawing normals to T_1 and T_2 as shown in Fig. Ex. S3.13(a) (Solution).

This is because every little element can be approximated a being that of a circle, even though the centre of the circle keeps changing by the additional elements.

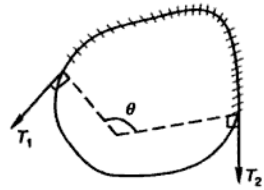


Fig. Ex. S3.13(a) ***(Solution)***

Supposing it was a belt going over a shaft instead of a rope going over a post. No difference. The T_1/T_2 ratio for the impending slip of the belt over the shaft would be the same provided the belt was massless. Therefore, for a belt drive,

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

It may, however, be noted that a belt is usually required to transmit power from a driver pulley to a driven pulley which creates a tight side with tension T_2 and a slack side with tension T_1 as shown in Fig. Ex. S3.13(b) (Solution). It may be observed that the angle of lap on the bigger pulley is greater than 180 degrees and that on the smaller pulley is less than 180 degrees. It is, therefore, more likely to slip on the smaller pulley.

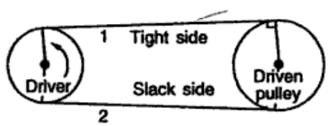


Fig. Ex. S3.13(b) ***(Solution)***

***Example* S3.14** A horizontal drum of a belt drive carries the belt over a semicircle around it. It is rotated anticlockwise to transmit a torque of 300 Nm. If the coefficient of friction between the belt and the rope is 0.3, calculate the tensions in the limbs 1 and 2 of the belt shown in Fig. Ex. S3.14 and the reaction on the bearings. The drum has a mass of 20 kg and the belt is assumed to be massless.

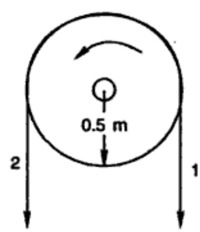


Fig. Ex. S3.14

Solution If the drum was assumed to be stationary, the impending motion of the belt over the drum would be from 2 to 1. Clearly, T_1 should be more than T_2.

According to this condition,

$$\frac{T_1}{T_2} = e^{0.3\pi} = 2.57$$

or $$T_1 = 2.57\, T_2 \qquad \text{(i)}$$

From the knowledge of the torque transmitted,

$$(T_1 - T_2) \times 0.5 = 300$$

$$T_1 - T_2 = 600 \text{ N} \tag{ii}$$

from (i) and (ii)

$$T_1 = 982 \text{ N}; \ T_2 = 382 \text{ N}$$

The downward force exerted by the belt on the drum is

$$T_1 + T_2 = 982 + 382 = 1364 \text{ N}$$

Since the weight of the drum is

$$W = 20 \times 9.81 = 196 \text{ N}$$

the vertical reaction on the bearing should be the sum, i.e.,

$$1364 + 196 = 1560 \text{ N}$$

***Example* S3.15** Two pulleys, one 450 mm in diameter and the other 200 mm in diameter, are on parallel shafts 1.95 m apart and are connected by a crossed belt as shown in Fig. Ex. S3.15. Find the length of the belt required and the angle of contact between the belt and each pulley.

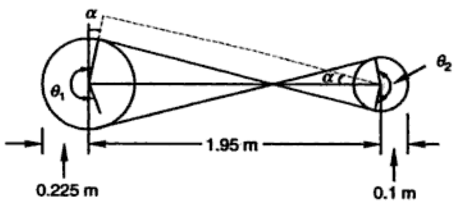


Fig. Ex. S3.15

What power can be transmitted by the belt when the larger pulley rotates at 200 rev/min, if the maximum permissible tension in the belt is 1 kN and the coefficient of friction between the belt and the pulley is 0.25?

Solution From the geometry of the problem,

$$\sin\alpha = \frac{0.225 + 0.1}{1.95} = 0.167; \qquad \alpha = 9.6°$$

$$\theta_1 = 180° + 2 \times 9.6 = 199.2°$$

$$= 199.2 \times \frac{\pi}{180} = 3.477 \text{ rad}$$

$$\theta_2 = \theta_1 = 3.477 \text{ rad}$$

Length of the belt is given by

$$L = 0.225(3.477) + 0.1(3.477) + 2 \times 1.95 \cos 9.6°$$

$$= 4.98 \text{ m}$$

According to the condition of impending slippage of the belt over the pulleys,

$$\frac{T_1}{T_2} = e^{\mu\theta} = e^{0.25 \times 3.477} = 2.385$$

or $$T_1 = 2.385\ T_2$$

If the greater of the tensions is allowed to the maximum permissible, i.e.,

$$T_1 = 1000 \text{ N}$$

then $$T_2 = 419 \text{ N}$$

The power transmitted must be given by

$$\text{Power} = (T_1 - T_2) \times \text{speed of the belt}$$

$$= (1000 - 419) \times \frac{\pi \times 0.45 \times 200}{60}$$

$$= 2738 \text{ W} = 2.738 \text{ kW}$$

***Example* S3.16** A mass of 500 kg is to be maintained in position by pulling a rope taken over a half barrel and wrapped twice around a capstan as shown in Fig. Ex. S3.16. If the coefficient of static friction is 0.2 for all contact surfaces, calculate the minimum force F required to maintain the load.

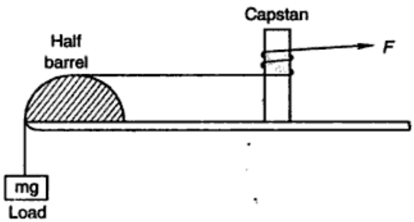


Fig. Ex. S3.16

Solution Consider the equilibrium of the segment of the rope taken over the half barrel. According to the condition of impending slippage of the rope downwards,

$$\frac{T_1}{T_2} = e^{0.2 \times \pi/2} = 137$$

or $$T_1 = 1.37\ T_2$$

whence $$T_2 = \frac{T_1}{1.37} = \frac{500 \times 9.81}{1.37} = 3583 \text{ N}$$

Now consider the equilibrium of the rope wrapped around the capstan. For the state of impending motion of the rope towards left, i.e., in lowering the load,

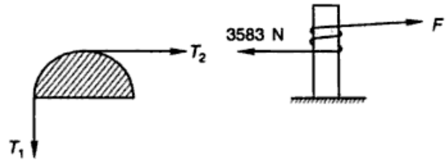


Fig. Ex. S3.16 ***(Solution)***

$$\frac{3583}{F} = e^{0.2 \times 4\pi} = 12.35$$

whence $$F = 290 \text{ N}$$

It may be seen that the load of 4905 N can be maintained in position against falling down, by the application of a force which is merely 20% of its magnitude through frictional devices.

Let us also work out the force F which would be required to initiate an upward motion of the load. The impending motion of the rope would then be upwards and towards right and by equilibrium considerations,

$$F = 500 \times 9.81 \times 1.37 \times 12.35$$

$$= 82{,}990 \text{ N} = 82.99 \text{ kN}$$

which is over 16 times the load itself. It follows that the rope passed over the half barrel and wound round the Capstan is a good device if the load is to be maintained in position but very inefficient if the load is to be lifted.

Example **S3.17** Two blocks A and B are to be held in position by means of an inextensible rope passing over a fixed drum as shown in Fig. Ex. S3.17. The coefficient of friction between the blocks, between the block and the inclined surface, and between the rope and the drum is 0.2. The mass of B is 500 kg. Determine the minimum weight of A so that B is prevented from moving downwards. The drum cannot rotate.

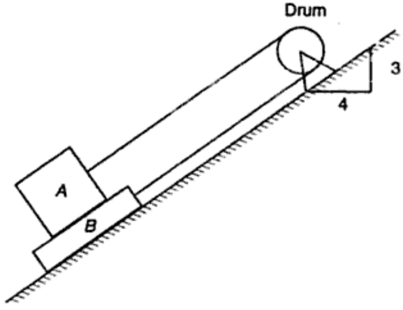


Fig. Ex. S.3.17

Solution A tendency of B to slip down the incline implies that the tendency of the upper block A must be to slip upward. The friction force is, therefore, shown accordingly. Friction on the non-rotatable drum implies that the tensions T_A and T_B on either side of the inextensible string are unequal and from the knowledge of impending motion

$$T_B > T_A$$

The inclination of the slope is specified by

$$\tan\theta = \frac{3}{4} \text{ or } \sin\theta = 0.6 \text{ and } \cos\theta = 0.8$$

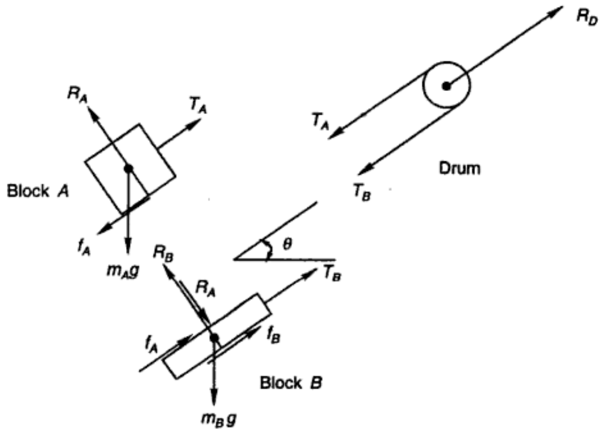


Fig. Ex. 3.17 ***(Solution)***

With reference to the free-body diagram for the block A, as shown in Fig. Ex. 3.17 (Solution) the conditions of equilibrium are

$$T_A - f_A - m_A\, g \sin\theta = 0$$

$$R_A - m_A\, g \cos\theta = 0$$

and the condition of impending motion is

$$f_A = \mu R_A$$

Substituting for the coefficient of friction and θ

$$T_A - 0.2R_A - 0.6\, m_A\, g = 0 \qquad \text{(i)}$$

$$R_A - 0.8\, m_A\, g = 0 \qquad \text{(ii)}$$

For block B,

$$T_B + f_A + f_B - m_B\, g \sin\theta = 0$$

$$R_B - R_A - m_B\, g \cos\theta = 0$$

and $$f_B = \mu R_B$$

whence $$T_B + 0.2\, R_A + 0.2\, R_B - (500 \times 9.81) \times 0.6 = 0$$

$$R_B - R_A - (500 \times 9.81) \times 0.8 = 0$$

or $$T_B + 0.2R_A + 0.2R_B = 2943 \quad \text{(iii)}$$

$$R_B - R_A = 3924 \quad \text{(iv)}$$

For the drum,

$$R_D - T_A - T_B = 0$$

Since $T_B > T_A$

$$T_B / T_A = e^{\mu\phi} = e^{0.2\pi} = 1.874 \quad \text{(v)}$$

We have now set up five equations, (i) to (v), for the five unknowns T_A, T_B, R_A, R_B and m_A of which only the last one is desired. Even so, the simultaneous set of equations must be solved.

Obtaining $T_B = 1.874\, T_A$ from Eq. (v),

$R_A = 0.8\, m_A\, g$ from Eq. (ii),

$$R_B = R_A + 3924 = 0.8\, m_A\, g + 3924$$

from Eq. (iv) and substituting in Eq. (i)

$$T_A - 0.16\, m_A\, g - 0.6\, m_A\, g = 0$$

or $$T_A = 0.76\, m_A\, g$$

and now substituting in Eq. (iii),

$$1.874(0.76\, m_A\, g) + 0.16\, m_A\, g + 0.16\, m_A\, g + 785 = 2943$$

whence $$m_A\, g = 1237$$

and $$m_A = 126 \text{ kg}$$

The minimum mass of block *A* should, therefore, be 126 kg; for m_A less than 126, the lower block would slide down the incline and the upper block up the incline.

S4.5 LIFTING BY A SCREW JACK

A screw jack consists of a square-threaded central rod called a screw fitted into the internally-threaded collar of a jack. The load *W* is placed on the screw and the effort *F* is applied horizontally at the end of a lever of arm *L* as shown in Fig. S3.6. The lifting action of the screw jack takes place through the normal and frictional forces developed at the threaded surface of contact within the collar. Considering a typical

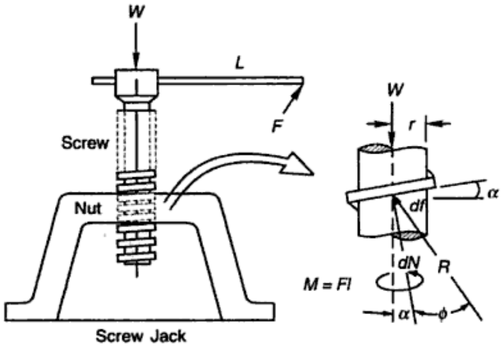


Fig. S3.6 ***Analysis of a Screw Jack***

element of the threaded surface in contact, the normal force is dN and the friction force is df opposing the input moment

$$M = FL$$

constituted by the effort F applied at an arm L. The total reaction force at the contact surface must balance the vertical force W and horizontal component H given by

$$H = M/r = FL/r$$

At the state of impending motion, or during the lifting action of the screw, the resultant reaction force makes an angle ϕ with the normal where $\phi = \tan^{-1} \mu$ and μ is the static coefficient or kinetic coefficient of friction as the case may be.

If α is the pitch angle of the screw, then for equilibrium,

$$R \cos (\alpha + \phi) = W$$

$$R \sin (\alpha + \phi) = M/r = FL/r$$

and from these equations,

$$F = W \tan (\alpha + \phi)\, r/L \qquad \text{(S3.4)}$$

If the load W was being lowered instead of being raised, the friction force would be directed up the incline and the resultant reaction R would be inclined to the line of action of W by an angle $(\alpha - \phi)$. The force F required to lower the load would be given by

$$F = W \tan (\alpha - \phi) \times r/L \qquad \text{(S3.5)}$$

which is indeed much smaller than that required to raise the load.

The efficiency of a screw jack is defined as the ratio of the work output to the work input over the same period of time. Work input for one revolution of the effort

$$= 2\, \pi LF = W \cdot 2\pi r \tan (\alpha + \phi)$$

Work output in the same duration

$$= W \times \text{pitch of the screw}$$

$$= W 2\pi r \tan\alpha$$

Efficiency
$$\eta = \frac{W \cdot 2\pi r \tan\alpha}{W \cdot 2\pi r \tan(\alpha + \phi)}$$

or
$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)} \tag{S3.6}$$

The efficiency can alternatively be interpreted as the ratio of the force required to lift a load in the absence of friction to the actual force required to lift the load.

$$F\text{ (without friction)} = W \tan(\alpha + 0) \cdot r/L$$

$$F\text{ (actual)} = W \tan(\alpha + \phi) \cdot r/L$$

$$\eta = \frac{F\text{ (without friction)}}{F\text{(actual)}} = \frac{\tan\alpha}{\tan(\alpha + \phi)}$$

An expression for the maximum efficiency of a screw jack can be obtained by setting

$$d\eta/d\alpha = 0$$

or
$$\frac{1}{\cos^2\alpha \tan(\alpha + \phi)} - \frac{\tan\alpha}{\sin^2(\alpha + \phi)} = 0$$

or
$$\sin 2(\alpha + \phi) = \sin 2\alpha$$

whence
$$\alpha = \pi/4 - \phi/2$$

The maximum efficiency is given by

$$\eta_{max} = \frac{\tan(\pi/4 - \phi/2)}{\tan(\pi/4 + \phi/2)} = \frac{1 - \sin\phi}{1 + \sin\phi} \tag{S3.7}$$

Assuming that the coefficient of friction is 0.15, $\phi = \tan^{-1} 0.15 = 8.53$ degrees; $\alpha = \pi/4 - 8.53/2 = 40.73$ degrees $\eta_{max} = 74.2\%$.

Let us also examine the condition for a screw jack to be self-locking. A screw jack is called self-locking if, in the absence of the applied moment FL, the screw jack does not unwind to lower the load. Since

$$F \times L = W \tan(\alpha - \phi) \times r$$

Equating $F \times L$ to zero or making it negative implies

$$\tan(\alpha - \phi) \le 0$$

or
$$\tan\alpha - \tan\phi \le 0 \tag{S3.8}$$

or
$$\alpha \le \phi$$

$$\tan\alpha \le \mu$$

The screw jack will be self-locking if tan α equals or is less than μ.
From the expression for its efficiency

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

it follows that, if $\alpha = \phi$

$$\eta = \frac{\tan \alpha}{\tan 2\alpha} = \frac{\tan \alpha(1 - \tan^2 \alpha)}{2 \tan \alpha}$$

$$= 0.5 - 0.5 \tan^2 \alpha \qquad \text{(S3.9)}$$

which must be less than 50%.

***Example* S3.18** A screw jack requires a force F applied at a radius a on a handle to lift a load W on top of it. Determine the M/F for raising the load for a helix angle α for the screw and coefficient of friction μ. Would the M/F be different for lowering the load?

Solution For an elemental area ds of the surface of the screw, the normal reaction is pds where p is the normal pressure on it. The vertical force is (ref. Fig. Ex. S3.18 (a) (Solution))

$$pds \cos \alpha - \mu pds \sin \alpha$$

For the screw jack

$$W = \int (p \cos \alpha - \mu\, p \sin \alpha)\, ds = (p \cos \alpha - \mu\, p \sin \alpha)\, A$$

and $$M = \int r\,(p \sin \alpha + \mu p \cos \alpha) ds = F \cdot a = r\,(p \sin \alpha + \mu \phi \cos \alpha)\, A$$

$$\frac{M}{F} = \frac{r\,(\tan \alpha + \mu)}{1 - \mu \tan \alpha}$$

Now, for lowering the load, the screw is turned in the reverse direction; the frictional force acts in the opposite direction (see Fig. Ex. S3.18(b) (Solution)).

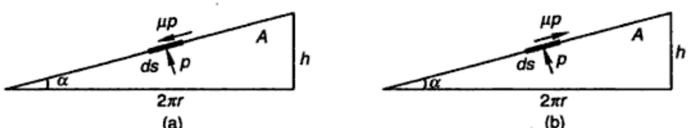


Fig. Ex. S3.18 ***(Solution)***

We get $W = (p \cos \alpha + \mu\, p \sin \alpha)\, A$ (ref. Fig. Ex. S3.18(b) (Solution))

$$M = r\,(p \sin \alpha - \mu\, p \sin \alpha)$$

and hence $$M/F = \frac{r\,(\mu - \tan \alpha)}{1 + \mu \tan \alpha}$$

It is, of course, less than the M/F value for raising the load.

A screw jack is said to be self-locking if it does not let the load come down even if the applied moment or force at the lever arm is zero, i.e., it does need a moment or a force at the lever arm to lower the load,

$$\mu > 0$$

or $$r(\mu - \tan\alpha) > 0$$

or $$\tan\alpha < \mu$$

or $$\tan\alpha < \tan\phi$$

or $$\alpha < \phi$$

i.e., when the helix angle is less than the friction angle.

***Example* S3.19** A screw thread of a screw jack has a mean diameter of 10 cm and a pitch of 1.25 cm. The coefficient of friction between the screw and its nut-housing is 0.25. Determine the force F that must be applied at the end of a 50 cm lever arm to raise a mass of 5000 kg. Is the device self-locking? Also determine its efficiency.

Solution From the definition of the pitch of a screw,

$$\tan\alpha = \frac{p}{2\pi\gamma} = \frac{1.25}{2\pi \times 5} = 0.04$$

$$\alpha = 2.28°$$

Also, $$\mu = 0.25 = \tan\phi;\ \phi = 14.04°$$

The force required at the end of 50 cm long lever is given by

$$F = W\tan(\alpha + \phi) \times r/L$$

$$= 5000 \times 9.81 \tan(2.28 + 14.04) \times 5/50$$

$$= 1436 \text{ N}$$

For the screw jack

$$\alpha < \phi$$

Hence, the screw jack must be self-locking.

The efficiency of the jack is given by

$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)} = \frac{0.04}{0.293} = 0.137$$

$$= 13.7\%$$

The force required at the end of the lever to lower the load may be determined as follows:

$$F = W\tan(\alpha - \phi) \times r/L$$

$$= 5000 \times 9.81 \tan(2.28 - 14.04) \times 5/50$$

$$= -1021 \text{ N}$$

The force required to lower the load is thus 1021 N in a direction opposite to that required to raise it.

S3.6 DISC AND BEARING FRICTION

It is often necessary to estimate the torque required to overcome the frictional resistance offered by a surface to the rotation of the other surface. It may also be of interest to estimate the power lost in friction at a given speed of rotation.

Consider, for example, the end of a rotating cylindrical shaft resting on a flat surface. Such a pair of surfaces constitute a thrust bearing. Let the axial force transmitted by the shaft on the flat pad be P. The force may be distributed uniformly or non-uniformly over the area of contact. The problem is dealt with by considering an elementary area of contact subtending angle $d\theta$ at the centre and of width dr at a radius r as shown in Fig. S3.7. The area of the strip is

$$dA = r dr\, d\theta$$

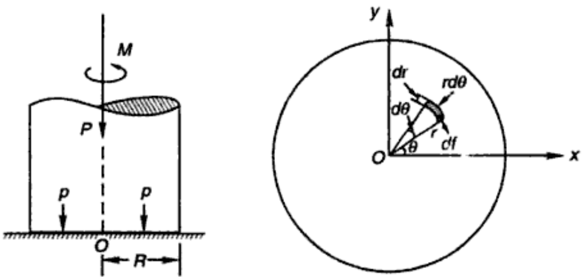


Fig. S3.7 ***Analysis of Disc Friction***

The normal force and hence the normal reaction at the strip is

$$dF_n = pr\, dr\, d\theta$$

where p is the intensity of normal force over the small area. The elementary frictional force over the area, acting tangent to the radius, must be given by

$$df = \mu_d\, dF_n$$
$$= \mu_d pr\, dr\, d\theta$$

where μ_d is the coefficient of dynamic friction between the contacting surfaces.

The elementary moment exerted by the elementary frictional forces acting at a radius r from the centre and at an angle θ from the x-axis about the axis of the shaft is

$$dM = df\, r$$
$$= \mu_d pr^2\, dr\, d\theta$$

and the moment exerted by the entire frictional effect at the contacting surfaces is

$$M = \int_0^R dM = \int_0^{2\pi}\int_0^R \mu_d p r^2\, dr\, d\theta = \int_0^R \mu_d \times p 2\pi r^2\, dr$$

$$= 2\pi\mu_d \int_0^R p r^2 dr \qquad \text{(S3.10)}$$

Integration of this expression may be carried out for an assumed or given distribution of p with r. Let us consider two cases:

Case (a)

If $$p = \text{constant} = \frac{P}{\pi R^2}$$

then $$M = 2\pi \mu_d\, p \int_0^R r^2\, dr$$

or $$M = 2\pi\, \mu_d p\, R^3/3 = 2/3\ P\ R\mu_d \qquad \text{(S3.11)}$$

Case (b)

If the pressure intensity is assumed to be inversely proportional to the radius,

$$p = \frac{a}{r}$$

$$P = \int_0^R 2\pi r p\, dr = 2\pi a R$$

whence $$a = \frac{P}{2\pi R}$$

and $$p = \frac{P}{2\pi Rr}$$

With this expression, the moment is determined as

$$M = 2\pi\mu_d \int_0^R \frac{P}{2\pi Rr} r^2 dr$$

$$= 1/2\ PR\mu_d \qquad \text{(S3.12)}$$

***Example* S3.20** A shaft of radius R with an axial load F acting on it rests on a flat thrust bearing of radii R_1 and R_2 as shown in Fig. Ex. S3.20. Determine the torque, M for impending rotation of the shaft.

Solution Consider an elementary area

$$dA = r\ \delta\theta \times \delta r$$

at a radial distance r, subtending an angle $d\theta$ at the centre as shown in Fig. Ex. S3.20 (Solution).

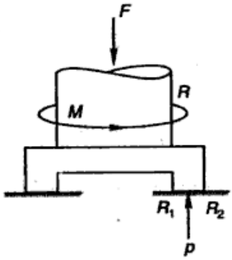


Fig. Ex. S3.20

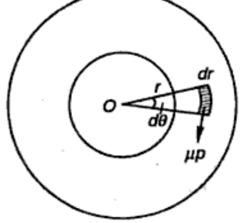


Fig. Ex. S3.20 *(Solution)*

For a pressure p, the moment is given by

$$M = \int_{R_1}^{R_2} \int_{0}^{2\pi} r \cdot \mu \, p \cdot r \, \delta\theta \, \delta r$$

and
$$F = \int_{R_1}^{R_2} \int_{0}^{2\pi} p \cdot r \, \delta\theta \, \delta r$$

If
$$p = \text{constant}$$

$$M = \frac{2}{3} \mu \pi (R_2^3 - R_1^3) p$$

$$F = \pi (R_2^2 - R_1^2) p$$

and
$$M/E = \frac{2}{3} \mu \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$$

If p varies inversely as r, $p = \dfrac{C}{r}$

then
$$M/F = \mu \frac{R_1 + R_2}{2}$$

***Example* S3.21** A shaft of radius R with axial load F is provided with a conical thrust bearing of radii R_1 and R_2 and cone angle α as shown in Fig. Ex. S3.21. Determine the torque needed to rotate it.

Solution Consider an elementary area $dA = r \cdot \delta\theta \cdot \delta r/\sin \alpha$ at a radial distance r, subtending a small angle $\delta\theta$ at the centre.

For a constant pressure p the frictional moment is given by

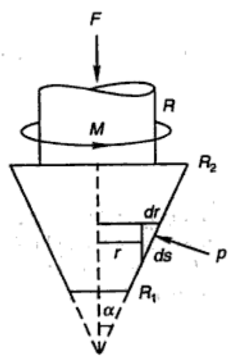


Fig. Ex. S3.21

$$M = \int_{R_1}^{R_2} \int_{0}^{2\pi} r \cdot \mu \cdot p \cdot r\delta\theta \cdot \delta r/\sin \alpha = \mu p \frac{R_2^3 - R_1^3}{3} \sin \alpha$$

and the thrust is

$$F = \int_{R_1}^{R_2} \int_{0}^{2\pi} p \cdot r\delta\theta \cdot \delta r/\sin\alpha \cdot \sin\alpha = p\frac{R_2^2 - R_1^2}{2}$$

$$M/F = \frac{2}{3}\mu/\sin\alpha\left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right)$$

If pressure p varies inversely as the radius,

$$p = \frac{C}{r}$$

then,
$$M/F = \frac{\mu}{\sin\alpha}\frac{R_1 + R_2}{2}$$

***Example* S3.22** The conical end of a shaft of diameter 5 cm rests in a conical bearing of cone angle 60° as shown in Fig. Ex. S3.22. If the coefficient of dynamic friction is 0.3, calculate the frictional torque and power required to rotate the shaft at 1000 revolutions per minute, if the axial load on the shaft is 5 kN.

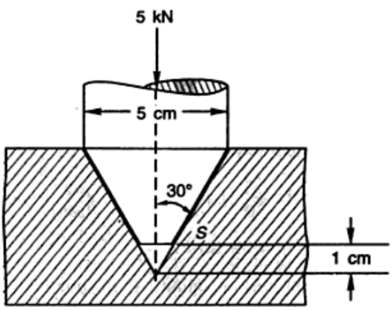


Fig. Ex. 3.22

Solution Consider a strip of the conical surface in contact. For an inclined length ds, the area of the strip is

$$dA = 2\pi(s \sin 30°)\, ds$$

where s is the inclined distance on the surface measured from the apex.

The axial force due to pressure p on the strip is

$$dP = dA p \sin 30°$$

$$= \frac{\pi}{2} ps\, ds$$

and the total axial force is

$$P = \int dp$$

The limits of integration are from

$$S_1 = \frac{1}{\cos 30^\circ} \text{ or } 1.155 \text{ cm to } S_2 = 5 \text{ cm}$$

$$P = \int_{0.01155}^{0.05} \frac{\pi}{2} ps\, ds$$

$$= \frac{\pi}{2} p\, s^2/2 \Big|_{0.0115}^{0.05}$$

Since $\qquad P = 5$ kN

$$5 = 0.001\ 86\, p$$

whence $\qquad p = 2690$ kN/m^2

The frictional force df on the strip must be given by

$$df = \mu_d p\, dA$$

$$= 0.3 \times 2690 \times 2\pi s \sin 30^\circ\, ds \text{ kN}$$

and the frictional moment dN is such that

$$dM = df\, s \sin 30^\circ$$

$$= 1267\, s^2\, ds \text{ kN m}$$

The total frictional moment is given by

$$M = \int dM = 1267 \left| s^3/3 \right|_{0.0115}^{0.05} \text{ kN m}$$

$$= 52 \text{ Nm}$$

The speed of 1000 revolutions per minute corresponds to

$$\omega = \frac{2\pi \times 1000}{60} = 104.7 \text{ rad/s}$$

The frictional power is given by

$$\text{Power} = M\omega = 52 \times 104.7 = 5445 \text{ W}$$

$$= 5.445 \text{ kW}$$

***Example* S3.23** A simple disc brake consists of four shoes, each subtending an angle α with the inner radius R_1 and outer radius R_2. The shoes are pressed against a coaxial rotating disc with a force $4P$. Determine the torque exerted on the disc if the coefficient of static frictions is μ.

Solution The area of contact of a single shoe is (ref. Fig. Ex. S3.23 (Solution))

$$A = \frac{\alpha}{2}(R_2^2 - R_1^2)$$

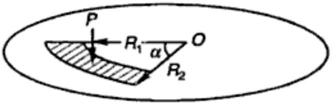


Fig. Ex. S3.23 *(Solution)*

and the pressure intensity is given by

$$p = \frac{4P}{4}\frac{1}{A} = \frac{2P}{\alpha(R_2^2 - R_1^2)}$$

The elementary frictional moment on an element of area $rd\theta\, dr$ is

$$dM = r\,df = r\mu\, dF_n = r\mu pr\, d\theta \times dr = \frac{2P\mu}{\alpha(R_2^2 - R_1^2)} \times r^2\, d\theta\, dr$$

and the total moment for one shoe is given by

$$M = \int dM$$

$$= \frac{2P\mu}{\alpha(R_2^2 - R_1^2)} \int_{R_1}^{R_2} \int_0^{\alpha} r^2\, dr\, d\theta$$

$$= \frac{2\mu P(R_2^3 - R_1^3)}{3(R_2^2 - R_1^2)}$$

For the shoe brake with four shoes, the moment is

$$M' = \frac{8\mu P(R_2^3 - R_1^3)}{3(R_2^2 - R_1^2)}$$

It may be noted that the resulting torque is independent of the angle α subtended by each shoe at the centre of the disc brake.

S3.7 INPUT/OUTPUT OF SIMPLE MACHINES

A machine is a device which enables us to employ the input to advantage for achieving a desired output. A machine may consist of a single element or an assemblage of elements. Machines are classified as electrical, mechanical, electromechanical and others depending upon the nature of the input and output. We confine ourselves to the consideration of a simple mechanical machine where the input may be a small force or a moment at a convenient point and the output may be a larger load being lifted or moved against resistance.

It is usual to employ the terms mechanical advantage, velocity ratio and mechanical efficiency to describe the features of a lifting machine. The *mechanical advantage* is the ratio of the load lifted to the effort applied:

$$\text{Mechanical advantage} = \frac{\text{Load lifted}}{\text{Effort applied}} = \frac{W}{P} \qquad \text{(S3.13)}$$

The *velocity ratio* refers to the ratio of the velocities of the points of application of the effort and load. Assuming the lifting process to take place steadily, i.e., at the constant rate, the velocity ratio over an interval of time may also be defined as

$$\text{Velocity ratio} = \frac{\text{Distance moved by the point of application of effort}}{\text{Distance moved by the load}} = \frac{y}{x} \quad \text{(S3.14)}$$

where the distances are measured along the directions of the respective forces as shown in Fig. S3.8.

The *efficiency* of a machine is defined as the ratio of the work output to the work input, i.e.,

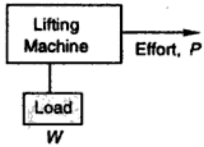


Fig. S3.8 ***Input and Output of a Lifting Machine***

$$\text{Efficiency, } \eta = \frac{\text{Work output}}{\text{Work input}} = \frac{Wx}{Py}$$

$$= \frac{W}{P} \Big/ \frac{y}{x}$$

$$= \frac{\text{Mechanical advantage}}{\text{Velocity ratio}} \quad \text{(S3.15)}$$

A machine is said to be *ideal* if the efficiency is 100% which may be so in the absence of dissipation actions, such as friction. For an ideal machine, therefore,

$$\text{Mechanical advantage} = \text{Velocity ratio}$$

In actual practice, some energy must be lost in dissipative action and the efficiency is consequently less than 100%. The mechanical advantage must, therefore, be less than the velocity ratio. It follows that

$$\frac{W}{P} \le \frac{y}{x} \quad \text{(S3.16)}$$

depending upon whether a machine is ideal or not. Let us consider the reversed operation of a machine. The reversed operation refers to the movement of the machine components under the application of the load only when the effort is removed. It is possible when the work done by the load overcomes the frictional work over the same time interval,

$$Wx \ge Py - Wx$$

or

$$2Wx \ge Py$$

or

$$\frac{W}{P} \Big/ \frac{y}{x} \ge 0.5$$

or

$$\eta \ge 0.5 \quad \text{(S3.17)}$$

A machine may, therefore, operate in the reverse direction on the removal of the effort if its efficiency exceeds or equals 50%. It is often necessary to stop the reversed operation of a machine. This may be done by reducing the efficiency to less than 50%. The machine is then said to be *self-locking*. The condition of self-locking is contradictory to the condition of an ideal operation.

The relationship between the load lifted and effort required, sometimes called *law of the machine* is such that

$$P = aW + b \tag{S3.18}$$

which shows that the effort bears a linear relationship with the load as shown in Fig. S3.9. The relationship shows that a minimum effort equal to the intercept b is required to lift a load, however small, because the frictional resistance is to be overcome. The effective frictional resistance at the point of application of the effort

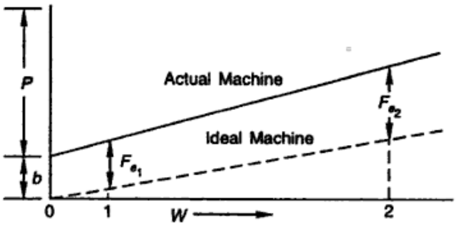


Fig. S3.9 ***Law of the Machine***

is defined as the difference between the ideal effort for a frictionless machine and the actual effort for a real machine. The effective frictional resistance F_c increases with the load as also shown in Fig. S3.9. From the expression for mechanical efficiency,

$$\eta = \frac{W}{P} \Big/ \frac{y}{x}$$

and the law of the machine

$$P = aW + b$$

the efficiency may be written as

$$\eta = \frac{W}{aW + b} \Big/ \frac{y}{x} = \frac{x/y}{a + b/W} \tag{S3.19}$$

This expression for η shows that the efficiency of a machine must be zero at zero load and that the efficiency increases as the load increases.

As $$W \to \infty, \eta \to \frac{x}{ay}$$

The maximum efficiency is thus given by

$$\eta_{\max} = \frac{1}{a \times \text{velocity ratio}} \tag{S3.20}$$

as shown in Fig. S3.10, and the corresponding mechanical advantage is given by

$$\text{Mechanical advantage} = \eta \times \text{velocity ratio} = \frac{1}{a} \qquad \text{(S3.21)}$$

Lifting machines may employ one or more of elements such as levers, pulleys, gears, inclined planes and screw-threads.

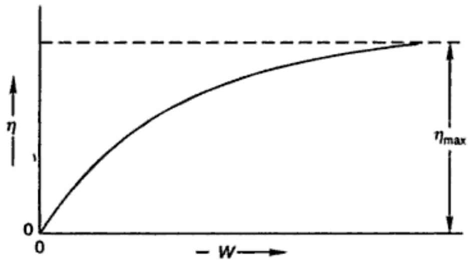


Fig. S3.10 ***Variation of Efficiency with Load***

Example S3.24 A drum of mass 6 kg holding water of mass 40 kg is to be raised from a well by the application of a 120 N human force. Would you recommend the use of a single pulley of diameter 10 cm, a simple wheel and axle of diameters 40 cm and 10 cm or a differential wheel and axle of diameters 40 cm, 10 cm and 5 cm? (Ref. Fig. Ex. 3.24)

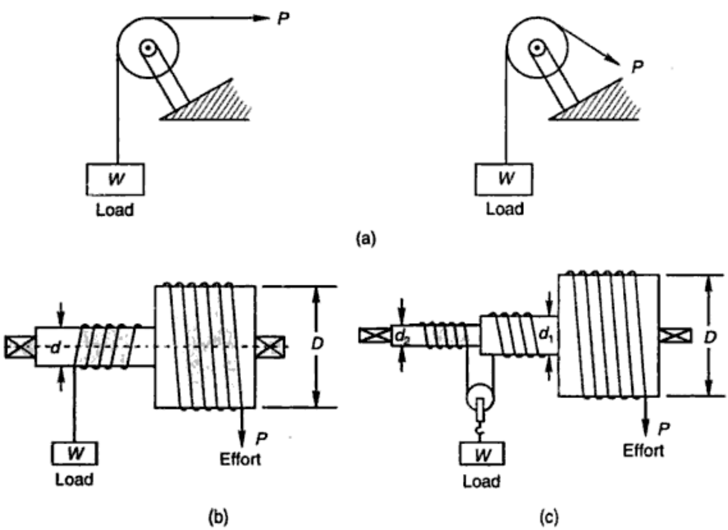


Fig. Ex. 3.24

Solution The requirement of the lifting machine to lift a load

$$W = (6 + 40) \times 9.81 = 451.3 \text{ N}$$

by an effort $P = 120$ N

The mechanical advantage is

$$\frac{W}{P} = \frac{451.3}{120} = 3.76$$

A single pulley has a velocity ratio of 1.0. If it operates without friction, its efficiency may be 100% and the mechanical advantage is then 1.0. If the mechanical efficiency is short of 100% due to friction, etc., the mechanical advantage would be less than 1.0. Thus a single pulley may have a mechanical advantage of 1.0 under ideal conditions. It is usefully employed if it is desired to change the direction of the effort applied with respect to the direction of the load being lifted as shown in Fig. Ex. S3.24(a). It is unsuitable for the present problem where the mechanical advantage desired is far beyond 1.0.

A simple wheel and axle of diameters D and d as shown in Fig. Ex. S3.24(b) operates such that the ratio of the distance moved by the effort to the distance moved by the load, i.e.,

$$\text{Velocity ratio} = \frac{\pi D}{\pi d} = \frac{D}{d}$$

which equals 40/10 = 4.0 in the given case. The efficiency of the machine would be

$$\eta = 3.76/4.0 = 0.94 = 94\%$$

A differential wheel and axle of diameters D, d_1 and d_2 as shown in Fig. Ex. S3.24(c) operates such that

$$\text{Velocity ratio As} = \frac{\pi D}{\pi d_1 - \pi d_2} = \frac{D}{d_1 - d_2}$$

which, in the present case, would be 40/(10 – 5) = 8.0. The efficiency of the machine would then be

$$\eta = 3.76/8.0 = 0.47 = 47\%$$

In order to decide the preference in favour of the simple or differential wheel and axle, it may be noted that both are adequate as far as the lifting action is concerned. A simple wheel and axle is more efficient but the device is capable of operating in the reversed direction if the effort is removed. On the other hand, the differential wheel and axle has less than 50% efficiency, a fact which ensures self-locking when the effort is removed.

For the simple wheel and axle, the effort lost in overcoming the frictional resistance equals the actual effort minus the ideal effort

$$120 - \frac{451.3}{4.0} = 7.2 \text{ N}$$

whereas the effort lost in overcoming friction in the differential wheel and axle is

$$120 - \frac{451.3}{8.0} = 63.6 \text{ N}$$

One would prefer the simple wheel and axle arrangement unless the need for a self-locking arrangement can justify the additional effort lost in friction.

***Example* S3.25** Determine the effort required at the end of an arm 40 cm long to lift a load of 5 kN by means of a simple screw jack with screw threads of pitch 1 cm if the efficiency at this load is 45%. Also calculate the effort needed if the jack is converted into a differential screw jack with internal threads of pitch 7 mm and the efficiency of operation is 30% (ref. Fig. Ex. 3.25).

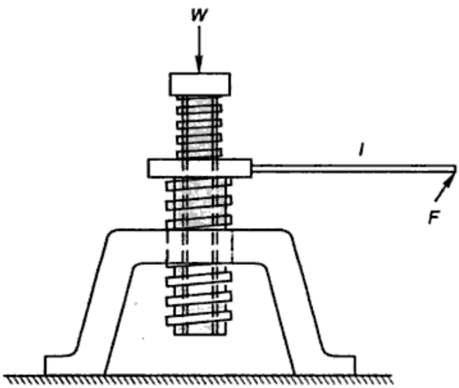


Fig. Ex. S3.25

Solution The velocity ratio of the simple screw jack is given by the ratio of the distance moved by the point of application of the effort, i.e., $2\pi \times 0.40 = 0.8\pi$ m and the distance moved by the load, i.e., 0.01 m if the arm is turned by one revolution

$$\text{Velocity ratio} = \frac{0.8\pi}{0.01} = 251.3$$

The mechanical advantage must be

$$W/P = 251.3 \times 0.45 = 113.1$$

whence $$P = W/113.1 = 5000/113.1 = 44.2 \text{ N}$$

If the jack is converted into a differential screw jack, the distance moved by the point of application of the effort over one revolution remains the same, i.e., 0.8π but the corresponding distance through which the load moves becomes

$$0.01 - 0.007 = 0.003 \text{ m}$$

and the velocity ratio becomes

$$0.8\pi/0.003 = 837.8$$

The mechanical advantage must now be

$$W/P = 837.8 \times 0.30 = 251.3$$

whence $$P = 5000/251.3 = 19.9 \text{ N}$$

***Example* S3.26** A carpenter's hand drill consists of a spindle AB which has a drill at A and a bearing at B as shown in Fig. Ex. 3.26. A rope is wrapped round the spindle 4 turns and the ends of the rope are tied to a handle CD. During operation, the handle is applied an effort P with one hand while the block at B is kept pressed with a force W with the other hand. Calculate the maximum torque which can be produced at the drill if the spindle has a diameter of 5 cm and the coefficient of friction between the rope and the spindle surface is 0.15. The tension in the slack side of the rope may be assumed to be 5 N.

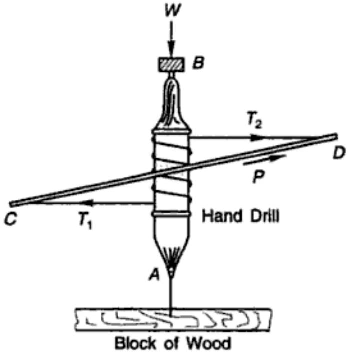


Fig. Ex. S3.26

Solution The torque at the drill would be maximum when the torque applied at the spindle is maximum, the two being equal. In the limiting case,

$$\frac{T_2}{T_1} = e^{\mu\theta} \tag{i}$$

and the torque is given by

$$M = (T_2 - T_1)\, r \tag{ii}$$

with the usual rotation.

From the data,

$$T_1 = 5 \text{ N} \qquad \mu = 0.15$$

$$r = 5/2 = 2.5 \text{ cm} = 0.025 \text{ m}$$

$$\theta = 4 \text{ turns} = 4 \times 2\pi = 8 \text{ rad}$$

From (i),

$$T_2 = 5 \times e^{0.15 \times 8\pi} = 5e^{1.2\pi} = 217 \text{ N}$$

From (ii),

$$M = (217 - 5) \times 0.025 = 5.3 \text{ Nm}$$

Experiment E7

Performance of Simple Lifting Machines

Objective

To determine the effort required to lift a load and efficiency of lifting by some simple machines.

Apparatus

Simple lifting machines, such as a screw jack, wheel and differential axle, worm and worm wheel, *winch crab*, as shown in Figs. E7.1 to E7.4; metre rod and standard weights.

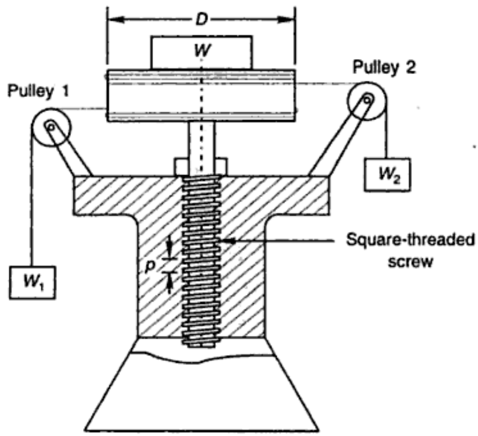


Fig. E7.1 ***Screw Jack***

Background Information

A lifting machine is employed to lift a larger load W at a point by employing a smaller force P at some other point. During the process, the distance y moved by the effort may be much more than the distance x, moved by the load:

$$\text{Work input} = Py$$

$$\text{Work output} = Wx$$

$$\text{Efficiency of the machine, } \eta = \frac{Wx}{Py} = \frac{W/P}{y/x}$$

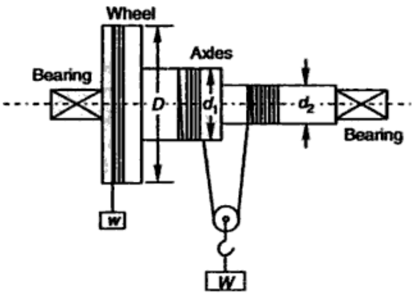


Fig. E7.2 ***Wheel and Differential Axle***

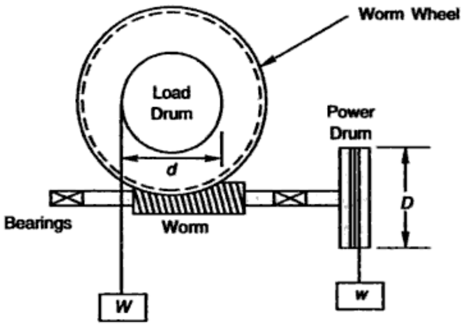


Fig. E7.3 ***Worm and Worm Wheel***

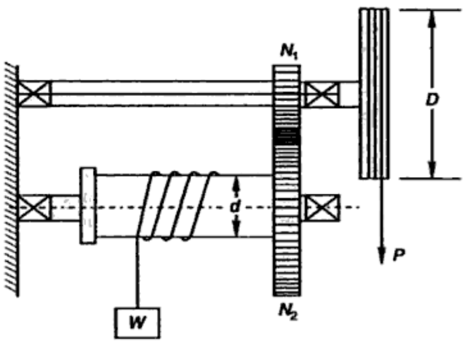


Fig. E7.4 ***Winch Crab***

$$\eta = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}}$$

where the term *mechanical advantage* stands for the ratio of the load to the effort applied to lift the load at a constant velocity and the term *velocity ratio* stands for the ratio of the distance moved by the effort to the distance moved by the load. Conventionally, the load is lifted vertically up and the point of application of effort is moved vertically down. The larger the load, the more the effort required. The plot of P vs. W is usually linear, whereas that between η and W is non-linear as shown in Fig. E7.5.

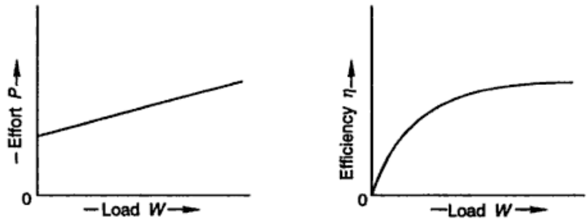


Fig. E7.5 ***Variation of Effort and Efficiency with Load***

While the effort applied to lift a given load at a constant velocity is recorded by measurement, the velocity ratio is determined from a consideration of the distances moved by the points of application of the load and effort.

For a screw jack, the distance moved by the effort is πD for the load to be lifted by a distance p where D is the diameter of the drum and p is the pitch of the screw

$$VR = \frac{\pi D}{p}$$

For a wheel and differential axle, the distance moved by the effort is πD for the load to be lifted by a distance $(\pi d_1 - \pi d_2)$ where D is the diameter of the wheel and d_1 and d_2 are the diameters of the axles on which the string is wound in the opposite directions:

$$VR = \frac{\pi D}{\pi d_1 - \pi d_2} = \frac{D}{d_1 - d_2}$$

For a *worm and worm wheel*, the distance moved by the effort is πD for the load to be lifted by a distance πd divided by the number of teeth N on the worm wheel where D and d are the diameters of the power drum and the load drum respectively

$$VR = \frac{\pi D}{\pi d / N} = \frac{ND}{d}$$

For a winch crab or lift, the distance moved by the effort is πD for the load to be lifted by a distance πd divided by the ratio N_2/N_1 of the teeth of the wheel and pinion where D and d are the diameters of the power drum and load drum respectively

$$VR = \frac{\pi D}{\pi d / (N_2 / N_1)} = \frac{N_2 D}{N_1 d}$$

Observations and Calculations

It is necessary to note the parameters which enter into the determination of the velocity ratio for a given machine. The velocity ratio *VR* is thus obtained. The effort required to lift a load at a constant speed is determined practically and the experiment is repeated for various loads in convenient steps.

S. No.	*W*	*P*	*W/P*	*VR*
1				
2				
.				
.				
.				

Result

Plots of P vs. W and η vs. W for the lifting machine may be made from the measured and calculated values. One may like to express the P vs. W plot in the form of a linear relationship.

$$P = aW + b$$

often called the *law of the machine*. The values of a and b are obtained from the plot.

Points for Discussion

1. A machine is said to be self-locking if the load stays in position even though the effort is removed. What is the condition, in terms of efficiency, for a machine to be self-locking? Which of the machines tested by you are self-locking?
2. How should the effort vary with the load for an ideal, i.e., a frictionless machine? How does the plot alter due to friction?
3. Assuming that the law of the machine is linear, what should be the maximum mechanical advantage and maximum efficiency of the machine?
4. Of the various lifting machines known to you, how would you decide which one to choose for a particular situation? For example, which lifting machine is best suited, in your opinion, for the following jobs:
   (a) lifting a drum of water from a well?
   (b) lifting a heavy consignment from a ship?
   (c) lifting the body of a truck for the purpose of changing a wheel?

Concept Review Questions

1. Comment on the nature of friction between two surfaces and the concept of impending motion.
2. A block of base dimensions $a \times b$ and height h is subjected to a horizontal force F at its mid-height. Draw the free-body diagram of the block
   (a) for small F
   (b) for large F

both being before the state of impending action. Examine the difference in the point of application of the resultant of the normal reaction and the frictional force at the base of the block.

3. Under what conditions can a cylinder roll down and under what conditions can it slide down an inclined plane?
4. In the analysis of a belt-drive, the relationship between the tensions in the tight side and the slack side incorporates the coefficient of static friction μ. Explain why it is not the coefficient of dynamic friction instead.
5. Whenever a rope or a wire under tension is to be held, it is wrapped round a tree-trunk or a pole by giving it a number of turns. Why?
6. Explain why the lifting action of a screw jack is likened to pushing up an incline.
7. From the expression for the efficiency of a screw jack,

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

where α is the equivalent inclination and ϕ is the friction angle, obtain the expression for the maximum efficiency and also the efficiency for it to be self-locking.

8. Define and relate the mechanical advantage velocity ratio and efficiency of a machine. What is meant by the law of the machine.

Tutorial Problems

S3.1 Figure Prob. S3.1 shows the location of the centre of gravity in a model of a car which can be driven either by its rear or front wheels. If the inertia of the wheels, rolling resistance and aerodynamic drag are ignored, what is the ratio of the maximum accelerations in the two cases when the coefficient of friction between the tyres of the car and ground is 0.7?

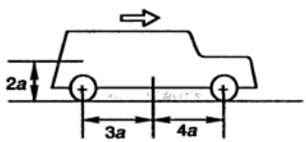


Fig. Prob. S3.1

S3.2 In Fig. Prob. S3.2, the coefficients of friction between the weight W and the wedge is μ_1 and between the wedge and the lower

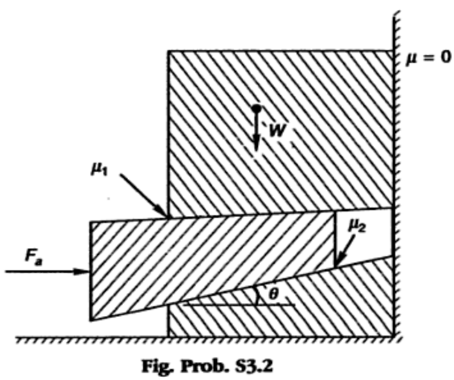


Fig. Prob. S3.2

block μ_2. Ignoring the weight of the wedge and assuming no friction, determine the applied force F required to raise W by forcing the wedge to the right.

$$\left[\textbf{Ans. } W\left(\mu_1 + \frac{\mu_2 + \tan\theta}{1 - \mu_2 \tan\theta}\right)\right]$$

S3.3 A homogeneous ladder 6 m long and weighing 400 N rests against a smooth wall. The angle between it and the floor is 70°. The coefficient of friction between the floor and the ladder is 0.25. How far up the ladder can a 80 kg man walk before the ladder slips?

(**Ans.** 4.41 m)

S3.4 A horizontal force, F_a is applied to a block which rests on an inclined plane, as shown in Fig. Prob. S3.4. Find the force required to initiate motion up the plane.

$$\left(\textbf{Ans. } F_a = W\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)$$

S3.5 The block of weight W is on a surface (the coefficient of friction is μ) which is inclined at angle θ with the horizontal as shown in Fig. Prob. S3.5. This surface is part of a triangular block of weight W_1. A horizontal force P causes the system to have an acceleration a to the right. What value of P will cause the top block to move relative to the surface? Assume no friction of the bottom surface. What is the acceleration?

$$\left(\textbf{Ans. } a = \frac{g(\mu - \tan\theta)}{(1 + \mu\tan\theta)},\ P = \frac{(W + W_1)(\mu - \tan\theta)}{1 + \mu\tan\theta}\right)$$

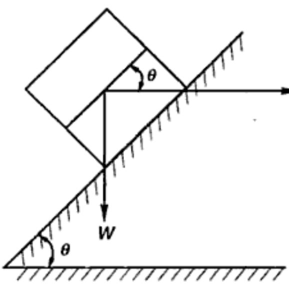


Fig. Prob. S3.4

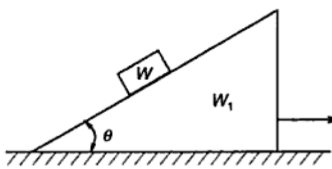


Fig. Prob. S3.5

S3.6 A uniform rod of mass m and length L is lying on a rough horizontal table. A horizontal force P is applied to the rod perpendicular to its axis at a distance $kL(k > \frac{1}{2})$ from one end so that it just moves. Show that the rod rotates about a point distance hL from the same end and that $P = \mu mg\,(1 - 2h)$, where $h = k - (k^2 - k + \frac{1}{2})^{1/2}$.

S3.7 A bar rests on two pegs and makes an angle β with the horizontal. The coefficients of friction are μ_1 at one peg, which is at a distance a from G, the centre of gravity of the bar, and μ at the other peg at distance b from G. Show that for an equilibrium condition to exist.

$$\tan \beta \leq \frac{\mu_1 b + \mu_2 a}{(a+b)}$$

S3.8 At what height above the surface of a billiard table should a ball of radius r be struck by a horizontal force F in order to have no sliding at the point of contact? (as shown in Fig. Prob. S3.8).

S3.9 A cylinder of diameter d weighing W rests at a corner of two surfaces as shown in Fig. Prob. S3.9. Prove that the maximum force P that can be applied as shown without causing the cylinder to rotate is 3/8 W. Take the coefficient of friction for each pair of contacting surfaces as 0.5.

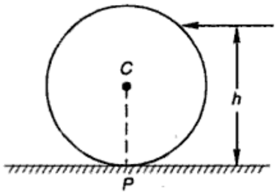


Fig. Prob. S3.8

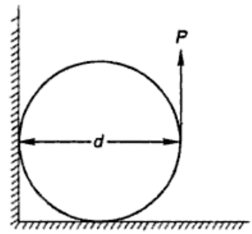


Fig. Prob. S3.9

S3.10 Find the minimum weight W of the triangular block such that it remains in equilibrium under the action of the force 1 kN applied to it as shown in Fig. Prob. S3.10. Take $\mu = 0.25$.

(**Ans.** 2.12 kN)

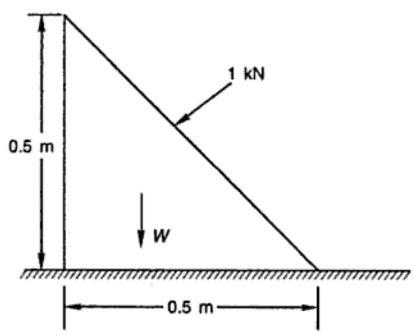


Fig. Prob. S3.10

S3.11 A cylinder of diameter 0.3 m and mass 25 kg rests on a rough surface as shown in Fig. Prob. S3.11, with $\mu_s = 0.4$ and $\mu_d = 0.35$. Determine the force P to be applied shown to roll the cylinder without slip over the step.

(**Ans.** P = 70.5 N)

S3.12 A cylinder weighing 2670 N, 1.2 m in diameter is acted upon by a force of 445 N, as shown in Fig. Prob. 3.12 with the help of a cord wrapped around it. Determine the

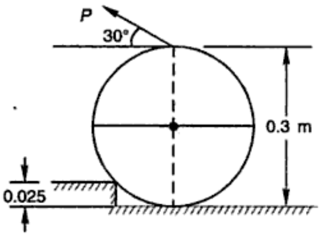


Fig. Prob. S3.11

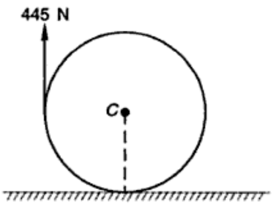


Fig. Prob. S3.12

coefficient of friction required to prevent slipping. What would happen if the coefficient of friction is reduced?

(**Ans.** $\mu = 0.21$; it would roll with slip at the point of contact)

S3.13 A rope is wrapped three and a half times around a cylinder, as shown in Fig. Prob. S3.13. Determine the force T_1 exerted on the free end of the rope, that is required to support a 1 kN weight. The coefficient of friction between the rope and cylinder is 0.25.

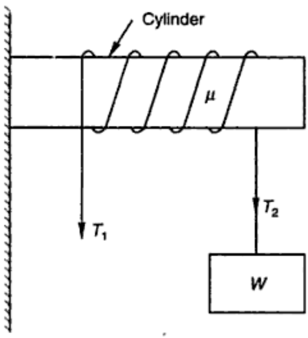


Fig. Prob. S3.13

S3.14 A belt passes over a number of fixed cylinders, as shown in Fig. Prob 3.14. Find the tension T for a given tension T_0 in terms of the coefficients of frictions and angle.

$\left(\textbf{Ans. } T = T_0 e^{-(\mu_1\beta_1 + 1\mu_2\beta_2 + \ldots)}\right)$

S3.15 A square-threaded screw jack has a pitch of 1 cm and a mean diameter of 7.5 cm. The mean diameter of the bearing surface between the cap and the screw is 9 cm. The coefficient of friction between all surfaces is 0.10. What force is required at the end of a lever 90 cm long to raise 40 kN?

(**Ans.** 440 N)

S3.16 Four turns of a rope around a horizontal post are just able to hold a 450 kg mass with a pull of 45 N. Determine the coefficient of friction between the rope and the post.

(**Ans.** $\mu = 0.18$)

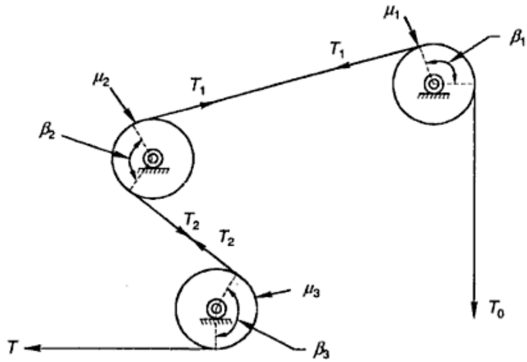


Fig. Prob. 3.14

S3.17 Two equal pulleys, each of diameter 75 cm, are connected by a belt. The tension in the tight side of the belt is 200 N. If the coefficient of friction is 0.25, determine the tension in the slack side of the belt at the instant of impending slip.

(Ans. 91 N)

S3.18 A screw jack has a pitch of 6 mm and the mean radius of the threads is 60 mm. The mean diameter of the bearing surface under the cap is 80 mm. What is the turning moment necessary to lift a 680 kg box? Take $\mu = 0.06$.

(Ans. 46.4 Nm)

S3.19 An axial force P presses the disc brake of radius R onto a flexible elastic surface so that the contact pressure decreases parabolically from p_0 at the centre to zero at the periphery of the disc. Show that the pressure distribution can be expressed as

$$p = \frac{2P}{\pi R^2}(1 - r^2/R^2)$$

Also show that the torque required to cause the impending motion of the disc brake is given by $M = 8/15\ \mu_s PR$.

S3.20 Find the lifting force P required to raise the load 100 N supported as shown in Fig. Prob. S3.20. Take $\mu = 0.3$. What would happen if the force P is less than this value? What is the minimum force P required to just hold the load in position?

(Ans. 658.6 N; No lifting; 15.2 N)

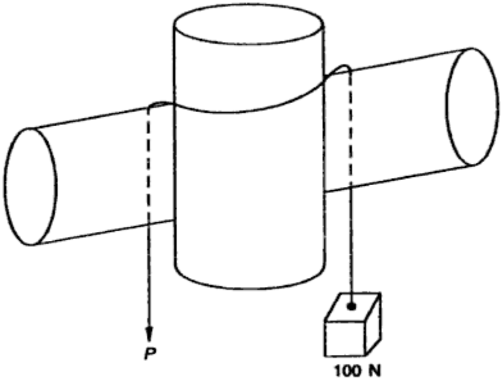


Fig. Prob. S3.20

S3.21 A drum brake consists of a drum with brake lining over an angle 4α as shown in Fig. Prob. S3. 21. The brake shoes are pressed on it with a force P to produce a braking couple C.

Show that

$$2h\frac{P}{C} = \frac{h(\alpha + \sin\alpha\cos\alpha)}{2\mu R\sin\alpha} \pm 1$$

where the sign depends upon the direction of motion of the drum.

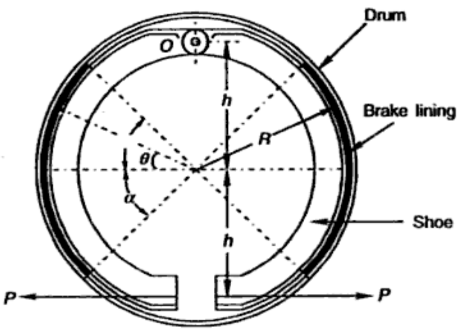


Fig. Prob. S3.21

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. The force of friction between two bodies in contact is
   (a) a function of the relative velocity between them.
   (b) dependent on the areas of contact.
   (c) always normal to the surface of contact.
   (d) never shown in the free-body diagram of the system of these two bodies.
2. The frictional force is independent of
   (a) the area of contact
   (b) the coefficient of friction
   (c) the normal reaction
   (d) the angle of friction
3. The coefficient of friction depends upon
   (a) the normal reaction
   (b) the surface roughness
   (c) the tangential force applied
   (d) the speed of movement
4. Once a body just begins to slide, it continues to slide because
   (a) the body has inertia
   (b) inertia force acts on the body
   (c) the body accelerates
   (d) the frictional force becomes less.
5. The frictional force on a body acted upon by a force on a rough horizontal surface is
   (a) always equal and opposite to the horizontal component of the force.
   (b) equal and opposite to the applied force.
   (c) equal and opposite to the horizontal component of the applied force if the body is at rest or moving with a constant velocity.
   (d) independent of the vertical component of the force.
6. The coefficient of friction between two surfaces is the constant of proportionality between the applied tangential force and the normal reaction
   (a) at the instant of application of the force.
   (b) at any instant when the body is at rest.
   (c) at the instant of impending motion.
   (d) at an instant after the motion takes place.
7. The ratio between the tensions in the tight side and slack side of a flat belt drive increases
   (a) in direct proportion to the angle of lap.
   (b) exponentially as the angle of lap increases.
   (c) in direct proportion to the coefficient of friction.
   (d) proportional to the width of the belt.
8. The condition for a screw jack to be self-locking is that
   (a) its efficiency should be the maximum possible.
   (b) its efficiency should be the minimum possible.
   (c) its efficiency should be more than 50%.
   (d) it should not unwind to lower the load if left to itself.
9. The maximum efficiency of a machine
   (a) should be 100% under ideal conditions.
   (b) is directly proportional to the velocity ratio.

(c) is given by mechanical advantage divided by velocity ratio.
(d) should occur when the load is 50% of maximum permissible load.

Answers to Multiple-Choice Questions

1 (d),	2 (a),	3 (b),	4 (d),	5 (c),	6 (c),
7 (b),	8 (d),	9 (c)			

4 Central Points: Centroid, Centre of Mass and Centre of Gravity

4.1 INTRODUCTION

It is often necessary to define a point such that the entire length of a curve, area of a surface or volume, mass or gravitational force for a body should be representable to act at that point for some purposes. The point should serve as a convenient origin for the coordinate axes moving with the body. It can be regarded as a convenient base point for the application of the principle of moment of momentum. It can also be taken as the fundamental base for the moment of inertia computations. In view of so many good reasons it is natural to define a central point for a given physical entity, more so, if a single point can satisfy all the above criteria.

Mathematically, *a central point* is that point about which the summation of the first moments of the elements of the body results in zero. Alternatively, if an origin is chosen arbitrarily, then the central point is that where the entire physical quantity may be assumed to be concentrated for the purpose of calculating moment about the origin. This definition, as will be seen later, makes the central point very meaningful since it is unique and invariant with the choice of the origin and the orientation of the set of axes with respect to the body.

The terminology of the central point for different physical entities is as follows:

Terminology	*Physical Entity*
Centroid	Length of a curve
Centroid	Area of a surface
Centroid	Volume of a body
Centre of Mass	Mass of a body
Centre of Gravity	Gravitational force on a body

4.2 CONCEPT OF FIRST MOMENT

The *first moment* of an 'element' about an origin is defined as the product of its position vector with the element itself. The element may belong to any physical quantity, such as continuous length, area, volume, mass or distributed gravitational force. The element is correspondingly termed as

Length element	dl	for length	(i)
Area element	dA	for area	(ii)
Volume element	dV	for volume	(iii)
Mass element	dm	for mass	(iv)
Force element	$g\,dm$	for gravity	(v)

It may be seen that the element can be a scalar quantity as in (i) to (iv) and a vector quantity as in (v). The product definition of the first moment implies a magnitude multiplication for the scalar elements and a cross product for the vector elements.

Consequently, if the element is located by a position vector **r** with respect to an origin O as shown in Fig. 4.1, the first moments are expressed as follows:

$\mathbf{r}\,dl$	for the length element	
$\mathbf{r}\,dA$	for the area element	
$\mathbf{r}\,dV$	for the volume element	
$\mathbf{r}\,dm$	for the mass element	
$\mathbf{r} \times (-\,gdm\,\mathbf{k})$	for the gravitational force element	(4.1)

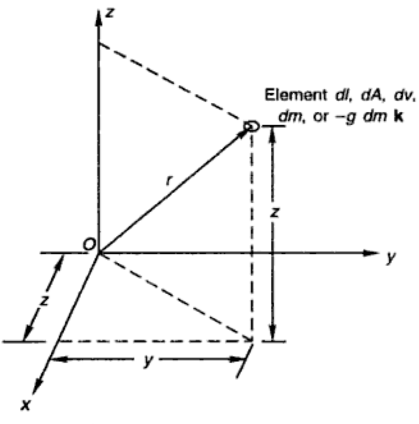


Fig. 4.1 ***Location of an Element***

In general, the position vector **r** has three rectangular components:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The first moments for the length, area, volume, mass and force elements can be expressed as follows:

$xdl\,\mathbf{i} + ydl\,\mathbf{j} + zdl\,\mathbf{k}$	for length	(i)	
$xdA\,\mathbf{i} + ydA\,\mathbf{j} + zdA\,\mathbf{k}$	for area	(ii)	
$xdV\,\mathbf{i} + ydV\,\mathbf{j} + zdV\,\mathbf{k}$	for volume	(iii)	
$xdm\,\mathbf{i} + ydm\,\mathbf{j} + zdm\,\mathbf{k}$	for mass	(iv)	
$ygdm\,\mathbf{i} - xgdm\,\mathbf{j}$	for gravity	(v)	(4.2)

The first moment of any element can be positive or negative, depending upon the choice of the origin and the contribution of the element. For example, the component of the first moment of an area element, i.e.,

$$x\, dA$$

will acquire the sign of x if the area actually exists and it will acquire a sign opposite to that of x if the area in question is a void from the total area counted as positive. The sign and magnitude of x will, of course, depend upon the choice of the origin.

4.3 CENTRAL POINTS: DEFINITIONS OF CENTROID, CENTRE OF MASS AND CENTRE OF GRAVITY

The *central point* is defined as a point where the entire physical quantity can be assumed to be concentrated to give the same first moment as that obtained by considering the elements of the body. The central points for a length, an area and a volume are called the centroids whereas the central points for the distribution of mass and gravitational force are termed as the centre of mass and centre of gravity respectively.

The central point is denoted by C and its position vector by

$$\mathbf{r}_c = x_c\,\mathbf{i} + y_c\,\mathbf{j} + z_c\,\mathbf{k}$$

Thus, for a line in space whose total length is l the first moment is written as $\mathbf{r}_c\, l$. Equating it to the summation of the first moments of the length elements over the entire length.

$$\mathbf{r}_c\, l = \int \mathbf{r}\, dl^*$$

whence

$$\mathbf{r}_c = \left(\int \mathbf{r}\, dl\right)/l \tag{4.3}$$

Similarly, denoting the total area of a surface by A, its centroid is given by

$$\mathbf{r}_c = \left(\int \mathbf{r}\, dA\right)/A \tag{4.4}$$

Likewise, the centroid of a volume V is given by

$$\mathbf{r}_c = \left(\int \mathbf{r}\, dV\right)/V \tag{4.5}$$

and the centre of mass of a body of mass m is located by

$$\mathbf{r}_c = \left(\int \mathbf{r}\, dm\right)/m$$

$$= \left(\int \mathbf{r}\,\rho\, dv\right)/m \tag{4.6}$$

where ρ is the mass density.

*In general, the limits for such definite integrals are not written for the sake of convenience.

The centre of gravity of a body acted upon by a parallel and uniform gravitational force $-mg\,\mathbf{k}$ is given by

$$\mathbf{r}_c \times (-mg\,\mathbf{k}) = \int \mathbf{r} \times (-g\,dm\,\mathbf{k})$$

$$\mathbf{r}_c \times m\,\mathbf{k} = \int (\mathbf{r} \times dm\,\mathbf{k}) \tag{4.7}$$

The central point, i.e., the centroid, centre of mass or centre of gravity may or *may not lie on the body itself* but it is a point fixed with respect to the body. The choice of the coordinate axes and origin is arbitrary; the centroid is a unique point for the given physical quantity which may be a length, an area, a volume, a mass or a distributed force. The central point, therefore, represents the given physical quantity so far as its first moment about any origin is concerned.

4.4 CENTROID OF A LENGTH

The centroid of length l of a curve in space is given by Eq. (4.3), i.e.,

$$\mathbf{r}_c = \left(\int \mathbf{r}\,dl\right)/l$$

which implies that the coordinates of the centroid are

$$x_c = \left(\int x\,dl\right)/l$$

$$y_c = \left(\int y\,dl\right)/l \tag{4.8}$$

and

$$z_c = \left(\int z\,dl\right)/l$$

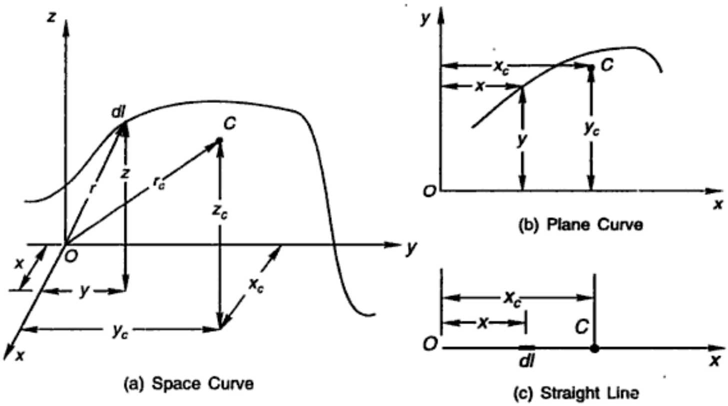


Fig. 4.2 ***Centroid of a Length***

For a plane curve, the x and y coordinate axes being chosen in the plane of the curve,

$$x_c = \int (x\, dl)/l \tag{4.9}$$

and

$$y_c = \int (y\, dl)/l$$

For a straight line, the x-axis being chosen along the length,

$$x_c = \left(\int x\, dl\right)/l \tag{4.10}$$

It may be noted that the centroid of a curve may or may not lie on the curve for space and plane curves but it must lie at the mid-point of straight line. Further, the centroid for a curve is independent of the choice of coordinates and the origin. In other words, the centroid is a point for a curve and is not concerned with the choice of the origin O and the orientation of the axes.

4.5 CENTROIDS OF A COMPOSITE LENGTH

When the length of a curve can be decomposed into simpler shapes, such as straight lines, arcs of circles, etc., the centroid of the length can be determined by employing the knowledge for these simpler shapes. The principle is:

The first moment of the total length must equal the algebraic sum of the first moments of the lengths of its parts which indeed follows from the definition of the centriod.

If a composite length consists of component lengths l_1, l_2, l_3, ... with centroids at (x_{c1}, y_{c1}), (x_{c2}, y_{c2}), etc., respectively, then the centroid of the composite area is located by

$$X_c = \frac{x_{c1}\, l_1 + x_{c2} l_2 + \ldots}{l_1 + l_2 + l_3 + \ldots}$$

$$= \frac{\Sigma x_c\, l}{\Sigma l} \tag{4.11}$$

and

$$Y_c = \frac{y_{c1} l_1 + y_{c2} l_2 + \ldots}{l_1 + l_2 + l_3 + \ldots}$$

$$= \frac{\Sigma y_c\, l}{\Sigma l} \tag{4.12}$$

Care must be taken to regard the component lengths as positive or negative depending upon their contribution to the composite length. The coordinates of their individual centroids, referred from the same origin, can be positive or negative.

***Example* 4.1** Locate the centroid of an arc of a circle of radius R and subtended angle 2α. Hence determine the coordinates of the centroid of

(i) a quarter circular arc
(ii) a semicircular arc
(iii) a complete circular arc.

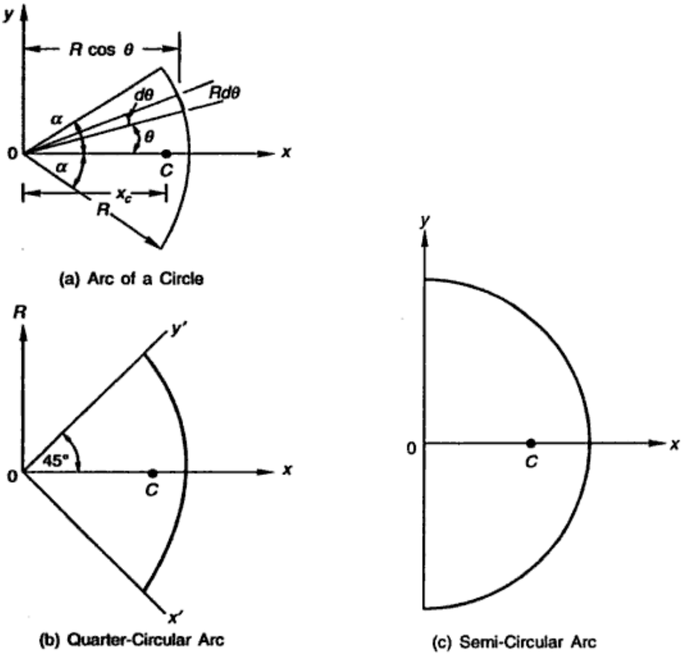


Fig. Ex. 4.1 *(Solution)*

Solution Consider an arc of a circle with reference to the coordinate axes at the centre of the circle drawn symmetrically with respect to the arc. Symmetry of the x axis ensures that the centroid is located on the x axis or

$$y_c = 0$$

In order to determine x_c, consider a length element

$$dl = R\,d\theta$$

at an angle θ from the x-axis as shown in Fig. Ex. 4.1. (Solution)

By definition,

$$x_c = \left(\int x\,dl\right) / l = \left(\int x\,dl\right) / \int dl$$

$$= \left(\int_{-\alpha}^{\alpha} x\, R\, d\theta \right) \Big/ \left(\int_{-\alpha}^{\alpha} R\, d\theta \right)$$

Substituting $x = R \cos \theta$,

$$x_c = \left(\int_{-\alpha}^{\alpha} R^2 \cos \theta\, d\theta \right) \Big/ \left(\int_{-\alpha}^{\alpha} R\, d\theta \right)$$

$$= \frac{R^2 (\sin \alpha - \sin(-\alpha))}{2R\alpha}$$

$$= \frac{R \sin \alpha}{\alpha}$$

(i) For a quarter circular arc,

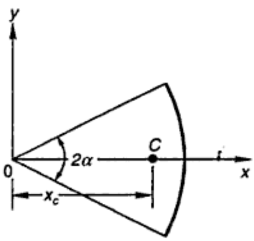
$$2\alpha = \pi/2; \ \alpha = \pi/4$$

Consequently,

$$x_c = \frac{R \sin \pi/4}{\pi/4} = \frac{2\sqrt{2}}{\pi} R$$

and $\quad y_c = 0$

Table 4.1 Centroid of Length of an Arc

Description	*Shape*	*Length*	x_c
(Arc of a circle)		$2R\alpha$	$\frac{R \sin \alpha}{\alpha}$
(Quarter circular arc) $2\alpha = \pi/2$		$\frac{R}{2}$	$\frac{2\sqrt{2}\, R}{\pi}$
(Semicircular arc) $2\alpha = \pi$		πR	$\frac{2R}{\pi}$
(Circular arc) $2\alpha = 2\pi$		$2\pi R$	0

(ii) For a semicircular arc

$$2\alpha = \pi, \quad \alpha = \pi/2$$

$$x_c = \frac{R \sin \pi/2}{\pi/2} = \frac{2R}{\pi}$$

$$y_c = 0$$

(iii) For a complete circular arc,

$$2\alpha = 2\pi; \quad \alpha = \pi$$

$$x_c = \frac{R \sin \pi}{\pi} = 0$$

$$y_c = 0$$

The centre of the circle must be the centroid of the complete circular arc by virtue of its symmetry about the diameteral axes.

***Example* 4.2** A metal wire is bent into the form shown in Fig. 4.2. *EF* is along $-x$ direction and *BA* is along $+x$ direction and *ED* and *BC* are parts of same circle joined by *DC*. Find the centroid of the combined length of the wire.

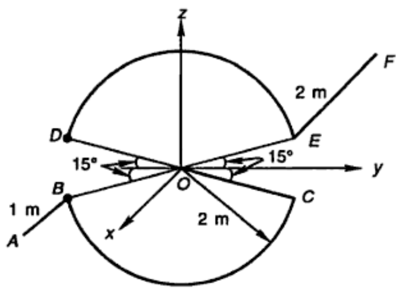


Fig. Ex. 4.2

Solution One way of finding the centroid is to start from one end and use integration in parts, i.e., *A* to *B*, *BC*, *CD*, *DE* and *E* to *F*. Another, better way of doing this is to consider the line as a complete ring with diameter *DC* minus *DB* and *EC* plus *EF* and *AB*. The ring with diameter *DC* has its centroid at *O* due to symmetry.

The centroids and lengths of different parts of the line are given below.

Part	*Length*	*Centroid*		
		x_c	y_c	z_c
Complete ring and DC	$4\pi + 4$	0	0	0
EC	$-\frac{\pi}{3}$	0	1.977	0
BD	$-\frac{\pi}{3}$	0	– 1.977	0
EF	2	– 1	1.932	.518
AB	1	0.5	– 1.932	– .518

Lengths *EC* and *BD* are negative because they are to be subtracted from the ring to get the given shape.

Length of the combined line = 4 + 2 + 1 + 4π – 2π/3 = 17.472

Thus the centroid of the combined length is

$$x_c = \frac{(4\pi + 4)\times 0 + \left(\frac{-\pi}{3}\right)\times 0 + \left(\frac{-\pi}{3}\right)\times 0 + 2\times(-1) + 1\times 0.5}{17.42}$$

$$= \frac{-1.5}{17.472} = 0.086 \text{ m}$$

$$y_c = \frac{(4\pi + 4)\times 0 + \left(-\frac{\pi}{3}\right)\times 1.977 + \left(-\frac{\pi}{3}\right)\times(-1.977) + 2\times 1.932 - 1\times 1.932}{17.472}$$

$$= 0.111 \text{ m}$$

Similarly $z_c = 0.03$ m

A glance at the procedure reveals that if one had followed the earlier stated procedure of integration, the solution would have become cumbersome and also involving equations for every part of the line.

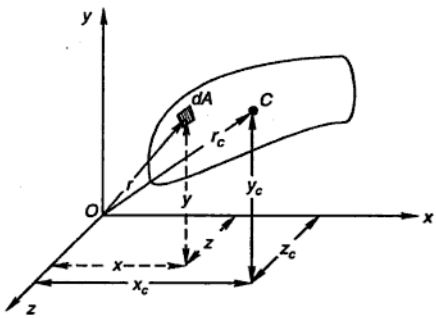
4.6 CENTROID OF AN AREA

From the definition of the centroid of an area (Eq. (4.4))

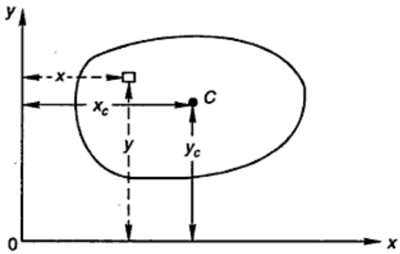
$$\mathbf{r}_c = \left(\int \mathbf{r}\, dA\right)/A$$

the x, y and z coordinates for a set of axes, with an arbitrary O as shown in Fig. 4.3, are given by

$$\boxed{\begin{aligned} x_c &= \left(\int x\, dA\right)/A \\ y_c &= \left(\int y\, dA\right)/A \\ z_c &= \left(\int z\, dA\right)/A \end{aligned}} \tag{4.13}$$



(a) Curved Surface



(b) Plane Surface

Fig. 4.3 ***Centroid of Area***

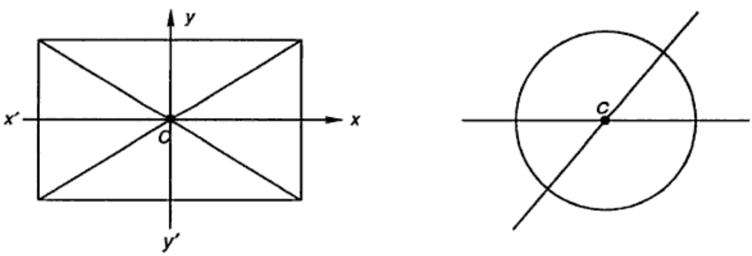
For a plane area, the x and y coordinate axes being chosen in the plane of the area as shown in Fig. 4.3(b),

$$x_c = \left(\int x\,dA\right)/A$$

$$y_c = \left(\int y\,dA\right)/A \tag{4.14}$$

It may be added to clarify that the centroid of an area may or may not lie on the area in question and that it is a unique point for a given area regardless of the choice of the origin and the orientation of the axes about which we take the first moments.

If an area has an axis of symmetry, the centroid must lie on that axis. If an area has two axes of symmetry, then the point of intersection of the axes must be the centroid. For example, a rectangle shown in Fig. 4.4 is symmetrical about the axes $x - x'$ and $y - y'$, the point of intersection of $x - x'$ and $y - y'$ is, therefore, the centroid. Similarly, for a circle, the point of intersection of any two diameters is the centroid; this is the centre of the circle itself. The centroid of a composite area may be determined from the knowledge of the centroids of the constituent areas in the same way as is done for a composite length.



(a) Rectangle (b) Circle

Fig. 4.4 ***Location of Centroid***

***Example* 4.3** Locate the centroid of a right-angled triangle with base b and height h.

Solution Let us consider an area element dA with respect to the x and y axes drawn from the right-angled vertex O as shown in Fig. Ex. 4.3(a)

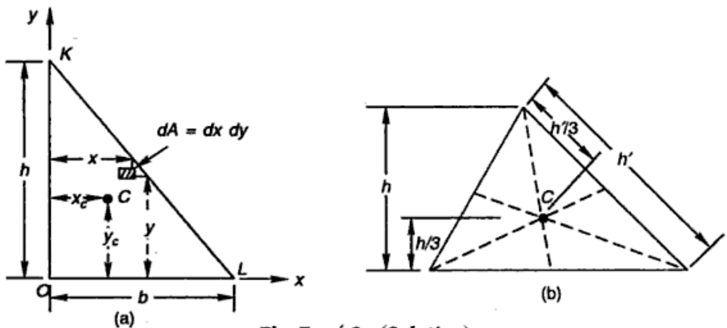


Fig. Ex. 4.3 (***Solution***)

$$x_c = \left(\int x\, dA\right) / A \text{ and } y_c = \left(\int y\, dA\right) / A$$

introducing $\qquad dA = dx\, dy = dy\, dx$

and recognising the equation of the straight line *KL* as

$$y = -\frac{h}{b} x + h$$

or

$$x = -\frac{b}{h} y + b$$

and that

$$A = \frac{1}{2} bh$$

$$x_c = \int_0^h \int_0^{\left(-\frac{b}{h} y + b\right)} x\, dx\, dy / A$$

$$= \left[\int_0^h \left(-\frac{b}{h} y + b\right)^2 / 2\, dy\right] / A$$

$$= \left|\frac{b^2}{h^2} \frac{y^3}{3} + b^2 y - \frac{b^2}{h} y^2\right|_0^h / bh$$

$$= \frac{b}{3}$$

Similarly,

$$y_c = \int_0^b \int_0^{\left(-\frac{h}{b} x + h\right)} y\, dy\, dx / A$$

$$= \left[\int_0^b \left(-\frac{h}{b} x + h\right)^2 / 2\, dx\right] / A$$

$$= \left| \frac{h^2}{b^2} \frac{x^3}{3} + h^2 y - \frac{h^2}{b} x^2 \right|_0^b \Big/ A$$

$$= \frac{h}{3}$$

The centroid of a right-angled triangle is, therefore, located at one-third the distance along the base and one-third the distance along the height. It may be noted that, in general, for any triangle, the centroid is located at one-third the height of the triangle from the base. *Obviously, any side can be taken as the base and the rule of one-third height applies to any orientation.* The centroid of a triangle can, therefore, be located easily by this consideration.

It is also interesting to note that the centroid of a triangle is coincident with the point of intersection of its medians, as shown in Fig. 4.3(b) (Solution).

***Example* 4.4** Locate the centroid of the area of a circular sector. Hence, obtain the coordinates of the centroid of a quarter circle and a semi-circle with reference to a set of axes at the centre of the circle.

Solution Consider the sector of a circle as shown in Fig. Ex. 4.4 (Solution) with reference to the coordinate axes at the centre of the circle drawn symmetrically with respect to the sector. Symmetry about the x-axis ensures that the centroid lies on the x-axis or

$$y_c = 0$$

In order to determine x_c, consider an area element

$$dA = \frac{1}{2} R \cdot R \, d\theta = \frac{1}{2} R^2 \, d\theta$$

at an angle θ from the x-axis. The centroid of the elementary area must be at 2/3 R from the centre such that

$$x = \frac{2}{3} R \cos \theta$$

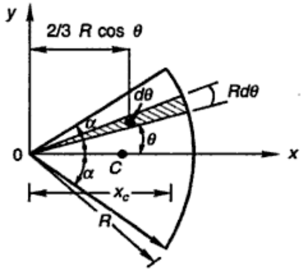
By the definition of centroid,

$$x_c = \left(\int x \, dA \right) \Big/ \int dA$$

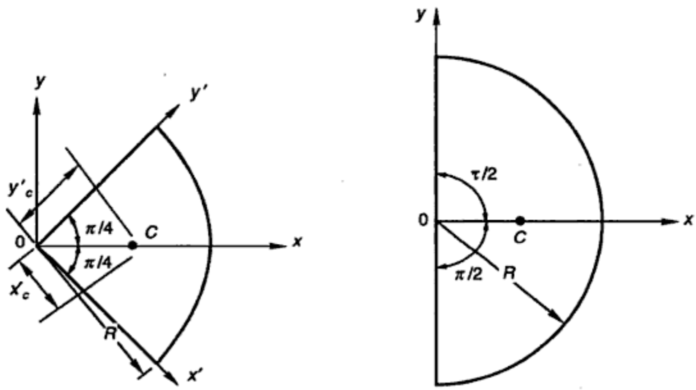
$$= \left[\int_{-\alpha}^{\alpha} \frac{2}{3} R \cos\theta \cdot \frac{1}{2} R^2 \, d\theta \right] \Bigg/ \int_{-\alpha}^{\alpha} \frac{1}{2} R^2 \, d\theta$$

$$= \frac{2}{3} R \frac{|\sin\theta|_{-\alpha}^{\alpha}}{|\theta|_{-\alpha}^{\alpha}}$$

$$= \frac{2}{3} R \frac{\sin\alpha}{\alpha}$$



(a) Sector of a Circle



(b) Quarter-Circle (c) Semi-Circle

Fig. Ex. 4.4 *(Solution)*

For a quarter-circle,

$$2\alpha = \frac{\pi}{2}; \alpha = \frac{\pi}{4}$$

Hence, the limits of integration become

$$-\pi/4 \text{ to } +\pi/4$$

The x-coordinate of the centroid is

$$x_c = \frac{2}{3} R \frac{\sin \pi/4}{\pi/4} = \sqrt{2}\, \frac{4R}{3\pi}$$

Generally, the centroid of a quarter circle is referred to the x' and y' axes:

$$x_c' = y_c' = x_c \cos \frac{\pi}{4} = \frac{4R}{3\pi} = 0.424\, R$$

For a semicircle,

$$2\alpha = \pi;\ \alpha = \pi/2$$

and the limits of integration become

$$-\pi/2 \text{ to } +\pi/2$$

The x coordinate of the centroid is

$$x_c = \frac{2}{3}R\frac{\sin \pi/2}{\pi/2} = \frac{4R}{3\pi} = 0.424\ R$$

Example 4.5 Two non-viscous, incompressible and immiscible liquids of densities ρ and $1.5\ \rho$ are poured into the two limbs of a circular tube of radius R and small cross-section kept fixed in a vertical plane as shown in Fig. Ex. 4.5. Each liquid occupies one-fourth the circumference of the tube.

Find the angle θ that the radius vector to the interface makes with the vertical in equilibrium position.

Solution Centroids C_1 and C_2 of the liquids in the circular tube are located by

$$OC_1 = \frac{2\sqrt{2}}{\pi}R = OC_2$$

Masses of the two liquids are $\rho a\ \pi R/2$ and $1.5\ \rho a\ \pi R/2$ respectively.

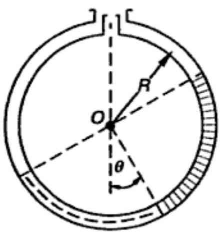


Fig. Ex. 4.5

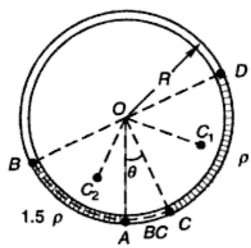


Fig. Ex. 4.5 ***(Solution)***

The centroid of the composite column BD lies on the vertical line OA. Taking moments, for equilibrium,

$$\rho a\frac{\pi R}{2}\cdot g \times \frac{2\sqrt{2}}{\pi}R\sin(45+\theta)$$

$$= 1.5\ \rho a\frac{\pi R}{2}\cdot g \times \frac{2\sqrt{2}}{\pi}R\sin(45-\theta)$$

whence, $$\sin(45+\theta) = 1.5\sin(45-\theta)$$

$$\sin 45\cos\theta + \cos 45\sin\theta = 1.5\ [\sin 45\cos\theta - \cos 45\sin\theta]$$

$$\frac{1}{2}\sin 45 \cos\theta - \frac{3}{2}\cos 45 \sin\theta = 0$$

$$\cos\theta - 3\sin\theta = 0;\ \tan\theta = \frac{1}{3};\ \theta = 18.43°$$

***Example* 4.6** Determine the centroid of the area bounded by the x-axis, the line $x = a$ and the parabola $y^2 = kx$ as shown in Fig. Ex. 4.6.

Solution We may choose to consider a differential area element

$$dA = dx\,dy$$

located by x, y coordinates, as in the case of a triangle or we may prefer to deal with strip elements, as in the case a circular sector (Fig. Ex. 4.6 (a) and (b)) (Solution). In this case, let us do it both ways and see the equivalence of the procedures.

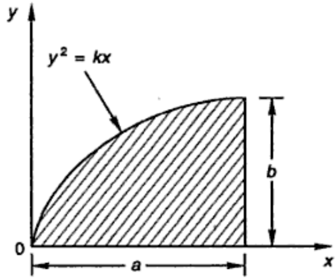


Fig. Ex. 4.6

Method I

Choosing a differential area element

$$dA = dx\,dy$$

at a location (x, y), the centroid is given by

$$x_c = \left(\int x\,dA\right)/A,\ y_c = \left(\int y\,dA\right)/A$$

Noting that

$$A = \int dA = \int_0^b \int_{y^2/k}^a dx\,dy$$

$$= \int_0^b (a - y^2/k)\,dy$$

$$= \left| ay - \frac{y^3}{3k} \right|_0^b = ab - \frac{b^3}{3k}$$

and

$$k = \frac{y^2}{x} = \frac{b^2}{a}$$

therefore,

$$A = ab - \frac{b^3 a}{3b^2} = \frac{2ab}{3}$$

$$x_c = \left(\int_0^b \int_{y^2/k}^a x\,dx\,dy \right) \Bigg/ A$$

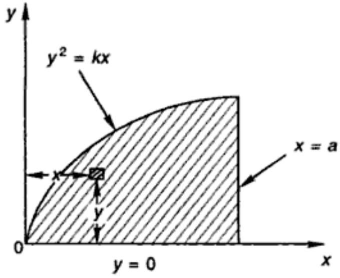


Fig. Ex. 4.6(a) ***(Solution)***

$$= \left(\int_0^b (a^2 - y^4/k^2)/2\, dy \right) \Big/ A$$

$$= \frac{1}{2}\left(a^2 b - \frac{b^5 a^2}{5b^2} \right) \Big/ A$$

$$= \frac{2}{5} a^2 b \frac{3}{2ab} = \frac{3}{5} a$$

$$y_c = \int_0^a \int_{y=0}^{\sqrt{kx}} y\, dy\, dx / A$$

$$= \left(\int_0^a \frac{kx}{2} dx \right) / A = \left| \frac{kx^2}{4} \right|_0^a / A$$

$$= \frac{b^2}{a} \cdot \frac{a^2}{4} \frac{3}{2ab} = \frac{3}{8} b$$

The centroid is, hence, at $\left(\frac{3}{5} a, \frac{3}{8} b \right)$.

Method II

Choose an elementary strip of area

$$dA = y\, dx$$

at a distance x from the y-axis, the centroid of the strip being at $y/2$ from the base, the centroidal ordinate for the given area is

$$y_c = \left(\int y/2 \cdot y\, dx \right) \Big/ A$$

$$= \left(\int_0^a \frac{kx}{2} dx \right) \Big/ A$$

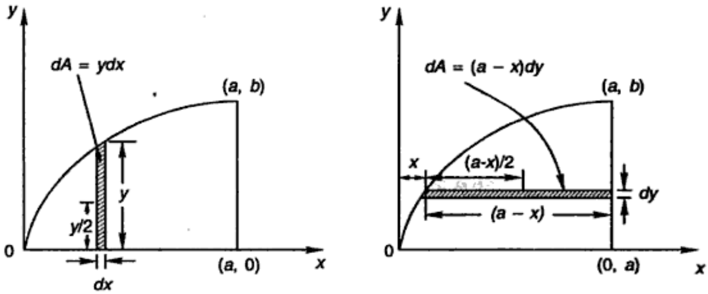


Fig. Ex. 4.6(b) ***(Solution)***

$$= \frac{ka^2}{4} \bigg/ A$$

$$= \frac{b^2}{a} \cdot \frac{a^2}{4} \cdot \frac{3}{2ab} = \frac{3}{8} b$$

Similarly, by taking a strip of area

$$dA = (a - x)\, dy$$

with its centroid at

$$x + \frac{a - x}{2} = \frac{x + a}{2}$$

from the y-axis, the abscissa is given by

$$x_c = \left(\int_0^b \frac{x + a}{2} \cdot (a - x)\, dy \right) \bigg/ A$$

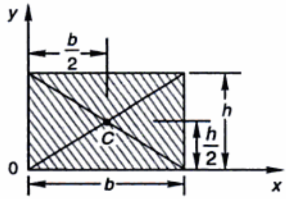
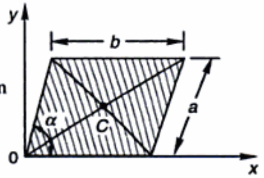
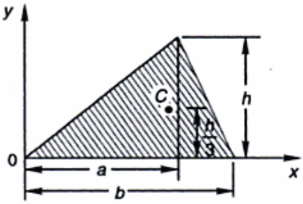
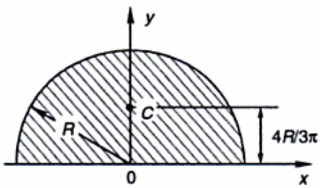
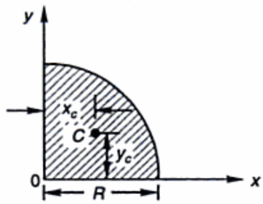
$$= \left(\int_0^b \frac{1}{2} \left(a^2 - \frac{y^4}{k^2} \right) dy \right) \bigg/ A$$

$$= \frac{1}{2} \left| \left(a^2 y - \frac{y^5 \cdot a^2}{5b^4} \right) \right|_0^b \bigg/ A$$

$$= \frac{2}{5} a^2 b \cdot \frac{3}{2ab} = \frac{3}{5} a.$$

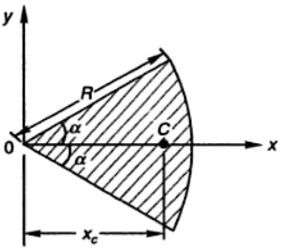
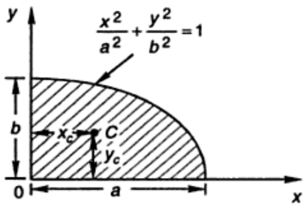
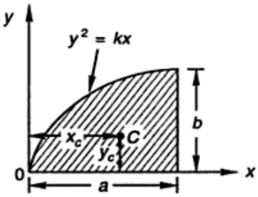
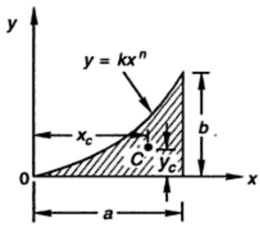
The centroid is again observed to be at the point $\left(\frac{3}{5} a, \frac{3}{8} b \right)$

Table 4.2 Centroids of Geometrical Shapes

Description	*Shape*	*Area*	x_c	y_c
Rectangle		bh	$b/2$	$h/2$
Square ($h = b = a$)		a^2	$a/2$	$a/2$
Parallelogram		$ab \sin \alpha$	$\frac{b + a\cos\alpha}{2}$	$\frac{a\sin\alpha}{2}$
Rectangle ($\alpha = \pi/2$)		ab	$b/2$	$a/2$
Triangle		$\frac{1}{2}bh$	$1/3\ (a + b)$	$h/3$
Semi-circle		$\frac{\pi R^2}{2}$	0	$\frac{4R}{3\pi} = 0.424R$
Quarter circle		$\frac{\pi R^2}{4}$	$\frac{4R}{3\pi} = 0.424\ R$	$\frac{4R}{3\pi} = 0.424\ R$

(Contd.)

Table 4.2 (*Contd.*) **Centroids of Geometrical Shapes**

Description	*Shape*	*Area*	x_c	y_c
Sector of a circle		$R^2\alpha$	$\frac{2}{3}\frac{R\sin\alpha}{\alpha}$	0
Quarter ellipse		$\frac{\pi ab}{4}$	$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$
Quarter parabola		$\frac{2ab}{3}$	$\frac{3}{5}a$	$\frac{3}{8}b$
General spandrel		$\frac{ab}{n+1}$	$\frac{n+1}{n+2}a$	$\frac{n+1}{2n+1}\cdot\frac{b}{2}$
Parabolic spandrel ($n = 2$)		$\frac{ab}{3}$	$\frac{3a}{4}$	$\frac{3b}{4}$

***Example* 4.7** Locate the centroid of the given composite area shown in Fig. Ex. 4.7.

Solution The given area can be considered to comprise a rectangle 40 cm × 50 cm plus a semicircle of 20 cm radius minus a circle of 10 cm radius. With respect to the x-y axes with the origin at O, we proceed by preparing a table:

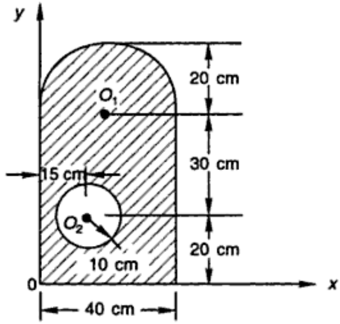


Fig. Ex. 4.7

Component	*Area* (cm^2)	x_c (*cm*)	y_c (*cm*)	x_cA (cm^3)	y_cA (cm^3)
Rectangle	2000	20	25	40 000	50 000
Semicircle with centre O_1	$\frac{\pi \times 20^2}{2} = 628.3$	20	$50 + 0.424 \times 20 = 58.5$	12 566	36 756
Circle with centre O_2 (void area)	$-\pi \times 10^2 = -314.2$	15	20	−4712	−6284
Total	**2314.1** ΣA	—	—	**47 854** $\Sigma x_c A$	**80 472** $\Sigma y_c A$

Employing the relations for the centroidal point,

$$X_c = \frac{\Sigma x_c A}{\Sigma A} \quad \text{and} \quad Y_c = \frac{\Sigma y_c A}{\Sigma A}$$

we obtain,

$$X_c = \frac{47\,854}{2314.1} = 20.68 \text{ cm}$$

$$Y_c = \frac{80\,472}{2314.1} = 34.77 \text{ cm}$$

4.7 THEOREMS OF PAPPUS-GULDINUS

There are two very important theorems initially due to the Greek geometer Pappus and later restated by the Swiss mathematician Guldinus which deal with the surfaces and volumes of revolution.

Theorem 1

The area of a surface of revolution is equal to the length of the generating curve

times the distance travelled by the centroid of the generating curve while the surface is generated.

Proof of Theorem 1

A variety of generating curves can be employed to generate the surfaces of revolution as shown in Fig. 4.5.

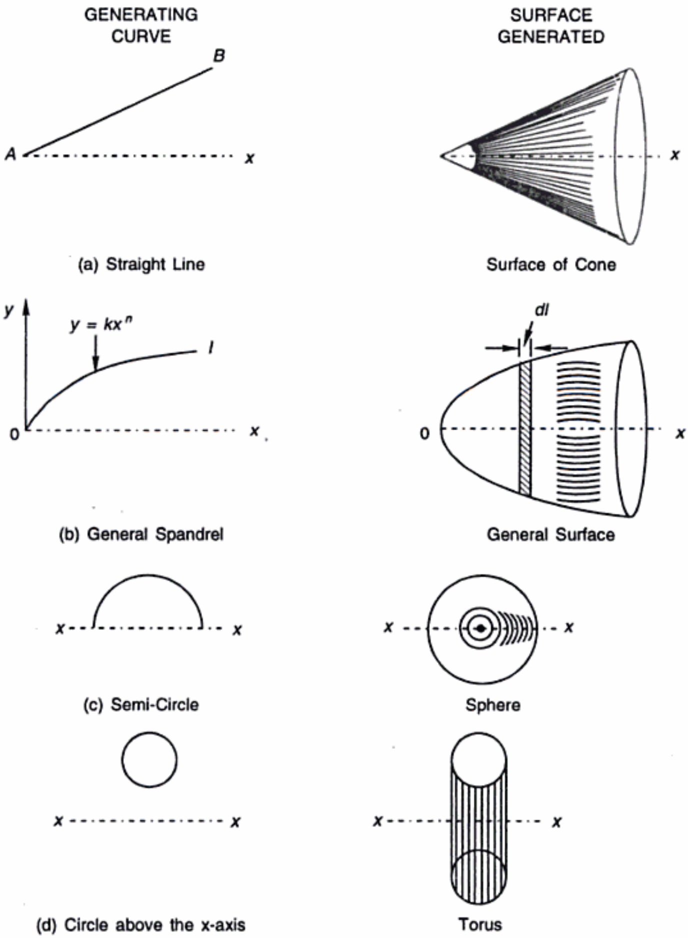


Fig. 4.5 ***Surfaces of Revolution***

Consider a generating curve, say the general spandrel to generate a surface of revolution as shown in Fig. 4.5(b). An elementary length dl of the curve of length l generates a surface area

$$dA = 2\pi y \cdot dl$$

The total surface area generated by the given curve is

$$A = \int dA = \int_0^l 2\pi y\, dl$$

$$= 2\pi \int_0^l y\, dl$$

$$= 2\pi y_c \cdot l \qquad (4.15)$$

Since $\qquad y_c l = \int y\, dl$

by the definition of the centroid of a curve.

Hence, the area of the surface generated is given by the product of $2\pi y_c$ and the length of the surface l; as if the entire length of the generating curve were concentrated at the radius y_c. In other words, the area of the surface generated equals the area of a cylindrical surface of radius y_c and length l.

Theorem 2

The volume of a body of revolution is equal to the generating area times the distance travelled by the centroid of the area while the body is generated.

Proof of Theorem 2 and its scope

Consider a surface area A bounded by a curve, $y = 0$ and $x = a$ lines as shown in Fig. 4.6. An area element dA of the surface, when revolved about the x-axis generates a volume

$$dV = 2\pi y \cdot dA$$

The entire volume generated by the total area is

$$V = \int dV = \int 2\pi y\, dA$$

$$= 2\pi \int y\, dA = 2\pi y_c\, A \qquad (4.16)$$

since $\qquad y_c A = \int y\, dA$

by the definition of the centroid of an area.

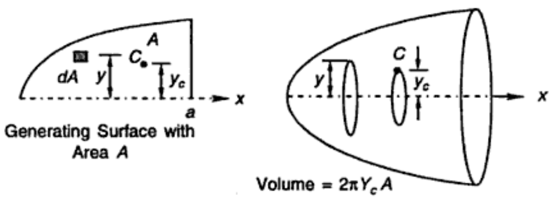


Fig. 4.6 ***Volume Generated by a Surface***

Hence the volume of the body generated is given by the product of $2\pi y_c$ and the magnitude of area A of the surface as if the entire area of the generating surface was concentrated at its centroid at radius y_c. In other words, the volume of the body generated equals the volume of a right circular cylinder of radius y_c and cross-sectional area A.

The theorems of Pappus-Guldinus provide simple means of relating the areas and volumes of the surfaces and bodies of revolution to the lengths and areas of the generating curves and surfaces. *These relationships can be used with advantage both ways, i.e., to determine the areas and volumes of the surfaces and bodies of revolution from the given curves and surfaces to generate these or to locate the centroids of the curves and surfaces from the knowledge of the areas and volumes of the surfaces and bodies generated.*

***Example* 4.8** A semicircle is rotated about its diameter to generate a sphere. Calculate the volume of a sphere of radius R.

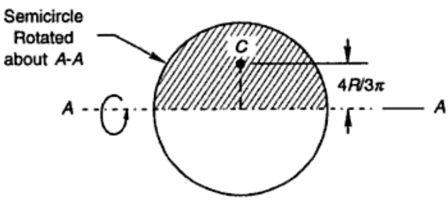


Fig. Ex. 4.8

Solution The theorems of Pappus-Guldinus can be used to great advantage in this case. The centroid of the semicircle is $4R/3\pi$ above the diametral axis. The area of the semicircle is $\pi R^2/2$. According to the Pappus-Guldinus, the volume of the body of revolution generated should be same as that which would be obtained if the entire area $\pi R^2/2$ were concentrated at a radius $4R/3\pi$. The volume of the sphere is, therefore,

$$2\pi \frac{4R}{3\pi} \times \frac{\pi R^2}{2} = \frac{4}{3}\pi R^3$$

***Example* 4.9** Determine the centroid of a quadrant of a circle using the theorems of Pappus and Guldinus.

Solution It is an interesting application of the theorem to locate the centroid of an area if the volume generated by revolving the area over an axis is known. In this case, the volume generated by revolving the quadrant of a circle about either the x-axis or the y-axis is that of a hemisphere, i.e.,

$$\frac{2}{3}\pi R^3$$

Since the area of the quadrant of a circle is

$$\frac{\pi R^2}{4}$$

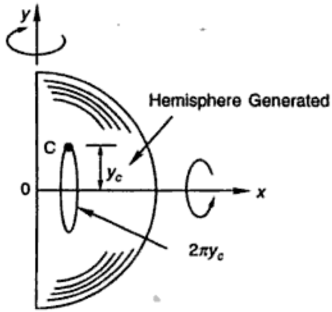


Fig. Ex. 4.9

The distance travelled by the centroid of the quadrant of a circle in generating the hemispherical volume must be

$$\frac{2}{3}\pi R^3 \Big/ \frac{\pi R^2}{4} = \frac{8R}{3}$$

Equating it to $2\pi y_c$ for rotation about the x-axis

$$\boxed{y_c = \frac{4R}{3\pi}}$$

Similarly, the x-coordinate of the centroid may be determined by rotating the quadrant about the y axis

$$\boxed{x_c = \frac{4R}{3\pi}}$$

4.8 CENTROID OF A VOLUME

The centroid of a volume, by definition, is given by Eq. (4.5), i.e.,

$$\mathbf{r}_c = \left(\int \mathbf{r}\, dv\right) \Big/ V$$

whence the coordinates of the centroid are obtained as

$$x_c = \left(\int x\, dV\right) \Big/ V$$

$$y_c = \left(\int y\, dV\right) \Big/ V \qquad (4.17)$$

$$z_c = \left(\int z\, dV\right) \Big/ V$$

as are shown for the volume of a *matka* in Fig. 4.7.

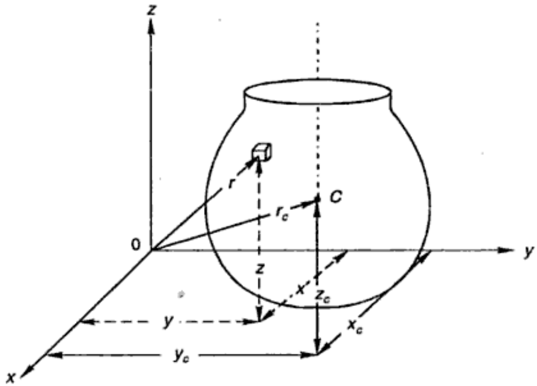
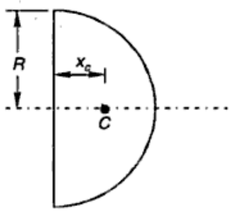
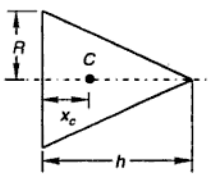
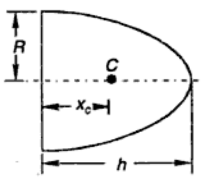


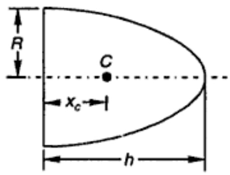
Fig. 4.7 ***Centroid of a Volume***

Table 4.3 Centroids of Volumes of Revolution

Description	*Shape*	x_c	*Volume*
Hemisphere		$\frac{3}{8}R$	$\frac{2}{3}\pi R^3$
Right circular cone		$\frac{h}{4}$	$1/3\pi R^2 h$
Semi-ellipsoid of revolution		$\frac{3}{8}h$	$2/3\pi R^2 h$

(*Contd.*)

Table 4.3 (*Contd.*) **Centroids of Volumes of Revolution**

Description	*Shape*	x_c	*Volume*
Paraboloid of revolution		$\frac{h}{3}$	$1/2\pi R^2 h$

The *centroid of a volume* is a point about which the first moment of the volume equals the summation of the first moment of the distributed volume elements. It is a unique point for the volume and its location is the body is independent of the choice of the origin or the orientation of the axes.

When a volume V possesses a plane of symmetry, the centroid of the volume must lie in that plane. When a volume possesses two planes of symmetry, the centroid must lie on the line of intersection of the two planes. When a volume has three or more planes of symmetry, the centroid must be located at the point of intersection of these planes. The centroid of the volume of a sphere, a cube, an ellipsoid or a rectangular parallelopiped can be located readily by considering their triple or multiple symmetry.

The *centroid of a volume* of revolution must lie on the axis of symmetry but the distance of the centroid from the apex must be determined by integration. It is cautioned that the centroid of a volume of revolution may not coincide with the centroid of its cross-section containing the axis. A summary of the centroids of some volumes of revolution is given in Table 4.3.

***Example* 4.10** Locate the centroid of the volume of a right circular cone of base radius R and height h.

Solution A right circular cone possesses an axis of symmetry; the centroid must be located on this axis. Let the origin be at O, the apex and x-axis along the axis of symmetry as shown in Fig. Ex. 4.10.

Consider an elementary disc of radius r and width dx at a distance x from the origin. The volume of the elementary disc is

$$dv = \pi r^2\, dx$$

and its centroid is at a distance x from O.

For the entire volume of the cone, by definition,

$$x_c = \left(\int x\, dV\right) / V$$

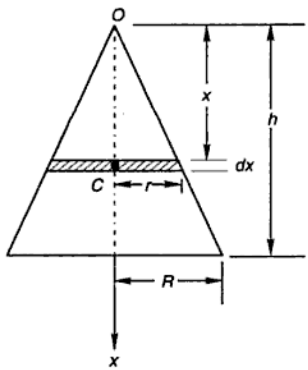


Fig. Ex. 4.10

$$= \left(\int_0^h x \pi r^2 \, dx \right) \Big/ V$$

Substituting $\quad r = \frac{R}{h} x \quad$ and $\quad V = \frac{1}{3} \pi R^? \, !$

$$x_c = \left(\int_0^h \frac{\pi R^2}{h^2} x^3 \, dx \right) \Big/ \left(\frac{1}{3} \pi R^2 h \right)$$

$$= \frac{\pi R^2 h^2}{4} \times \frac{3}{\pi R^2 h} = \frac{3}{4} h$$

The centroid of a right circular cone is, therefore, located at quarter height from its base.

***Example* 4.11** A right circular cone of base radius R and height h is attached to a hemisphere of radius R as shown in Fig. Ex. 4.11. Determine the ratio h/R for which the centroid of the composite volume is located in the plane between the cone and the hemisphere.

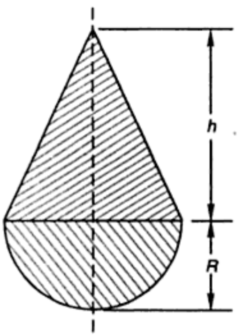


Fig. Ex. 4.11

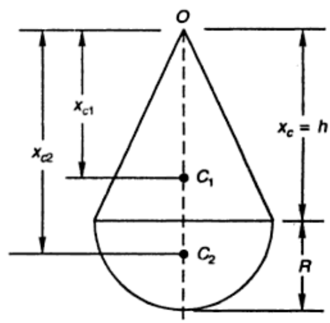


Fig. Ex. 4.11 *(Solution)*

Solution With reference to the origin O at the apex of the cone, as shown in Fig. Ex. 4.11 (Solution) the centroid of the cone alone lies at

$$x_{c1} = \frac{3}{4} h$$

and the centroid of the semicircle lies at

$$x_{c2} = h + \frac{3}{8} R$$

Recollecting that the volume of the cone is

$$V_1 = \frac{1}{3} R^2 h$$

and that the volume of the hemisphere is

$$V_2 = \frac{2}{3}\pi R^3$$

the centroid of the composite volume must be located such that

$$x_c(V_1 + V_2) = x_{c1}V_1 + x_{c2}V_2$$

$$= \frac{3}{4}h\,\frac{1}{3}\pi R^2 h + \left(h + \frac{3}{8}R\right)\frac{2}{3}\pi R^3$$

In order that the centroid lies in the plane between the cone and the hemisphere,

$$x_c = h$$

Then,

$$h\left(\frac{1}{3}\pi R^2 h + \frac{2}{3}\pi R^3\right) = \frac{1}{4}\pi R^2 h^2 + \frac{2}{3}\pi R^3 h + \frac{1}{4}\pi R^4$$

or $$\frac{h^2}{3} + \frac{2}{3}Rh = \frac{h^2}{4} + \frac{2}{3}Rh + \frac{R^2}{4}$$

or $$h^2 = 3R^2$$

and finally $$h/R = \sqrt{3}$$

4.9 CENTRE OF MASS

The *centre of mass* for a body of mass m is a point where the entire mass m can be assumed to be concentrated to give the same first moment as that obtained by considering the element of mass continuously distributed over the body:

$$\mathbf{r}_c = (\mathbf{r}\,dm)/m$$

In terms of an arbitrarily chosen set of x-y-z axes, as in Fig. 4.8(a),

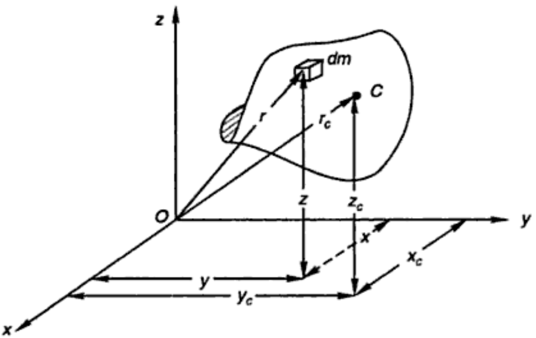


Fig. 4.8(a) ***Centre of Mass***

$$x_c = \left(\int x\, dm\right)/m$$

$$y_c = \left(\int y\, dm\right)/m \qquad (4.18)$$

$$z_c = \left(\int z\, dm\right)/m$$

The centre of mass is an important point in the study of dynamics. If the external forces acting on a body pass through its centre of mass then the body will behave, for all practical purposes, as if it were, a point mass concentrated at the centre of mass.

For a body with constant mass density ρ, also known as homogeneous body,

$$dm = \rho\, dv$$

and $$m = \rho v$$

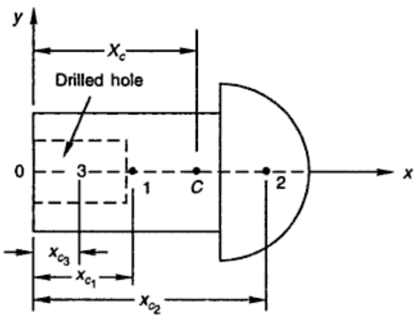
hence $$x_c = \left(\int x\rho\, dv\right)/\rho v = \left(\int x\, dv\right)/v$$

Similarly,

$$y_c = \left(\int y\, dv\right)/v \quad \text{and} \quad z_c = \left(\int z\, dv\right)/v \qquad (4.19)$$

which means that the centre of mass is coincident with the centroid of volume. It is indeed the variation in density which makes the centre of mass different from the centroid of volume for a body.

If the distribution of the mass of a body is symmetrical about an axis, the centre of mass must lie on that axis. If there are more than one axes of symmetry, the point of intersection of such axes corresponds to the centre of mass. For a body of revolution of uniform mass density, the centroid must lie on the axis of symmetry.



$$X_c = \frac{x_{c_1} m_1 + x_{c_2} m_2 - x_{c_3} m_3}{m_1 + m_2 - m_3}$$

Fig. 4.8 (b) ***Centre of Mass of a Composite Body***

The centre of mass for a composite mass consisting of masses m_1, m_2, m_3, ... with the mass centres at (x_{c1}, y_{c1}, z_{c1}) (x_{c2}, y_{c2}, z_{c2}), etc., respectively is located by

$$X_c = \frac{x_{c1}\, m_1 + x_{c2}\, m_2 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

$$= \frac{\Sigma x_c m}{\Sigma m} \tag{4.20}$$

$$Y_c = \frac{y_{c1} m_1 + y_{c2} m_2 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

$$= \frac{\Sigma y_c m}{\Sigma m} \tag{4.21}$$

$$Z_c = \frac{z_{c1} m_1 + z_{c2} m_2 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

$$= \frac{\Sigma z_c m}{\Sigma m} \tag{4.22}$$

Care must be taken, however, to ensure the proper sign of coordinate distances and masses of the component bodies. For example, the centre of mass of a composite body shown in Fig. 4.8(b) is determined by selecting the origin at O from where all x-distances are positive but the void mass 3 is considered negative because the mass 1 refers to the undrilled solid body.

***Example* 4.12** The density at any point of a slender rod varies with the first power of the distance of the point from one end of the rod. Locate the mass centre.

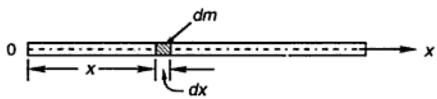


Fig. Ex. 4.12

Solution For the slender rod of length l shown in Fig. Ex. 4.12, the density at a distance x from the left end, say, is

$$\rho = kx$$

where ρ, the density is taken as the mass per unit length of the rod.

The mass dm of an element located at a distance x from the origin is, therefore,

$$dm = \rho\, dx = kx\, dx$$

From the definition of the centre of mass,

$$x_c = \left(\int x\, dm\right) \Big/ m = \left(\int x\, dm\right) \Big/ \int dm$$

$$= \left(\int_0^l kx^2\, dx\right) \Big/ \left(\int_0^l kx\, dx\right)$$

$$= \frac{kx^3}{3} \cdot \frac{2}{kx^2}\Big|_0^l = \frac{2}{3}l$$

The centre of mass for this rod is located at two-third of the length from the end chosen as a reference. This is obviously different from the centroid of volume of the slender rod which is at its mid-point.

***Example* 4.13** A thin wire of homogeneous material is bent to form an isosceles triangle as shown in Fig. Ex. 4.13. Determine the base angle α for which the centre of mass of the wire coincides with the centroid of the area enclosed by the wire.

Solution The centroid of the area of a triangle is at one-third the height of the triangle from the base. Further, by symmetry about the y-axis, the centroid must lie on it. Therefore,

$$X_c = 0 \quad \text{and} \quad Y_c = \frac{h}{3}$$

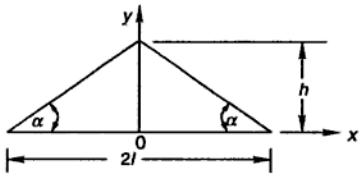


Fig. Ex. 4.13

The centre of mass of the wires of the triangle must also lie on the y-axis and the y-coordinate can be obtained by considering it as a composite mass:

Component	*Length*	*Mass*	y_c	$y_c \times$ *Mass*
Base	$2l$	$2\rho l$	0	0
Each side	$\sqrt{l^2+h^2}$	$\rho\sqrt{(l^2+h^2)}$	$\frac{h}{2}$	$\frac{\rho h}{2}\sqrt{l^2+h^2}$
Totals for the triangle	—	$2\rho(l+\sqrt{l^2+h^2})$	Y_c	$\rho h\sqrt{l^2+h^2}$

$$Y_c = \frac{\rho h\sqrt{l^2+h^2}}{2\rho(l+\sqrt{l^2+h^2})}$$

Equating it to $\frac{h}{3}$ and cancelling ρ,

$$h\sqrt{l^2+h^2} = \frac{2h}{3}l + \frac{2h}{3}\sqrt{l^2+h^2}$$

or $$h\sqrt{l^2+h^2} = 2hl$$

or $$l^2 + h^2 = 4l^2$$

and $$h = \sqrt{3}\,l$$

whence $$\alpha = \tan^{-1}\frac{h}{l} = \tan^{-1}(\sqrt{3}) = 60°$$

***Example* 4.14** A hollow cylindrical component 15 cm outer diameter and 5 cm inner diameter is 30 cm long as shown in Fig. Ex. 4.14. Its mass density varies from 4000 kg/m^3 at the left end to 10 000 kg/m^3 at the right end. Locate the centre of mass of the component

(a) by assuming a linear variation of density over the length of the component
(b) by assuming that it consists of three components *A*, *B* and *C* of mass densities 4000 kg/m^3, 7000 kg/m^3 and 10 000 kg/m^3.

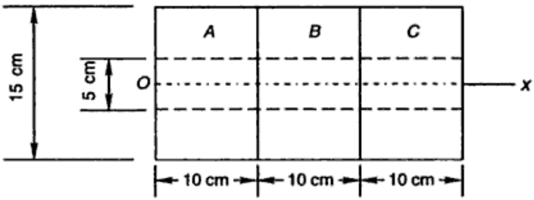


Fig. Ex. 4.14

Solution By virtue of axisymmetry of the component about x-axis as shown in Fig. Ex. 4.14, the mass centre must be located on the x-axis.

The area of the cross-section is

$$a = \frac{\pi(0.15^2 - 0.05^2)}{4}$$

(a) By definition of the centre of mass,

$$x_c = \left(\int x\,dm\right) \Big/ \int dm$$

$$= \left(\int x\rho\,dV\right) \Big/ \int \rho\,dV$$

$$= \left(\int x\rho\,dx\right) \Big/ \int \rho\,dx$$

For linear variation of density,

$$f = 4000 + (10\,000 - 4000) \times \frac{x}{0.3}$$

$$= 4000 + 20\,000\,x$$

where x is in m.

With reference to the origin at the left end,

$$x_c = \frac{\int_0^{0.3} (4000 + 20000x)\,x dx}{\int_0^{0.3} (4000 + 20000x)\,dx}$$

$$= \frac{\left|4000x^2/2 + 20\,000x^3/3\right|_0^{0.3}}{\left|4000x + 20\,000x^2/2\right|_0^{0.3}}$$

$$= \frac{360}{2100} = 0.1715 \text{ m} = 17.15 \text{ cm from the left end.}$$

(b) For the components of the composite mass,

Component	*Mass*	x_c	$x_c \times$ *Mass*
A	$4000 \times a \times 0.1$	0.05	20 a
B	$7000 \times a \times 0.1$	0.15	105 a
C	$10\,000 \times a \times 0.1$	0.25	250 a
Totals	**2100 a**		**375 a**

The centre of mass of the composite area is, therefore, located at

$$X_c = \frac{375a}{2100\,a} = 0.1786 \text{ m}$$

$$= 17.86 \text{ cm from the left end.}$$

It can be observed that the process of taking the averaged density over the segments yields the same mass, i.e., 2100 a kg as by assuming a linear variation of the density as expected but the location of the mass centre is different because the first moments of the variable-density mass involve second-order terms.

***Example* 4.15** A concentric hole of diameter 10 cm is drilled half way through a 20 cm diameter, 30 cm long solid cylinder of brass. The hole is then filled completely with gold and finished flush to make it a complete cylinder again as shown in Fig. Ex. 4.15. Locate the centre of mass of the finished cylinder.

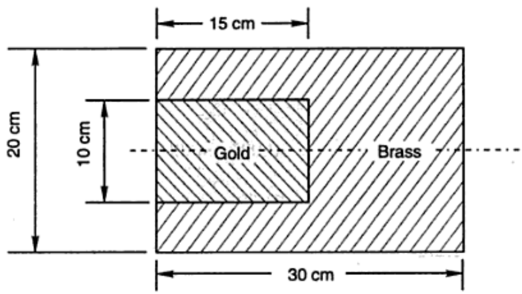


Fig. Ex. 4.15

Assume that the density of brass is 8500 kg/m^3 and density of gold is 19 500 kg/m^3.

Solution The composite mass may be visualised as being composed of:

a solid brass cylinder 20 cm diameter and 30 cm long
– a solid brass cylinder 10 cm diameter and 15 cm long
+ a solid gold cylinder 10 cm diameter and 15 cm long

This can also be interpreted as

1. a solid brass cylinder 20 cm diameter and 30 cm long plus
2. a solid cylinder 10 cm diameter and 15 cm long with a density equal to the difference of densities in gold and brass.

Consequently, taking the origin at the left end of the cylinder on the axis, as shown in Fig. Ex. 4.14.

Component	x_c *(cm)*	*m (kg)*	$x_c m$ *(kg cm)*
1	15	$\frac{\pi \times 20^2}{4} \times 30 \times \frac{8500}{10^6}$ $= 80.1$	1201.5
2	7.5	$\frac{\pi \times 10^2}{4} \times 15 \times \frac{(19500 - 8500)}{10^6}$ $= 12.96$	97.2
Total		**93.06**	**1298.7**

The distance of the centre of mass of the finished composite cylinder from the left end of the cylinder on the axis is

$$X_c = \frac{1298.7}{93.06} = 13.96 \text{ cm}$$

4.10 CENTRE OF GRAVITY

The *centre of gravity* for a body of mass m acted upon by a parallel and uniform gravitational force field is a point through which the resultant force due to gravity would act whatever the orientation of the body may be.

For a given orientation of the body, the gravitational force acts vertically downward. The line of action of the resultant force must also be a vertical line and the resultant force must be a summation of the forces acting on the mass elements, i.e.,

$$\mathbf{F} = \int d\mathbf{F} = -\int g\, dm\, \mathbf{k}$$

$$= -g \int dm\, \mathbf{k} = -mg\, \mathbf{k} \tag{4.23}$$

which is also the total gravitational pull on the body as shown in Fig. 4.9.

The line of action of the resultant gravitational force, i.e., $-mg\,\mathbf{k}$ should be such that the moment by the resultant about any arbitrary origin or about any set of axes must be the same as that exerted by the elemental forces distributed over the mass

$$\mathbf{r}_c \times \mathbf{F} = \int \mathbf{r} \times (-g\, dm\, \mathbf{k})$$

or $$\mathbf{r}_c \times (-mg\, \mathbf{k}) = \int \mathbf{r} \times (-g\, dm\, \mathbf{k})$$

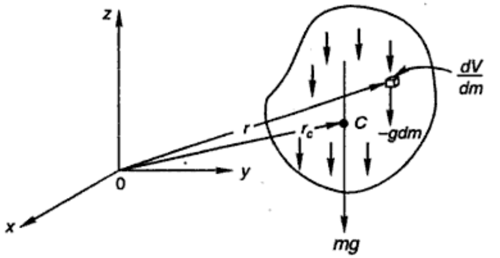


Fig. 4.9 *Concept of Centre of Gravity*

or $$\mathbf{r}_c \times m\,\mathbf{k} = \int \mathbf{r} \times dm\,\mathbf{k} \tag{4.24}$$

whence,

$$x_c = \left(\int x\,dm\right)/m = \left(\int x\rho\,dV\right)/m$$

and $$y_c = \left(\int y\,dm\right)/m = \left(\int y\rho\,dV\right)/m \tag{4.25}$$

If the body is now turned through some angle, say by 90°, about an axis other than the vertical axis, another line of action of the resultant gravitational force can be located and the point of intersection of the two lines of action locates the centre of gravity.

The coordinates of the centre of gravity are the same which locate the centre of mass of a body. It follows that the line of action of the gravitational force on a body of mass m must also pass through its centre of mass.

The centre of gravity can be different from the centre of mass only when the gravitational force field is not parallel and uniform, i.e., if there is a change in the magnitude or the direction of the gravitational force. It can be visualised that the centre of gravity of a large body with the dimensions not negligible in comparison with the radius of the earth or of a body of considerable width where the gravitational force must be taken directed towards the centre of the earth will differ from the centre of mass. For most practical purposes and unless otherwise stated, the centre of gravity and centre of mass are assumed to be identical. It is for this reason that many authors do not distinguish between them.

The centre of gravity of a material body may also become identical with the centroid of volume of the body if the material is homogeneous, i.e., if the density

$$\rho = \text{Constant}$$

Further, for a thin plate of constant thickness and homogeneous material, the centre of gravity may tend to coincide with the centroid of area of the plate. This, then, offers an experimental method of determining the centroid of area of a thin lamina. If the lamina is suspended by a thread from a point at its periphery, the line of suspension of the thread passes through the centroid. If now the lamina is suspended again from another point at its periphery, the new line of suspension of the thread also passes through its centroid. The point of intersection of the two lines locate the centroid as shown in Fig. 4.10. One may, however, suspend the lamina for a third time and ensure the location of the centroid.

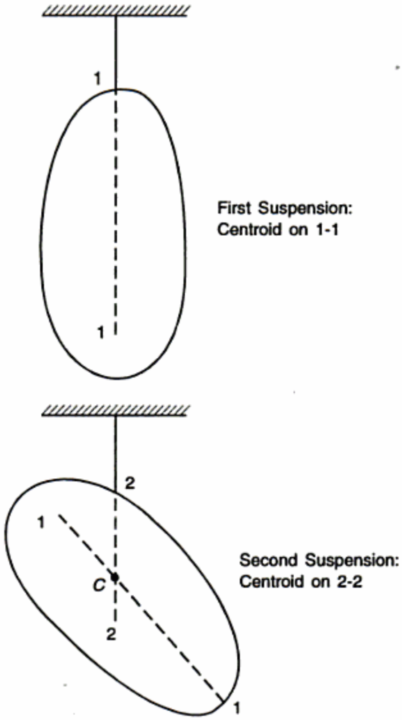


Fig. 4.10 ***Experimental Determination of the Centroid for a Thin Plate***

***Example* 4.16** Locate the centre of gravity of an idealised bullet of 1 cm diameter with a cone in the front and a hemisphere cut from the back as shown in Fig. Ex. 4.16. Assume the material to be homogeneous.

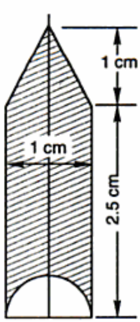


Fig. Ex. 4.16

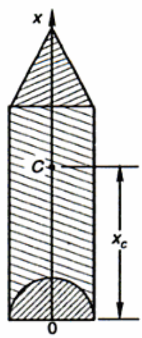


Fig. Ex. 4.16 ***(Solution)***

Solution By virtue of axisymmetry about the x-axis, the centre of gravity which is also the same as the centre of mass or centroid of volume due to homogeneity of the material must lie on the x-axis.

The bullet is idealised to consist of a cone plus a cylinder minus a hemisphere as shown in Fig. Ex. 4.16. The following table is made with reference to O as origin.

Component	*Volume* (cm^2)	x_c (cm)	$x_c \times$ *Volume* (cm^4)
Cone	$\frac{\pi}{3}(0.5)^2 \times 1$ $= 0.262$	$2.5 + 0.25$ $= 2.75$	0.7205
Cylinder	$\pi \times (0.5)^2 \times 2.5$ $= 1.963$	1.25	2.4544
Hemisphere	$-\frac{2}{3}\pi \times (0.5)^3$ $= -0.262$	$\frac{3 \times 0.5}{8}$ $= 0.1875$	-0.0491
Total	**1.963**		**3.1258**

The x coordinate of the centroid of the bullet is given by

$$X_c = \frac{3.1258}{1.963} = 1.592 \text{ cm from the base.}$$

***Example* 4.17** An isosceles triangle is to be cut out from one edge of a square piece of thin uniform sheet as shown in Fig. Ex. 4.17 such that the remaining sheet when suspended from the apex P of the cut will remain in equilibrium in any position. Find the area of the triangle cut-out.

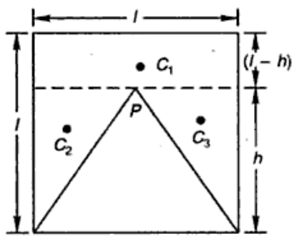


Fig. Ex. 4.17

Solution In order that a body remains in equilibrium in any orientation when suspended from a point, the point must be the centre of gravity. For the piece of thin uniform sheet, the point also qualifies to be the centre of area or the centroid.

The apex point P can, therefore, be located by the definition of the centroid:

$$(l-h)l \times \frac{(l-h)}{2} - 2 \times \frac{1}{2} \times \frac{h}{2} \times \frac{h}{3} = 0$$

or $$2h^2 - 6lh + 3l^2 = 0$$

whence, $$h = \frac{6l \pm \sqrt{36l^2 - 24l^2}}{4}$$

$$= [(3 \pm \sqrt{3})\, l]/2$$

$$= [(3 + \sqrt{3})l]/2 \qquad \text{or} \qquad [(3 - \sqrt{3})l]/2$$

Rejecting the first solution which requires h to be more than l,

$$h = [(3 - \sqrt{3})l]/2 = 0.634l$$

The area of the triangle cut out is, therefore,

$$0.634l \times l/2 = 0.317\, l^2$$

Alternatively, the apex point P can be located by the fact that this is arrived at by removing the cut-out triangle of area a from the given square piece of area l^2.

$$l^2 \times \left[\frac{1}{2} - l(l-h)\right] a \times \frac{2h}{3} = 0$$

Using the fact that $h = \dfrac{2a}{l}$

$$l^2 \times \left[\frac{1}{2} - \left(l - \frac{2a}{l}\right)\right] - a \times \frac{4a}{3l} = 0$$

or

$$8a^2 - 12l^2a + 3l^4 = 0$$

$$a = \frac{12l^2 \pm \sqrt{144l^4 - 96l^4}}{16}$$

$$= [(3 \pm \sqrt{3})/4]l^2$$

Again, rejecting the impermissible solution, the answer is

$$a = [(3 - \sqrt{3})/4]l^2 = 0.317\, l^2$$

***Example* 4.18** A wooden block of cross-section 10 cm × 10 cm is fixed on top of a semicircular steel cylinder of radius 5 cm as shown in Fig. Ex. 4.18. Determine the maximum height h of the wooden block so that the composite body will be in stable equilibrium at its base. It is given that the density of wood is one-tenth that of steel.

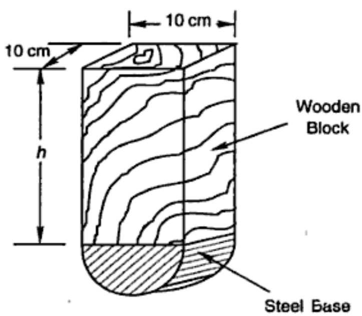


Fig. Ex. 4.18

Solution The composite body should be in stable equilibrium as long as the centre of gravity lies on the semi-cylindrical base. This is so because then a restoring couple will act on the body when it is tipped on either side.

In the parallel and uniform gravitational field of the earth, the centre of mass must be the centre of gravity. We proceed in a tabular form with reference to the mid-point on the base, to determine the centre of mass of the composite body.

Component	*Mass (kg)*	y_c *(cm)*	$y_c \times$ *Mass (kg cm)*
Wooden block	$10 \times 10 \times h \times \rho_1$ $= 100\, h\rho_1$	$5+\frac{h}{2}$	$\left(5+\frac{h}{2}\right)\times 100\, h\rho_1$
Semi-cylindrical base	$\frac{\pi \times 5^2}{2} \times 10\rho_2$ $= 390\rho_2$	$\frac{4\times 5}{3\times \pi}$ $= 2.1$	$820\, \rho_2$
Total	**$100\, h\rho_1 + 390\, \rho_2$**		**$500\, h\rho_1 + 50\, h^2\rho_1 + 820\, \rho_2$**

The centre of mass of the composite body is, therefore, situated at

$$Y_c = \frac{500\, h\rho_1 + 50\, h^2 \rho_1 + 820\, \rho_2}{100\, h\rho_1 + 390\, \rho_2}$$

which must be a maximum of 5 cm. Equating it to 5 cm to obtain the maximum height h of the wooden block

$$5 = \frac{500\, h\rho_1 + 50\, h^2 \rho_1 + 820\, \rho_2}{100\, h\rho_1 + 390\, \rho_2}$$

or $$1950\, \rho_2 = 50\, h^2\rho_1 + 820\, \rho_2$$

whence $$h = 4.75\sqrt{\frac{\rho_2}{\rho_1}}$$

Since $$\rho_2 = 10\, \rho_1$$

$$h = 4.75\sqrt{10} = 15 \text{ cm}$$

It may be noted that if the block and base are made of the same material, i.e., $\rho_2 = \rho_1$, then

$$h = 4.75 \text{ cm}$$

but if the block material is lighter than the base material, the length of the block can be made much longer for maintaining stable equilibrium. On the other hand, if the block material is heavier than the base material, the length of the block would be less than 4.75 cm.

Concept Review Questions

1. State why
   (a) The centroid of a curve, an area or a volume is independent of the choice of the origin or the orientation of the coordinate axes?

(b) The theorems of Pappus and Guldinus are valid for a complete revolution or a fraction of a revolution of the generating line or area?

2. Comment on the concept of the first moment of an element about an origin and about a line. Does the first moment have any relationship with the moment of a force?
3. Compare the location of the centroids of an arc of a circle and a sector of a circle subtending the same angle at the centre of the circle.
4. Under what conditions do the following coincide?
   (a) Centre of mass and centre of gravity.
   (b) Centroid of volume and centre of mass.
   (c) Centre of gravity and centroid of area.
5. Would you agree or disagree with the following statements and why?
   (a) The centroid of a body may or may not lie on a material point in the body.
   (b) The centroid of an area symmetrical about two axes must be the point of intersection of these axes.
   (c) The centroid of a parallelogram is located by the point of intersection of its diagonals because the diagonals are the axes of symmetry.
   (d) The vertical line of free suspension of a thin sheet of homogeneous or non-homogeneous material must contain the centre of gravity.

Tutorial Problems

4.1 Determine the moment of a semicircular arc about its diameter and hence locate its centroid.

$$\left(\textbf{Ans. } 2R^2; \frac{2R}{\pi}, 0\right)$$

4.2 Determine the y coordinate of the centroid of the area between the x-axis and the curves $y = \sin x$ between 0 and π. (**Ans.** $\pi/8$)

4.3 Locate the centroid of the area of a segment of a circle which subtends an angle 2θ at the centre. (**Ans.** 0, $2R \sin\theta/3\theta$)

4.4 Locate the centroid of a trapezium with the base b and the parallel sides h_1 and h_2.

$$\left(\textbf{Ans. } x_c = \frac{b(h_1 + 2h_2)}{3(h_1 + h_2)}, y_c = \frac{h_1^2 + h_2^2 + h_1 h_2}{3(h_1 + h_2)}\right)$$

4.5 Determine the location of the centroid of the area bounded by the x-axis and the sine curve $y = a \sin \frac{\pi x}{l}$ from $x = 0$ to $x = l$. (**Ans.** ($l/2$, $\pi a/8$))

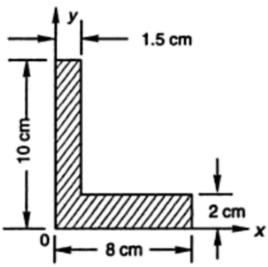


Fig. Prob. 4.6

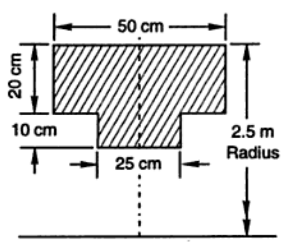


Fig. Prob. 4.7

4.6 Find the centroid of the *L*-section shown in Fig. Prob. 4.6.

(**Ans.** x_c = 2.6 cm, y_c = 3.17 cm)

4.7 A flywheel of outside diameter 5 m has a heavy rim of the cross-section shown in Fig. Prob. 4.7. Determine the mass of the rim if the density of the material of the rim is 7000 kg/m^3. (**Ans.** 13 030 kg)

4.8 From a circular area of diameter 2*d*, a smaller circle of diameter *d* is removed as shown in Fig. Prob. 4.8. Locate the centroid of the remaining area.

(**Ans.** *d*/6 left of 0)

4.9 Find the surface area of the annular torus formed by revolving the circle about the *x*-axis as shown in Fig. Prob. 4.9. (**Ans.** 4 $\pi^2 rR$)

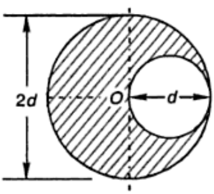


Fig. Prob. 4.8

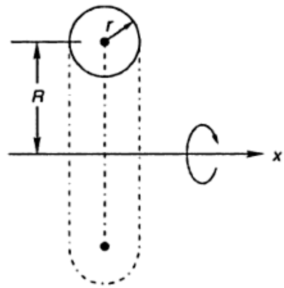


Fig. Prob. 4.9

4.10 An area is bounded by the curves $y^2 = 9x$ and $x^2 = 6y$. Sketch the area and find the coordinates of the centroid. (**Ans.** x_c = 3.09, y_c = 3.54)

4.11 A concentric hole of 10 cm diameter is drilled to a depth of 15 cm in a perspex cylinder of diameter 20 cm and 40 cm long as shown in Fig. Prob. 4.11. The hole is filled with lead to make it a complete cylinder again. Locate the centre of mass of this cylinder. Take the density of perspex as 1200 kg/m^3 and of lead as 12 000 kg/m^3.

(**Ans.** 5.72 cm below the centre of cylinder)

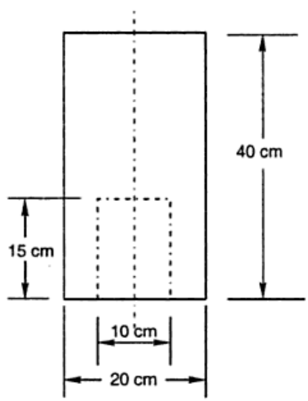


Fig. Prob. 4.11

4.12 A square hole is punched out of a thin circular lamina, the diagonal of the square being equal to the radius of the circle as shown in Fig. Prob. 4.12. Find the centre of gravity of the remaining lamina.

(**Ans.** 0.095 *R* left of 0)

4.13 A frustrum of a solid right circular cone of base diameter 2 m, top diameter 1 m and height 2 m has an axial hole of 0.5 m diameter in it. Locate the centroid of volume of the hollow cone. (**Ans.** 0.76 m from the base and on the axis)

4.14 Determine the maximum height *h* of a right circular cylinder mounted on a hemispherical base as shown in Fig. Prob. 4.14 so that the composite body may be in stable

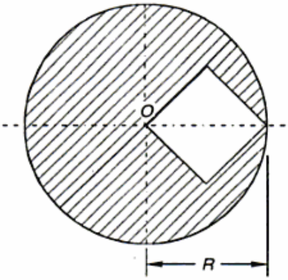


Fig. Prob. 4.12

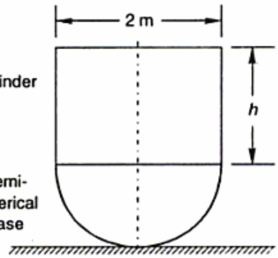


Fig. Prob. 4.14

block made of wood, the density of wood being one-tenth that of steel?

$\left(\textbf{Ans. } R/\sqrt{2},\ \sqrt{5}\ R\right)$

4.15 A thin rectangular plate of length a and width b of homogeneous material is suspended from a corner. Calculate the angle the longer side will make with the vertical in the equilibrium position?

$\left(\textbf{Ans. } \tan^{-1}\dfrac{b}{a}\right)$

4.16 Determine the length of a thin homogeneous wire which is bent into a semicircular arc of radius R together with extensions on either end as shown in Fig. Prob. 4.16 such that the centroid is located at the centre O. **(Ans.** $(\pi + 2)\sqrt{2}\ R$**)**

4.17 A thin semicircular bar of weight w is suspended from a hinge at A as shown in Fig. Prob. 4.17. Determine the angle between the diameter and the vertical line. What would the angle be if a weight W is suspended from point B?

$\left(\textbf{Ans. } \tan^{-1}\dfrac{2}{\pi} \text{ and } \tan^{-1}\dfrac{2w}{\pi(w + 2W)}\right)$

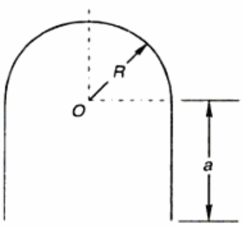


Fig. Prob. 4.16

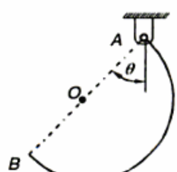


Fig. Prob. 4.17

4.18 Knowing that the surface and volume of a sphere of radius r are $4\pi r^2$ and $4/3\pi r^3$ respectively, deduce the centroid of a semi-circular arc and a semi-circular disc, each of radius r. **(Ans.** $2r/\pi$, $4r/3\pi$**)**

4.19 A frame consists of a wire bent into a rectangular shape 0.3 m by 0.2 m plus a length of the same wire bent into a semicircle of 0.3 m diameter fixed to a 0.3 m side. Find the distance of the mass-centre of the frame from the 0.3 base. **(Ans.** 0.163 m**)**

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. The length of a line can be considered to be concentrated at the centroid of that line for the purpose of calculating.
   (a) the area of the surface of revolution generated by revolving the line about any axis outside it.
   (b) the average distance of the line from any axis outside it.
   (c) the volume of the body generated by revolving the line about any axis outside it.
   (d) the weightage factor for all purposes.
2. The centre of volume and centre of mass of a body coincide.
   (a) if and only if the body is of uniform density.
   (b) if the body is geometrically symmetrical about the centre of mass.
   (c) if the density variation is symmetrical about the centroid.
   (d) if and only if the body is made of homogeneous material.
3. The centroid of a body
   (a) must be a point on that body.
   (b) is a point which can be made to lie on or outside the body by changing the coordinate system.
   (c) is a fixed point in space regardless of the orientation of the body.
   (d) is a unique point fixed with respect to the body.
4. The first moment of area of a semicircular area about its diameter d is given by
   (a) $d^3/12$.
   (b) $d^3/24$.
   (c) $d^3/6$.
   (d) $d^3/36$.
5. The first moment of a triangular area of base b and height h taken about an axis coincident with the base is given by
   (a) $bh^3/12$.
   (b) $b^2h/6$.
   (c) $bh^2/6$.
   (d) $h/3$.
6. Given that there is a rectangle and a triangle, each of base b and area A, the first moment of the area of the rectangle about its base
   (a) equals the first moment of the triangular area about its base b.
   (b) is more than the first moment of the triangular area about its base b.
   (c) is less than the first moment of the triangular area about its base b.
   (d) equals twice that of the triangular area about its base b.

Answers to Multiple-Choice Questions

1 (a), 2 (c), 3 (d), 4 (a), 5 (c), 6 (c)

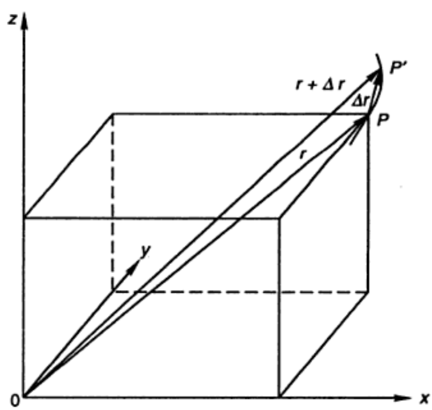
5 KINEMATICS OF A PARTICLE

and of a Point in General

5.1 KINEMATIC CONCEPTS

Kinematics refers to the study of motion of bodies without reference to mass or force. It deals with 'displacement, velocity and acceleration' of a point of interest at a particular time or with the passage of time.

A point may be displaced from its initial position in any direction, i.e., the displacement of a particle can have arbitrary components along the three mutually perpendicular directions. The particle is said to possess three degrees of freedom in a general motion. The number of the degrees of freedom of a particle in a given configuration equals the minimum number of coordinates required to describe its configuration. The minimum number of coordinates is often called the generalised coordinates. For example, a particle confined to move in a plane has two degrees of freedom and hence two generalised coordinates are required to describe its motion; the generalised coordinates may be x, y or r, θ or some other pair of independent coordinates not necessarily belonging to any particular coordinate system. A particle constrained to move along a curve, spatial or plane, has only one degree of freedom, and only one generalised coordinate, such as x, y, z, s, r or θ to describe its location at any instant.



(a) Space Motion

Fig. 5.1 ***Types of Motion***

5.2 MOTION REFERRED TO FIXED RECTANGULAR COORDINATES

Consider a point P moving in space with respect to a rectangular or cartesian frame of reference fixed in space as shown in Fig. 5.1(a). The *position vector* of the point is denoted by a vector $\mathbf{r}$.

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \tag{5.1}$$

at a certain instant of time t. In words, the point is located by tracing x along the x-axis, y along the y-axis and z along the z-axis.

Over a short interval of time Δt, the point moves over to a new position P' described by the position vector $\mathbf{r} + \Delta\,\mathbf{r}$,

$$\mathbf{r} + \Delta\,\mathbf{r} = (x + \Delta\,x)\,\mathbf{i} + (y + \Delta y)\,\mathbf{j} + (z + \Delta\,z)\,\mathbf{k}$$

The point is said to have been displaced by $\Delta\,\mathbf{r}$ such that

$$\begin{aligned}\Delta\,\mathbf{r} &= (\mathbf{r} + \Delta\,\mathbf{r}) - \mathbf{r}\\ &= (x + \Delta\,x)\,\mathbf{i} + (y + \Delta\,y)\,\mathbf{j} + (z + \Delta\,z)\,\mathbf{k}\\ &\quad - (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\\ &= \Delta\,x\,\mathbf{i} + \Delta\,y\,\mathbf{j} + \Delta\,z\,\mathbf{k}\end{aligned}$$

In words, the displacement $\Delta\,\mathbf{r}$ of a point is composed of displacements Δx along the x-axis, $\Delta\,y$ along the y-axis and $\Delta\,z$ along z-axis.

The *velocity* of the point is the rate of change of displacement:

$$\mathbf{V} = \underset{\Delta t \to 0}{\frac{\Delta\,\mathbf{r}}{\Delta\,t}} = \left(\underset{\Delta t \to 0}{\frac{\Delta x}{\Delta t}}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k}\right)$$

$$\begin{aligned}&= \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\\ &= V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}\end{aligned} \tag{5.2}$$

where $V_x = \dfrac{dx}{dt}$ = x-component of the velocity

$V_y = \dfrac{dy}{dt}$ = y-component of the velocity

$V_z = \dfrac{dz}{dt}$ = z-component of the velocity

and $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$ = speed, the magnitude of the velocity.

The *acceleration* of the point is the rate of change of velocity:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{V}}{\Delta t} = \frac{d\mathbf{V}}{dt}$$

$$= \frac{d^2 \mathbf{r}}{dt^2} = \frac{d^2 x}{dt^2}\mathbf{i} + \frac{d^2 y}{dt^2}\mathbf{j} + \frac{d^2 z}{dt^2}\mathbf{k}$$

$$= \frac{dV_x}{dt}\mathbf{i} + \frac{dV_y}{dt}\mathbf{j} + \frac{dV_z}{dt}\mathbf{k}$$

$$= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{5.3}$$

where $a_x = \frac{dV_x}{dt} = \frac{d^2 x}{dt^2}$ = x-component of the acceleration

$$a_y = \frac{dV_y}{dt} = \frac{d^2 y}{dt^2} = y\text{-component of the acceleration}$$

$$a_z = \frac{dV_z}{dt} = \frac{d^2 z}{dt^2} = z\text{-component of the acceleration}$$

and $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$, magnitude of the acceleration vector.

It is possible to rewrite Eqs. (5.2) and (5.3) in the integral form

$$\mathbf{r} = \int \mathbf{V}\, dt + \mathbf{C}$$

$$\mathbf{V} = \int \mathbf{a}\, dt + \mathbf{K}$$

where **C** and **K** are the vector constants of integration.

In terms of definite integrals, the change in velocity is given by

$$\mathbf{V} - \mathbf{V}_0 = \int_{t_0}^{t} \mathbf{a}\, dt \tag{5.4}$$

and the displacement is expressed as

$$\mathbf{r} - \mathbf{r}_0 = \int_{t_0}^{t} \mathbf{V}\, dt \tag{5.5}$$

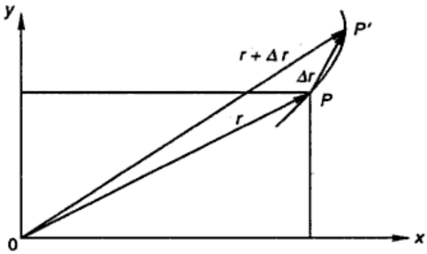
where $\mathbf{V}_0$ and $\mathbf{r}_0$ are the velocity and displacement respective at time t_0.

It may also be noted that a differential element of the arc-length is given by

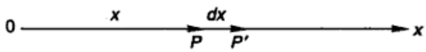
$$ds = (d\mathbf{r} \cdot d\mathbf{r})^{1/2} = [(dx)^2 + (dy)^2 + (dz)^2]^{1/2} \tag{5.6}$$

The distance travelled by a point along its trajectory can be calculated by integrating this expression over the appropriate time interval.

The plane motion of a particle may be studied by referring to two coordinaters as shown in Fig. 5.1(b).



(b) Plane Motion



(c) Rectilinear Motion

Fig. 5.1 ***Types of Motion*** (*Contd.*)

5.3 RECTILINEAR MOTION OF A POINT

In order to study the rectilinear motion of a point, we choose the x-axis along the line of motion as shown in Fig. 5.1(c). From the expressions for the position vector, displacement, velocity and acceleration, we write

$$\mathbf{r} = x\,\mathbf{i};\ \Delta\,\mathbf{r} = \Delta x\,\mathbf{i}$$

$$\mathbf{V} = \frac{dx}{dt}\mathbf{i} = V_x i$$

$$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} = \frac{dV_x}{dt}\mathbf{i} = a_x\ \mathbf{i}$$

Since all these vectors are directed along the x-axis only, the vector notation and suffices may be dropped.

Hence,

$$V = \frac{dx}{dt}, \qquad x = \int V\,dt + C$$

and

$$a = \frac{d^2x}{dt^2} = \frac{dV}{dt}, \qquad V = \int a\,dt + K$$

If the acceleration a is constant,

$$V = at + K$$

and if the initial velocity of the point at time $t = 0$ is U, then at any time t,

$$V = U + at \tag{5.7}$$

From the expression for x, the distance moved by the point

$$s = \int (U + at)\,dt$$

or $$s = Ut + \frac{1}{2}at^2 \tag{5.8}$$

Eliminating t in Eqs. (5.7) and (5.8)

$$V^2 - U^2 = 2\,as$$

Motion curves or motion diagrams are drawn to show the variation of displacement, velocity and acceleration with time for the rectilinear motion of a particle. An important aspect of the motion curves is that these are mutually related. Drawing of motion curves is useful to obtain a graphical picture of the distance traversed, velocity at any instant, the average velocity and the effect of acceleration particularly when the motion occurs in distinct phases.

The motion curves of some typical motions of a particle are shown in Fig. 5.2. In order to explore the relationship between the curves, consider the motion of the particle over a time interval $(t_2 - t_1)$.

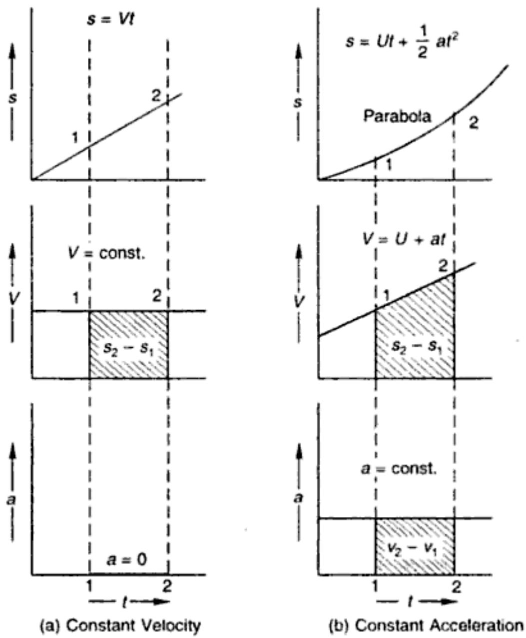


Fig. 5.2 ***Motion Curves***

From the relationship,

$$\Delta V = V_2 - V_1 = \int_1^2 a\,dt$$

it is clear that the area under the acceleration diagram for a time interval must equal the change of the velocity during that interval.

From the relationship,

$$\Delta s = s_2 - s_1 = \int_1^2 V\,dt$$

it follows that the area under the velocity diagram for a time interval must equal the displacement over the time interval.

Some comments on the shape of the motion curves can now be made:

1. The linear displacement diagram corresponds to uniform velocity, i.e., zero acceleration.
2. Constant acceleration implies linear velocity variation and a parabolic displacement diagram.
3. The point of zero acceleration must correspond to the point of maxima or minima or inflexion on the velocity curve and the point of inflexion on the displacement diagram.
4. The slope of the velocity curve must be the maximum at the point of the maximum acceleration.
5. The reversal of the motion of a particle corresponds to a drop in the displacement curve, reversal of velocity and maximum acceleration.

The *rectilinear motion of a particle due to gravitational acceleration of the earth* is of a special interest. In such a case,

$$a = g = 9.81 \text{ m/s}^2 \text{ directed towards the centre of the earth.}$$

A particle approaching the earth in this manner is said to be in a state of *free fall*. During a free fall, as shown in Fig. 5.3,

$$V = U + gt$$

and

$$V^2 - U^2 = 2gh$$

In particular, if a particle is dropped from rest, $U = 0$, then

$$V = gt$$

and

$$V^2 = 2gh \quad \text{or} \quad V = \sqrt{2gh}$$

whence

$$t = \sqrt{2h/g}$$

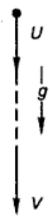


Fig. 5.3 Free Fall

For an upward motion of a particle in the gravitational field,

$$V = U - gt$$

and

$$V^2 - U^2 = 2gh$$

Since it is acted upon by a declaration $-g$.

When it comes to rest, $V = 0$

$$U = gt$$

$$U^2 = 2gh$$

and again, $t = \sqrt{2h/g}$

***Example* 5.1** A point moving in a straight line is observed to accelerate as

$$a = 12t - 20$$

It passed through a reference point at $t = 0$ and another point 20 cm away after an interval of 5 s. Calculate the displacement, velocity and acceleration of the point after a further interval of 5 s.

Solution

$$a = 12t - 20$$

Integrating with respect to time t,

$$V = 6t^2 - 20t + C_1$$

and integrating once again,

$$S = 2t^3 - 10t^2 + C_1t + C_2$$

Using the boundary conditions,

at $t = 0$; $S = 0$; $C_2 = 0$ and at $t = 5$ s, $S = 20$ cm, $C_1 = 4$

The expressions for the displacement, velocity and acceleration are, therefore,

$$S = 2t^3 - 10\,t^2 + 4t \text{ cm}$$

$$V = 6t^2 - 20t \text{ cm/s}$$

$$a = 12t - 20 \text{ cm/s}^2$$

Substituting $t = 5 + 5 = 10$ s

$$S = 1040 \text{ cm} = 10.4 \text{ m}$$

$$V = 400 \text{ cm/s} = 4 \text{ m/s}$$

$$a = 100 \text{ cm/s}^2 = 1 \text{ m/s}^2$$

***Example* 5.2** A particle, while at rest at the position (5, 6, 2) is accelerated at

$$\mathbf{a} = 6t\,\mathbf{i} - 24t^2\mathbf{j} + 10\,\mathbf{k} \text{ m/s}^2$$

Determine the acceleration; velocity and displacement of the particle after a lapse of one second.

Solution

At $t = 0$

$$\mathbf{r} = 5\,\mathbf{i} + 6\,\mathbf{j} + 2\,\mathbf{k}$$

and $\mathbf{V} = 0$

From the acceleration

$$\mathbf{a} = 6t\,\mathbf{i} - 24t^2\,\mathbf{j} + 10\,\mathbf{k} \tag{i}$$

the velocity at any instant must be

$$\mathbf{V} = \int \mathbf{a}\, dt + \mathbf{K}$$

$$= 3t^2\,\mathbf{i} - 8t^3\mathbf{j} + 10t\,\mathbf{k} + \mathbf{K}$$

where $\mathbf{K} = 0$ from the initial condition, $\mathbf{V} = 0$ at $t = 0$

Hence, $$\mathbf{V} = 3t^2\mathbf{i} - 8t^3\mathbf{j} + 10t\,\mathbf{k} \quad \text{(ii)}$$

Integrating Eq. (ii) with respect to time t again,

$$\mathbf{r} = \int \mathbf{V}\, dt + \mathbf{C}$$

$$= t^3\mathbf{i} - 2t^4\mathbf{j} + 5t^2\mathbf{k} + \mathbf{C}$$

From the initial condition

$$\mathbf{r} = 5\,\mathbf{i} + 6\,\mathbf{j} + 2\,\mathbf{k} \text{ at } t = 0$$

$$\mathbf{C} = 5\,\mathbf{i} + 6\,\mathbf{j} + 2\,\mathbf{k}$$

and

$$\mathbf{r} = (t^3 + 5)\mathbf{i} + (6 - 2t^4)\,\mathbf{j} + (5t^2 + 2)\mathbf{k} \quad \text{(iii)}$$

$$t = 1 \text{ s}$$

the acceleration is obtained from Eq. (i)

$$\mathbf{a} = 6\,\mathbf{i} - 24\,\mathbf{j} + 10\,\mathbf{k} \text{ m/s}^2$$

the velocity is obtained from Eq. (ii)

$$\mathbf{V} = 3\,\mathbf{i} - 8\,\mathbf{j} + 10\,\mathbf{k} \text{ m/s}$$

and the position vector is obtained from Eq. (iii),

$$\mathbf{r} = 6\,\mathbf{i} + 4\,\mathbf{j} + 7\,\mathbf{k} \text{ m}$$

The displacement from the initial position must be

$$\mathbf{S} = (6\,\mathbf{i} + 4\,\mathbf{j} + 7\,\mathbf{k}) - (5\,\mathbf{i} + 6\,\mathbf{j} + 2\,\mathbf{k})$$

$$= \mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k} \text{ m}$$

5.4 MOTION REFERRED TO CYLINDRICAL POLAR COORDINATES

Consider the motion of a point on a circular trajectory in the x-y plane, i.e., the r–θ plane as shown in Fig. 5.4.

The position of a point P at any time t can be specified by the x and y coordinates or the r and θ coordinates such that

$$x = r\cos\theta$$

$$y = r\sin\theta$$

or $$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

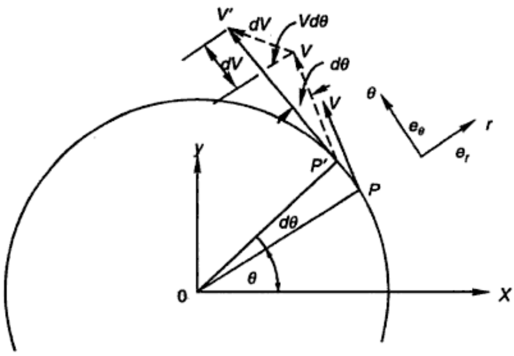


Fig. 5.4 *Circular Motion of a Point*

Let the angular displacement of the point be $\Delta\theta$ over a differential time interval Δt. The angular velocity is, therefore,

$$\omega = \lim_{\Delta t \to 0} \left(\frac{\Delta\theta}{\Delta t}\right) = \frac{d\theta}{dt}$$

directed about an axis normal to the plane; anticlockwise positive for the right-handed triad.

The linear velocity V of the point is given by

$$V = r\omega = r\frac{d\theta}{dt} \tag{5.9}$$

directed tangentially to the circular path.

Over a differential time interval dt, the velocity of the point changes both in magnitude and in direction as shown in the sketch.

Change	*Direction*	*Remarks*
dV	Tangentially forward; θ	$+ve$, if the speed increases
$Vd\theta$	Radially inwards; $-r$	$-ve$ always, because r is $+ve$ radially out

The acceleration of the point is the rate of change of velocity with time. The components of acceleration are

Tangential: $$\frac{dV}{dt} = r\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = r\frac{d^2\theta}{dt^2} = r\alpha \tag{5.10a}$$

Radial: $$-\frac{Vd\theta}{dt} = -V\omega = -\frac{V^2}{r} = -\omega^2 r \tag{5.10b}$$

The tangential component of acceleration dV/dt is by virtue of a change in speed only whereas the radial component called the centripetal acceleration is by virtue of the circular trajectory of radius r traced at a speed V. Obviously,

1. For a rectilinear motion, the 'radial' component is zero because the radius r of the 'circle' is infinite.
2. For a circular motion at a constant speed V, the tangential component $dV/dt = 0$ and the point undergoes only the centripetal acceleration $\omega^2 r$ directed radially inwards towards the centre or axis of rotation. For such a case,

$$\mathbf{a} = \omega^2 r\, \mathbf{e}_r$$

In terms of angular quantities alone,

angular displacement $= d\theta$

angular velocity $= \omega = \dfrac{d\theta}{dt}$

angular acceleration $= \alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

and in the integral form,

$$\boxed{\omega = \int \alpha\, dt + K'}$$

and

$$\boxed{\theta = \int \omega\, dt + C'}$$

If the angular acceleration α is constant,

$$\omega = \omega_0 + \alpha t \qquad (5.11)$$

where the initial angular velocity at time $t = 0$ is ω_0

and

$$\theta = \int (\omega_0 + \alpha t)\, dt + C'$$

or

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad (5.12)$$

where the angular displacement θ is referred to $\theta = 0$ at $t = 0$.

Eliminating t between the equations

$$\omega^2 - \omega_0^2 = 2\alpha\theta \qquad (5.13)$$

The relations for the angular quantities are similar to those for the linear quantities derived earlier. Table 5.1 brings out a systematic comparison.

Let us now consider the general motion of a point in space referred to cylindrical coordinates as shown in Fig. 5.5(a).

At any instant of time t, the position vector of the point P is

$$\mathbf{R} = r\,\mathbf{e}_r + z\,\mathbf{e}_z \qquad (5.14)$$

and at a later instant $t + \Delta t$, the position vector for the location Q of the point is

$$\mathbf{R} + \Delta\mathbf{R} = (r + \Delta r)(\mathbf{e}_r + \Delta\mathbf{e}_r) + (z + \Delta z)\,\mathbf{e}_z$$

The displacement of the point is, therefore, given by $\Delta\mathbf{R}$, which by subtraction is,

$$\Delta\mathbf{R} = \Delta r\,\mathbf{e}_r + r\Delta\mathbf{e}_r + \Delta z\,\mathbf{e}_z$$

neglecting the term $(\Delta r \Delta\mathbf{e}_r)$ which is smaller in order of magnitude than the other terms.

Table 5.1 Linear vs Angular Motion

	Linear Motion	*Angular Motion*	*Remarks*
Displacement	s	θ	$ds = r\,d\theta$
Velocity	$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$	$V = r\omega$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$	$a = r\alpha$
Initial velocity	u	ω_0	
Expressions relating the displacement, velocity, acceleration and time	$V = u + at$	$\omega = \omega_0 + \alpha t$	valid only for constant acceleration
	$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	
	$V^2 - u^2 = 2as$	$\omega - \omega_0 = 2\alpha\theta$	
	$s = \int V\,dt + C$	$\theta = \int \omega\,dt + C'$	
	$V = \int a\,dt + K$	$\omega = \int \alpha\,dt + K'$	

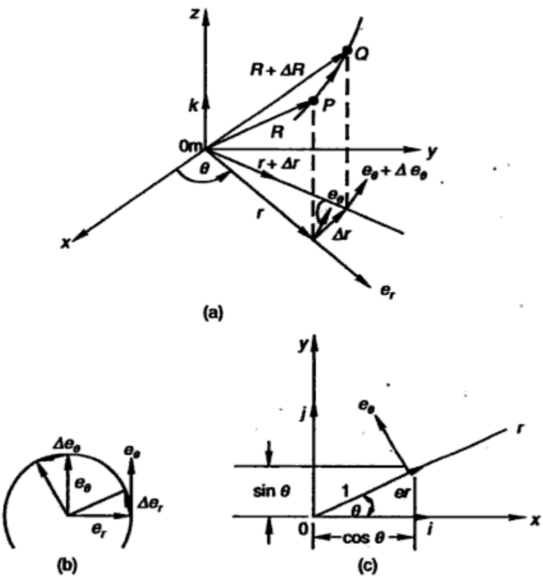


Fig. 5.4 ***Motion Referred to Cylindrical Coordinates***

The velocity of the point is expressed as

$$\mathbf{V} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{R}}{\Delta t} = \frac{d\mathbf{R}}{dt}$$

or
$$\mathbf{V} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} + \frac{dz}{dt}e_z \tag{5.15}$$

This expression requires the knowledge of the temporal derivative of the unit vector $\mathbf{e}_\theta$. It can be seen from Fig. 5.4(b) that a small change in $\mathbf{e}_r$ takes place in the direction of $(+\mathbf{e}_\theta)$ and a small change in $\mathbf{e}_\theta$ takes place only along the direction of $(-\mathbf{e}_r)$. The magnitude of the change in each case is given by the product of the magnitude of the unit vector, i.e., unity and the small angle $\Delta\theta$ through which the unit vector is turned;

$$1 \cdot \Delta\theta = \Delta\theta$$

Consequently,

$$\frac{d\mathbf{e}_r}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{e}_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}\mathbf{e}_\theta$$

$$= \omega\,\mathbf{e}_\theta \tag{5.16}$$

$$\frac{d\mathbf{e}_\theta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{e}_\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}(-\mathbf{e}_r)$$

$$= -\,\omega\,\mathbf{e}_r \tag{5.17}$$

The temporal derivatives of the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ may alternatively be obtained as follows with reference to Fig. 5.4(c). If, at a particular instant, the radial direction makes an angle θ with the x direction,

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

since the magnitude of $\mathbf{e}_r$ is unity.

Differentiating with respect to time t

$$\frac{d\mathbf{e}_r}{dt} = -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j}$$

$$= \frac{d\theta}{dt}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$$

$$= \omega\,\mathbf{e}_\theta$$

Similarly,

$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

$$\frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

$$= -\,\omega\,\mathbf{e}_r$$

Returning to the equation for the velocity of the point,

$$\boxed{\mathbf{V} = \dot{r}\,\mathbf{e}_r + r\omega\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z}$$

or
$$\mathbf{V} = V_r\,\mathbf{e}_r + V_\theta\,\mathbf{e}_\theta + V_z\,\mathbf{e}_z \tag{5.18}$$

Hence, the acceleration of the point,

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}(\dot{r}\,\mathbf{e}_r + r\omega\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z)$$

$$= \ddot{r}\,\mathbf{e}_r + \dot{r}\omega\,\mathbf{e}_\theta + \dot{r}\omega\,\mathbf{e}_\theta + r\omega\,\mathbf{e}_\theta - r\omega^2\,\mathbf{e}_r + z\,\mathbf{e}_z$$

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z} \qquad (5.19)$$

where dot (·) stands for derivative with respect to time.

Observation of the results for **V** and **a** lead us to some interesting and useful conclusions:

For plane motion in the x-y plane

$$z = \text{Const.},\ \dot{z} = O = \ddot{z}$$

the point can, therefore, have neither velocity nor acceleration in a direction perpendicular to the plane.

For constant speed of rotation ω

$$\omega = \dot{\theta} = \text{Const.},\ \omega = \ddot{\theta} = 0$$

the point can, therefore, not have any angular acceleration.

For plane circular motion in the x-y plane

$$z = \text{Const.},\quad \dot{z} = 0 = \ddot{z}$$

and $$r = \text{Const.},\quad \dot{r} = 0 = \ddot{r}$$

the expressions for the velocity and acceleration reduce to

$$\mathbf{V} = r\omega\,\mathbf{e}_\theta$$

$$\mathbf{a} = -r\omega^2\,\mathbf{e}_r + r\alpha\,\mathbf{e}_\theta$$

which shows that the velocity of the point is wholly tangential at any instant whereas the acceleration has two components:

$r\omega^2$ radially inwards, towards the centre of the circle, due to the rotational speed of the point about the centre

$r\alpha$ tangentially directed due to the angular acceleration α of the point.

In particular, for a plane circular motion at constant speed, the tangential acceleration vanishes and the point is only subjected to a tangential velocity together with a radially inwards acceleration often termed as the centripetal acceleration.

For helical motion, i.e., particle moving along a helix, as shown in Fig. 5.5, dealing in terms of cylindrical coordinates,

$$r = R, \text{ a constant}$$

$$z = kR\phi$$

where k is the tangent of the helix angle. Note that k = 0 corresponds to plane circular notion.

Now, let $\dot{\phi} = \omega$

and $\dot{z} = kR\dot{\phi} = kR\omega$

With these values, velocity $\mathbf{V} = \dot{\mathbf{r}} = R\omega \, \mathbf{e}_\phi + kR\omega \, \mathbf{e}_z$
and the acceleration

$$\mathbf{a} = \dot{\mathbf{V}} = \ddot{\mathbf{r}} = -R\,\omega^2 \, \mathbf{e}_r + R\,\dot{\omega}\,\mathbf{e}_\phi + kR\,\dot{\omega}\,\mathbf{e}_z$$

If the particle moves at a uniform speed along the helix, the angular acceleration $\dot{\omega}$ is zero and only the radial acceleration remains. Then, $\mathbf{a} = -R\,\omega^2\,\mathbf{e}_r$.

The radius of curvature ρ of the helix is given by

$$\rho = R\,(1 + k^2).$$

For a helix angle of 45°, $\rho = 2R$

For a helix angle of zero, $\rho = R$, i.e., the radius itself since it reduces to plane circular motion.

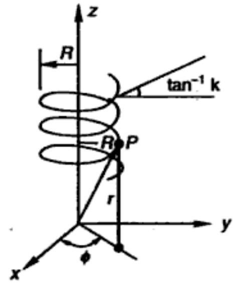


Fig. 5.5

***Example* 5.3** A Scotch yoke mechanism consists of a crank *CA* of radius *r* turning with a constant angular velocity ω rad/s and a reciprocating slotted sliding member *S* as shown in Fig. Ex. 5.3. Obtain the expression for the displacement, velocity and acceleration of the sliding member.

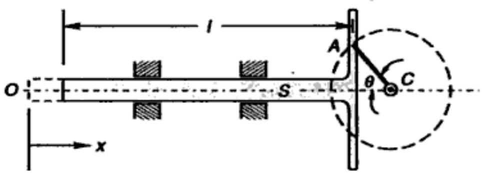


Fig. Ex. 5.3

Solution Let the motion of the slider be referred to its extreme left position *O* as the origin when the angle of the crank is zero at $t = 0$

$$x = OC - l - CA\cos\theta$$
$$= (l + r) - l - r\cos\theta = \rho\,(1 - \cos\theta)$$

Differentiating with respect to time *t* and recognising that

$$\frac{d\theta}{dt} = \omega \text{ the constant rotational speed}$$

and $\theta = \omega t$

$$V = \frac{dx}{dt} = r\omega \sin \omega t$$

and differentiating again,

$$a = \frac{dV}{dt} = r\omega^2 \cos \omega t.$$

Example 5.4 A 3 m long arm *OA* rotates in a plane such that $\theta = 0.15\, t^2$ where θ is the angle with *x*-axis in radius and *t* is in seconds. A slider collar *B* slides along the arm in such a way that its distance from the hinge *O* is given by $r = 3 - 0.4\, t^2$ where *r* is in metres. Determine the velocity and acceleration of the collar at an instant the arm has turned through 30°

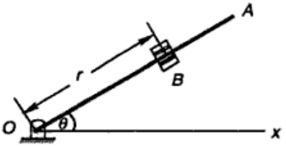


Fig. Ex. 5.4

Solution

For $\quad \theta = 30° \; \frac{\pi}{6}$ radians,

$$\frac{\pi}{6} = 0.5\, t^2$$

whence $\quad t = 1.87$ seconds

For $\quad r = 3 - 0.4\, t^2 = 3 - 0.4 \times 1.87^2 = 1.6$ m

$$\dot{r} = -0.8t = -0.8 \times 1.87 = -1.5 \text{ m/s}$$

$$\ddot{r} = -0.8 \text{ m/s}^2$$

Correspondingly, $\quad \theta = 0.15\, t^2 = \frac{\pi}{6} = 0.524$ rad

$$\dot{\theta} = 0.3\, t = 0.3 \times 1.87 = 0.561 \text{ rad/s}$$

$$\ddot{\theta} = 0.3 \text{ rad/s}^2$$

Velocity of the collar *B* is such that

$$V_r = \dot{r} = -1.5 \text{ m/s}$$

$$V_\theta = r\dot{\theta} = 1.6 \times 0.561 = 0.9 \text{ m/s}$$

which is 1.75 m/s at an angle α with the arm as shown in Fig. Ex. 5.4 (Solution) (a) such that $\alpha = \tan^{-1} \frac{0.9}{1.5} = 31°$

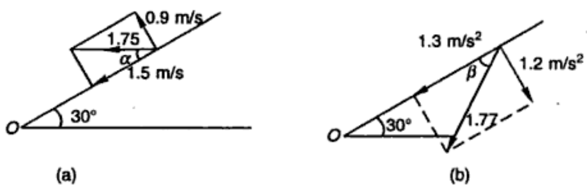


Fig. Ex. 5.4 *(Solution)*

The acceleration of the collar is such that

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$= -0.8 - 1.6 \times (0.561)^2 = -1.3 \text{ m/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= 1.6 \times 0.3 + 2 \times (-1.5) \times 0.561 = -1.2 \text{ m/s}^2$$

which is 1.77 m/s^2 at an angle β with the arm as shown in Fig. Ex. 5.4 (Solution) (b) such that $\beta = \tan^{-1}\dfrac{1.2}{1.3} = 42.7.$

Example 5.5 A wheel of radius 0.5 m is turned to advance up on a right-handed screw of pitch 1 cm as shown in Fig. Ex. 5.5. At an instant when the wheel is turned at a rotational speed of 2 rad/s, determine the velocity and acceleration of the hand held at A. If the wheel was accelerated rotationally at 0.6 rad/s^2, what would be the velocity and acceleration of the hand?

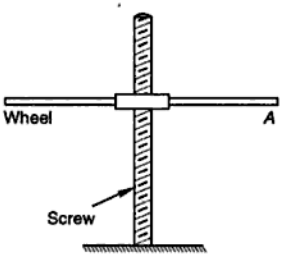


Fig. Ex. 5.5

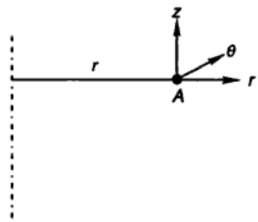


Fig. Ex. 5.5 *(Solution)*

Solution For the point A on the wheel (ref. Fig. Ex. 5.5 (Sol.))

$$r = 0.5 \text{ m} \quad \text{and} \quad \omega = 2 \text{ rad/s} = \dot{\theta}$$

The velocity is given by

$$\mathbf{V} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z \qquad \text{(i)}$$
$$= 0 + 0.5 \times 2\,\mathbf{e}_\theta + \frac{0.01 \times 2}{2 \times \pi}\,\mathbf{e}_z$$
$$= \mathbf{e}_\theta + 0.0032\,\mathbf{e}_z \text{ m/s}$$

The acceleration is obtained as

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z \qquad \text{(ii)}$$
$$= (0 - 0.5 \times 2^2)\mathbf{e}_r + (0.5 \times 0 + 2 \times 0 \times 2)\,\mathbf{e}_\theta + 0\,\mathbf{e}_z$$
$$= -2\,\mathbf{e}_r \text{ m/s}^2\text{, i.e., 2 m/s}^2 \text{ radially inwards}$$

For the second case,

$$r = 0.5 \text{ m}, \omega = \dot{\theta} = 2 \text{ rad/s and } \ddot{\theta} = 0.6 \text{ rad/s}^2$$

The velocity is given by Eq. (i) as

$$\mathbf{V} = \mathbf{e}_\theta + 0.3\, \mathbf{e}_z \text{ m/s}$$

the same as before and the acceleration from Eq. (ii) becomes,

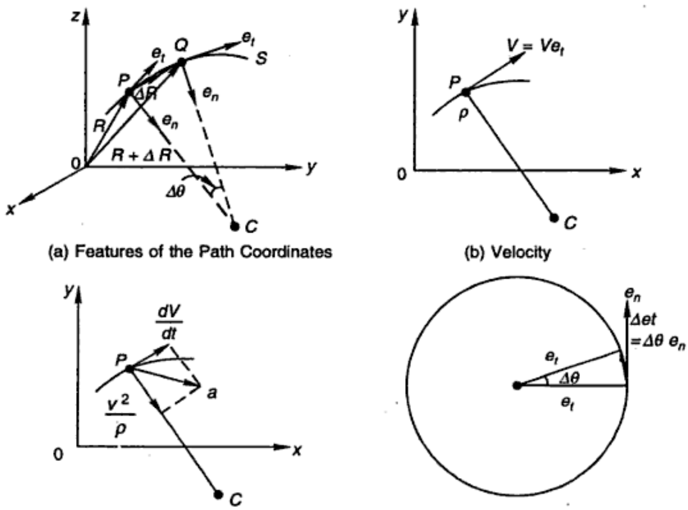
$$\mathbf{a} = (0 - 0.5 \times 2^2)\, \mathbf{e}_r + (0.5 - 0.6 + 2 \times 0 \times 2)\, \mathbf{e}_\theta + 0.0032 \times 0.6\, \mathbf{e}_z$$

$$= -2\, \mathbf{e}_r + 0.3\, \mathbf{e}_\theta + 0.0019\, \mathbf{e}_z \text{ rad s}^2$$

It may be noted that the speed and acceleration of advance of the wheel along the z-axis are related to the rotational speed and rotational acceleration respectively of the wheel. This fact has been used to evaluate $\dot{z}$ and $\ddot{z}$ terms in the analysis.

5.5 MOTION REFERRED TO PATH COORDINATES

It is sometimes very convenient to describe the kinematics of a point in terms of the path coordinates, i.e., the geometric features of the curve traced by the point as shown in Fig. 5.5(a). it may be appreciated that the path of a point may not be known *a priori* and hence the path coordinates cannot be specified until the point traces a curve in the vicinity of the position of interest. For this reason, the path coordinates are also known as *intrinsic coordinates*.



(a) Features of the Path Coordinates (b) Velocity

(c) Acceleration (d) Change in 'e_t' Unit Vector

Fig. 5.5 ***Motion Referred to Path Coordinates***

The orthogonal triad in the path coordinates consists of the following unit vectors:

$\mathbf{e}_t$ tangential to the path

$\mathbf{e}_n$ directed towards the instantaneous centre of curvature

$\mathbf{e}_b$ perpendicular to the plane containing $\mathbf{e}_t$ and $\mathbf{e}_n$ so as to form a right-hand triad.

In terms of the path coordinates, the velocity of a point is given by

$$\mathbf{V} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{R}}{\Delta t} = \frac{d\,\mathbf{R}}{dt} = \frac{d\,\mathbf{R}}{ds}\frac{ds}{dt} = V\,\mathbf{e}_t \tag{5.20}$$

which means that the velocity must be tangential to the path at any instant as shown in Fig. 5.5(b).

The acceleration of the point is expressed as

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}(V\mathbf{e}_t)$$

$$= \frac{dV}{dt}\mathbf{e}_t + V\frac{d\,\mathbf{e}_t}{ds}\frac{ds}{dt}$$

$$= \frac{d^2 s}{dt^2}\mathbf{e}_t + kV^2\,\mathbf{e}_n$$

$$\boxed{\mathbf{a} = \frac{d^2 s}{dt^2}\mathbf{e}_t + \frac{V^2}{\rho}\mathbf{e}_n} \tag{5.21}$$

where k, the curvature of the path $= \frac{d\theta}{ds}$ and ρ, the radius of curvature $= \frac{ds}{d\theta} = \frac{1}{k}$.

For a plane curve,

$$\rho = \left| \frac{(1 + (dy/dx)^2)^{3/2}}{d^2 y/dx^2} \right| = \left| \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \right| \tag{5.22}$$

The components of acceleration are shown in Fig. 5.5(c). The fact that

$$\frac{d\,\mathbf{e}_t}{d\theta} \cdot \frac{d\theta}{ds} = \frac{d\,\mathbf{e}_t}{ds} = k\,\mathbf{e}_n = \frac{1}{\rho}\mathbf{e}_n$$

used in the above derivation can be appreciated with reference to a circular diagram (Fig. 5.5(d)). The unit vector $\mathbf{e}_t$ suffers a small change $\Delta\mathbf{e}_t$ along the + $\mathbf{e}_n$ direction

$$\frac{d\,\mathbf{e}_t}{ds} = \lim_{\Delta s \to 0} \frac{1 \cdot \Delta\theta}{\Delta s}\mathbf{e}_n = k\,\mathbf{e}_n = \frac{1}{\rho}\mathbf{e}_n \tag{5.23}$$

Let us examine the relations for the simple case of motion of point on a circular trajectory. The curvature of a circle is the inverse of the radius

$$k = \frac{1}{r}$$

The velocity and acceleration of the point are

$$\mathbf{V} = V\mathbf{e}_t$$

and
$$\mathbf{a} = \frac{d\mathbf{V}}{dt}\mathbf{e}_t + \frac{V^2}{r}\mathbf{e}_n = a_t\mathbf{e}_t + a_n\mathbf{e}_n$$

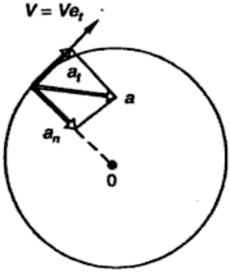


Fig. 5.6 ***Velocity and Acceleration for a Circular Trajectory***

It may be seen that the velocity is directed along a tangent to the circle at any instant. The acceleration, on the other hand, is made up of a tangential component, dV/dt; sense forward and normal component, V^2/r; sense inwards. This is shown in Fig. 5.6. In particular, if the point moves at a constant speed on a circular path, the tangential component of acceleration vanishes and only the radially inward centripetal acceleration remains.

Example **5.6** A particle is projected to move along a parabola

$$y^2 = 4x$$

At a certain instant, when passing through a point $P(4, 4)$ its speed is 5 m/s and the rate of increase of its speed is 3 m/s^2 along the path. Express the velocity and acceleration of the particle in terms of rectangular coordinates.

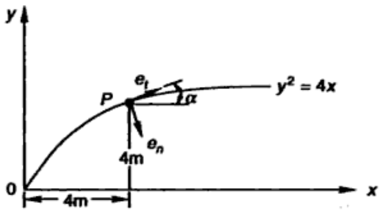


Fig. Ex. 5.6

Solution Since the data relate to the path of the particle, the path coordinates may be used to advantage. The unit vectors are related as follows:

$$\mathbf{e}_t = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j} \qquad \text{(i)}$$

$$\mathbf{e}_n = \sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j} \qquad \text{(ii)}$$

where
$$\tan\alpha = \frac{dy}{dx} \text{ at } P$$

From the equation of the path,

$$y^2 = 4x$$

differentiation with respect to x yields

$$2y\frac{dy}{dx} = 4$$

and $$\frac{dy}{dx} = \frac{2}{y} = \frac{1}{\sqrt{x}}$$

which, at P, is $\frac{1}{\sqrt{4}}$ or 0.5

and $\alpha = \tan^{-1} 0.5 = 26.57° = 0.464$ rad

Equations (i) and (ii) at point P become

$$\mathbf{e}_t = 0.894\ \mathbf{i} + 0.447\ \mathbf{j}$$

$$\mathbf{e}_n = 0.447\ \mathbf{i} - 0.894\ \mathbf{j}$$

The velocity of the particle is given by

$$\mathbf{V} = V\ \mathbf{e}_t = 5(0.894\ \mathbf{i} + 0.447\ \mathbf{j})$$

$$= 4.47\ \mathbf{i} + 2.235\ \mathbf{j}\ \text{m/s}$$

The tangential component of acceleration is

$$\mathbf{a}_t = 3(0.894\ \mathbf{i} + 0.447\ \mathbf{j}) = (2.68\ \mathbf{i} + 1.34\ \mathbf{j})\ \text{m/s}^2$$

The normal component of acceleration is

$$\mathbf{a}_n = \frac{V^2}{\rho}\mathbf{e}_n$$

The radius of curvature r is given by

$$\rho = \frac{(1 + (dy/dx)^2)^{3/2}}{d^2y/dx^2}$$

$$= \frac{(1 + 0.5^2)^{3/2}}{0.0625} = 22.36\ \text{m}$$

because $\frac{d^2y}{dx^2} = (-1/2x^{-3/2}) = 0.0625$

The normal component of acceleration is

$$\mathbf{a}_n = \frac{V^2}{\rho}\mathbf{e}_n$$

$$= \frac{5^2}{22.36} \times (0.447\ \mathbf{i} - 0.894\ \mathbf{j})$$

$$= 0.5\ \mathbf{i} - \mathbf{j}$$

The acceleration is, therefore, given by

$$\mathbf{a} = \mathbf{a}_n + \mathbf{a}_t$$

$$= 2.68\ \mathbf{i} + 1.34\ \mathbf{j} + 0.5\ \mathbf{i} - \mathbf{j}$$

$$= (3.18\ \mathbf{i} + 0.34\ \mathbf{j})\ \text{m/s}^2$$

***Example* 5.7** A particle moves in the *xy* plane with a velocity of 30 m/s directed at an angle of $\tan^{-1} 4/3$ as shown in Fig. Ex. 5.7. It accelerates as $a_x = -1.8$ m/s^2 and $a_y = -9$ m/s^2. Compute the radius of curvature of the path and the rate of change of speed along the path.

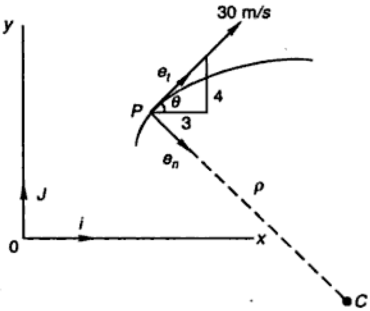


Fig. Ex. 5.7

Solution The unit vector along the velocity vector is

$$\mathbf{e}_t = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} = 0.6\,\mathbf{i} + 0.8\,\mathbf{j}$$

The unit vector along the inward normal is

$$\mathbf{e}_n = 0.8\,\mathbf{i} - 0.6\,\mathbf{j}$$

In terms of path coordinates, the acceleration is expressed as

$$\mathbf{a} = f\,\mathbf{e}_t + V^2/\rho\ \mathbf{e}_n$$

where f is the rate of change of speed along the path and ρ is the radius of curvature.

or $$-1.8\,\mathbf{i} - 9\,\mathbf{j} = f\,(0.6\,\mathbf{i} + 0.8\,\mathbf{j}) + \frac{900}{\rho}\,(0.8\,\mathbf{i} - 0.6\,\mathbf{j})$$

which results in two equations:

$$0.6\,f + 720/\rho = -1.8$$

and $$0.8\,f - 540/\rho = -9$$

whence $f = -8.3$ m/s and $\rho = 227$ m.

5.6 PLANE MOTION OF A POINT: GRAVITATIONAL FIELD

A point is said to be in plane motion if the point continues to move in one and the same plane, i.e., its trajectory is in a plane. Let us choose the *xz* plane to coincide with the plane of motion as shown in Fig. 5.1(b). From the expressions for the position vector, displacement, velocity and acceleration,

$$\mathbf{r} = x\,\mathbf{i} + z\,\mathbf{k}$$

$$\Delta\,\mathbf{r} = \Delta x\,\mathbf{i} + \Delta z\,\mathbf{k}$$

$$\mathbf{V} = \frac{dx}{dt}\mathbf{i} + \frac{dz}{dt}\mathbf{k} = V_x\mathbf{i} + V_z\mathbf{k}$$

$$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2z}{dt^2}\mathbf{k} = \frac{dV_x}{dt}\mathbf{i} + \frac{dV_z}{dt}\mathbf{k} = a_x\mathbf{i} + a_z\mathbf{k}$$

Of special interest here is the case of a particle projected at an angle α to the horizontal in the gravitational field of the earth close to its surface as shown in Fig. 5.7. Then,

$$a_x = 0 \quad \text{and} \quad a_z = -g$$

$$\mathbf{a} = -g\,\mathbf{k}; \text{ constant acceleration.}$$

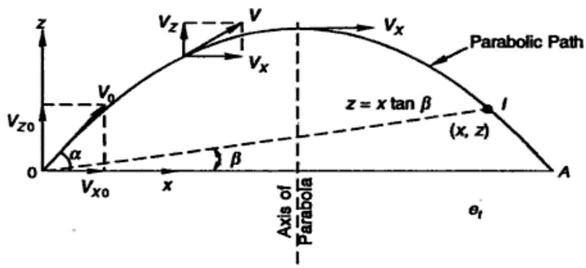


Fig. 5.7 ***Parabolic Trajectory of a Particle***

Starting with the equation

$$\mathbf{V} = \int \mathbf{a}\,dt + \mathbf{K}$$

and substituting $\mathbf{a} = -g\,\mathbf{k}$

$$\mathbf{V} = -gt\,\mathbf{k} + \mathbf{K}$$

$$V_x\mathbf{i} + V_z\mathbf{k} = -gt\,\mathbf{k} + V_{x0}\mathbf{i} + V_{z0}\mathbf{k}$$

$$= V_{x0}\mathbf{i} + (V_{z0} - gt)\,\mathbf{k}$$

where V_{x0} and V_{z0} are the initial components of velocity $\mathbf{V}_0$ at $t = 0$.

It follows that,

$$V_x = V_{x0}$$

i.e., the horizontal component of velocity remains the same during motion

and $V_z = V_{z0} - gt$

i.e., the vertical component of velocity undergoes a linear change.

Starting now with the equation

$$\mathbf{r} = \int \mathbf{V}\, dt + \mathbf{C}$$

and substituting $\quad \mathbf{V} = V_{x0}\, \mathbf{i} + (V_{z0} - gt)\, \mathbf{k}$

$$x\, \mathbf{i} + z\, \mathbf{k} = V_{x0} t\, \mathbf{i} + \left(V_{z0} t - \frac{1}{2} g t^2\right) \mathbf{k}$$

where x and z are measured from the initial point O as the origin at $t = 0$

It follows that,

$$x = V_{x0} t \tag{5.24}$$

$$z = V_{z0} t - \frac{1}{2} g t^2 \tag{5.25}$$

Eliminating t and using the relations

$$\frac{V_{z0}}{V_{x0}} = \tan \alpha \quad \text{and} \quad V_{x0} = V_0 \cos \alpha$$

$$\boxed{z = x \tan \alpha - \frac{g x^2}{2 V_0^2 \cos^2 \alpha}} \tag{5.26}$$

This is the equation of the trajectory. Since it represents a parabola, the path of a particle projected in some oblique direction must be a plane parabolic trajectory.

The assumptions in this analysis are:

1. *Air-resistance is negligible*
2. *The gravitational acceleration g is constant*
3. *The point stands for the particle or centre of mass of a body.*

These assumptions restrict the analysis to short-range and low-altitude motion of small objects in the atmosphere.

Let us now inspect the equation to provide some further information.

(a) *Maximum attainable height* z_{max} occurs when

$$V_z = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = 0$$

From the expression for z.

$$\frac{\partial z}{\partial x} = \tan \alpha - \frac{g x}{V_0^2 \cos^2 \alpha} = 0$$

whence ,

$$\alpha = \frac{1}{2} \sin^{-1} \left(\frac{2 g x}{V_0^2}\right)$$

$$x = V_0^2 \frac{\sin 2\alpha}{2g}$$

and the ordinate of the vertex is given by

$$z_{max} = \frac{V_0^2 \sin^2 \alpha}{2g} \tag{5.27a}$$

(b) *Range of the particle* on a horizontal plane equals twice the x coordinate for z_{max}.

$$\text{Range } OA = \frac{V_0^2 \sin 2\alpha}{g} \tag{5.27b}$$

This is a maximum for $\alpha = 45°$, Maximum range $= \dfrac{V_0^2}{g}$

(c) *Range of the particle on an incline plane OI* is determined by solving for (x, z), the point of interaction of the parabola and the inclined plane $z = x \tan \beta$.

(d) *Variation of z_{max} with α* is observed from the relationship

$$z_{max} = \frac{V_0^2 \sin^2 \alpha}{2g}$$

and z_{max} is the maximum possible when $\alpha = 90°$, i.e., when the particle is projected vertically upwards.

(e) For a particle projected horizontally,

$$\alpha = 0$$

$$z = -\frac{gx^2}{2V_0^2} \tag{5.28}$$

(f) For a particle projected vertically upwards,

$$\alpha = 90°, \quad \tan \alpha \to \infty \quad \text{and} \quad x = 0$$

the problem reduced to that of a rectilinear motion along the gravitational acceleration due to the earth.

(g) The time taken to reach a particular point on the trajectory can be estimated from the relations,

$$x = V_{x0}\, t$$

$$z = V_{z0}\, t - \frac{1}{2} g t^2$$

If the initial velocity $\mathbf{V}_0$ (in terms of, V_{x0} and V_{z0}) is known, the time taken can be estimated by knowing either the x or z coordinate of the point on the trajectory.

The time of flight t_{max} of the particle from the origin O to the end point A, at the initial level can be calculated by observing that

$$z = 0 \quad \text{at} \quad t = t_{max}$$

$$0 = V_{z0}\, t_{max} - \frac{1}{2} g t_{max}^2$$

whence
$$t_{max} = \frac{2V_{z0}}{g} = \frac{2\, V_0 \sin\alpha}{g} \tag{5.29}$$

Alternatively,
$$t_{max} = \frac{x_{max}}{V_{x0}}$$

$$= \frac{V_0^2 \sin 2\alpha}{g V_{x0}} = \frac{2V_0 \sin\alpha\, V_0 \cos\alpha}{g\, V_{x0}}$$

$$= \frac{2\, V_{z0}\ V_{x0}}{g V_{x0}} = \frac{2V_{z0}}{g} = \frac{2V_0 \sin\alpha}{g}$$

The time of flight on an inclined plane can be seen to be

$$t_{max} = \frac{2V_0 \sin(\alpha - \beta)}{g \cos\beta} \tag{5.30}$$

since the initial velocity component normal to the inclined plane is $V_0 \sin(\alpha - \beta)$ instead of $V_0 \sin\alpha = V_{z0}$ and the acceleration due to gravity is $g \cos\beta$ instead of g.

The range of flight on an inclined plane can be calculated by working out the horizontal coordinate x first,

$$x = V_{x0}\, t_{max} = V_0 \cos\alpha\, t_{max}$$

$$= \frac{2V_0^2 \cos\alpha \sin(\alpha - \beta)}{g \cos\beta}$$

The range s on the incline is given by

$$s = \frac{x}{\cos\beta}$$

or
$$s = \frac{2V_0^2 \cos\alpha \sin(\alpha - \beta)}{g \cos^2\beta} \tag{5.31}$$

It may be checked that when $\beta = 0$, it reduces to the expression

$$s = \frac{V_0^2 \sin 2\alpha}{g}$$

as obtained earlier for the horizontal plane.

(h) At any time t, the velocity of the particle can be computed from

$$V_x = V_{x0}$$

$$V_z = V_{z0} - gt$$

and hence, $$\mathbf{V} = V_x\,\mathbf{i} + V_z\,\mathbf{k} = V_{x0}\,\mathbf{i} + (V_{z0} - gt)\,\mathbf{k} \tag{5.32}$$

which is $\sqrt{(V_{x0}^2 + (V_{z0} - gt)^2)}$ in magnitude

and directed at an angle $\tan^{-1}\left(\dfrac{V_{z0} - gt}{V_{x0}}\right)$

(i) At a certain height h, the velocity of the particle can be computed from

$$V_x = V_{x0}$$

$$V_z = (V_{z0}^2 - 2gh)^{1/2}$$

and hence, $$\mathbf{V} = V_x\,\mathbf{i} + V_z\,\mathbf{k}$$

$$= V_{x0}\,\mathbf{i} + (V_{z0}^2 - 2gh)^{1/2}\,\mathbf{k} \tag{5.33}$$

(j) If the angle of projection α is negative, the z-coordinate continues to decrease, V_z continues to increase and the particle tends to drop down closer to the vertical line with the passage of time.

***Example* 5.8** A gun is fired, aimed at a ball, from a ground position simultaneously as the ball is let go vertically down. Show that the shot will hit the ball regardless of the initial velocity of the shot and the distances.

Solution Let the ball be at a horizontal distance x and be at an angle of elevation θ with respect to the gun as shown in Fig. Ex. 5.8(a) (Solution). Let the velocity of the shot at the instant of firing be V_0 at an angle of elevation of θ in the line of sight of the ball.

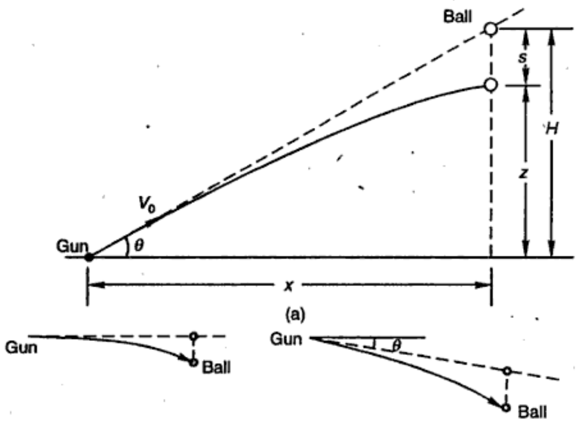


Fig. Ex. 5.8 *(Solution)*

The time taken for the shot to travel a horizontal distance x is

$$t = \frac{x}{V_{x0}} = \frac{x}{V_0 \cos\theta} = \frac{H/\tan\theta}{V_0 \cos\theta} = \frac{H}{V_0 \sin\theta} \tag{i}$$

The vertical coordinate z of the shot in that time is given by

$$z = V_{z0}t - \frac{1}{2}gt^2$$

$$= V_0 \sin\theta \frac{H}{V_0 \sin\theta} - \frac{1}{2}g\left(\frac{H}{V_0 \sin\theta}\right)^2$$

or

$$z = H - \frac{gH^2}{2V_0^2 \sin^2\theta} \tag{ii}$$

During the same time interval t given by Eq. (i), the ball drops through a distance s where

$$s = \frac{1}{2}gt^2 = \frac{1}{2}g\left(\frac{H}{V_0 \sin\theta}\right)^2 = \frac{gH^2}{2V_0^2 \sin^2\theta}$$

and the z coordinate of the ball becomes

$$z = H - \frac{gH^2}{2V_0^2 \sin^2\theta}$$

which is the same as that of the shot. The shot should, therefore, hit the ball irrespective of the initial coordinates of the ball with respect to the gun and initial velocity of the shot.

It may be noted that the drop of the ball before it is shot is

$$s = \frac{gH^2}{2V_0^2 \sin^2\theta} = \frac{g(x\tan\theta)^2}{2V_0^2 \sin^2\theta} = \frac{gx^2}{2V_0^2 \cos^2\theta}$$

which does vary with all the parameters. The drop is less if the initial velocity V_0 of the bullet is more, range angle θ is more or horizontal distance x is less.

It may also be noted that the shot will hit the ball whether the ball is initially at an angle of elevation, level, or an angle of depression with respect to the gun so long as the gun is fired at the initial line of sight of the ball. This is illustrated in Figs. Ex. 5.8(b) and (c) (Solution).

***Example* 5.9** The world records for the shot put and discus throw are 20 m and 70 m respectively. Assuming that their respective masses are 7 kg and 2 kg respectively, compare the work done by the champions in making their record throws if each trajectory starts at an elevation of 2 m and has an initial inclination of 45° with the horizontal. Neglect air resistance.

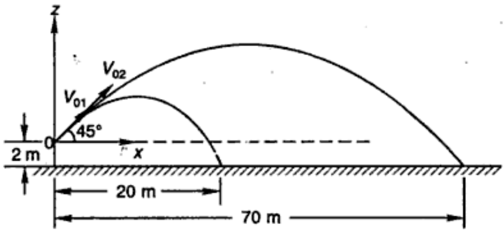


Fig. Ex. 5.9

Solution The equation of the trajectory of an object projected at an angle α with an initial speed V_0 from the point O in the gravitational field of the earth is

$$z = x \tan \alpha - \frac{gx^2}{2V_0^2 \cos^2 \alpha}$$

For the shot put

$$-2 = 20 \tan 45° - \frac{9.81 \times 20^2}{2V_{01}^2 \cos^2 45°}$$

whence $V_{01} = 13.36$ m/s

The kinetic energy at the instant of its projection is

$$\frac{1}{2} m_1 V_{01}^2 = \frac{1}{2} \times 7 \times 13.36^2 = 625 \text{ J}$$

For the discus throw

$$-2 = 70 \tan 45° - \frac{9.81 \times 70^2}{2V_{02}^2 \cos^2 45°}$$

whence $V_{02} = 26.2$ m/s

and the kinetic energy at the instant of projection is

$$\frac{1}{2} m_2 V_{02}^2 = \frac{1}{2} \times 2 \times 26.2^2 = 687 \text{ J}$$

Neglecting that the discus has a tendency of spinning during flight, the work done by a champion must equal the kinetic energy imparted to the object at the instant of projection, i.e., at the same datum. The champion throwing the discus, therefore, does more work than the champion putting the shot.

***Example* 5.10** A large balloon is rising up with a velocity of 9.81 m/s at an altitude of 39.2 m from the ground. At that instant, a stone of mass 5 kg is dropped from it. After how many seconds will the stone reach the ground?

Solution The acceleration due to gravity $g = 9.81$ m/s^2. Considering all quantities positive downwards,

$$U = -9.8 \text{ m/s}, \; g = +9.81 \text{ m/s}^2$$

and employing, $S = Ut + \frac{1}{2}gt^2$

$$39.2 = -9.81\, t + \frac{1}{2} \times 9.81\, t^2$$

whence $t^2 - 2t - 8 = 0$

$$t = 4 \quad \text{or} \quad t = -2$$

Rejecting the negative value of time, $t = 4$ s.

***Example* 5.11** A car A is travelling on a straight level road with a uniform speed of 60 km/hr. It is followed by another car B moving at a speed of 70 km/hr. When the distance between them is 2.5 km, the car B is decelerated at 20 km/hr^2. Will the car B catch up with A? If not, why not? If yes, at what distance and time?

Solution Let us suppose that car B catches up the car A in t hours. (If it doesn't, t will turn out to be negative or imaginary!)

In that time, A travels 60 t km. The distance travelled by B will be given by

$$S = ut + \frac{1}{2}at^2$$

$$= 70\, t - \frac{1}{2} \times 20\, t^2 = 70\, t - 10\, t^2$$

Since the car A is already leading by 2.5 km, the condition for B to catch up with A is

$$70\, t - 10\, t^2 = 60\, t + 2.5$$

whence, $10\, t^2 - 10\, t + 2.5 = 0,$ $\quad t = 0.5$ hr.

In that time, A travels $60 \times 0.5 = 30$ km and B travels by $70 \times 0.5 - 10 \times 0.5^2$ i.e., 33.5 km.

***Example* 5.12** A stone is dropped gently from the top of a tower. During its last one second of motion it falls through 64% of the height. Find the height of the tower.

Solution Assuming that the total time of fall is t seconds,

$$h = \frac{1}{2}\, gt^2 \qquad \text{(i)}$$

In $(t - 1)$ seconds it falls through a distance given by $\frac{1}{2}\, g(t-1)^2$ which is only 36% of its height,

$$0.36\, h = \frac{1}{2}\, g\, (t-1)^2 \qquad \text{(ii)}$$

Dividing (i) by (ii), $\frac{1}{0.36} = \frac{t^2}{t^2 + 1 - 2t}$

whence $0.64\, t^2 - 2t + 1 = 0$ and $t = 2.5$ seconds

From (i), $h = \frac{1}{2} \times 9.81 \times 2.5^2 = 30.65$ m.

***Example* 5.13** Two motor cars start from *A* simultaneously and reach *B* after 2 hours on the same road. The first travelled half the distance at a speed of 30 km/hr and the other half at a speed of 60 km/hr. The other car covered the entire distance with a constant acceleration. At what instants of time were the speeds of both the vehicles the same?

Solution Let the distance *AB* be 2*x*.

From the data for the first car,

$$\frac{x}{30} + \frac{x}{60} = 2 \text{ hrs}; x = 40, 2x = 80 \text{ km}.$$

For the second car,

$$80 = 0 + \frac{1}{2} at^2 = \frac{1}{2} az^2$$

$$a = 40 \text{ km/hr}^2.$$

$$V = 0 + 40 \times 2 = 80 \text{ km/hr at } B$$

At any instant of time, for car *B*

$$V = 40\, t$$

It becomes 30 km/hr at $t = 3/4$ hours and it becomes 60 km/hr at $t = 3/2$ hours after departure from *A*.

Let us check the state of the first car at these timings. At $t = 3/4$ hours, it was $30 \times 3/4$, i.e., 22.5 km from *A* running at 30 km/hr. At $t = 3/2$, it was indeed running at 60 km/hr, having crossed the 40 km mark. Hence, the two had the same speed at these twp timings. It can be checked that there was no overtaking!

***Example* 5.14** An elevator ascends with an upward acceleration of 1.2 m/s^2. At the instant when the upward speed is 2.4 m/s, a loose bolt drops from the ceiling of the elevator located 2.75 m from its floor. Calculate:

(a) the time of flight of the bolt from ceiling to floor of the elevator

(b) the displacement and the distance covered by the bolt during the free fall relative to the elevator shaft.

Solution

(a) The bolt, initially travelling up with a velocity of 2.4 m/s drops under gravity; its downwards displacement is

$$S_1 = -2.4\, t + \frac{1}{2} \times 9.81\, t^2$$

The floor of the elevator is displaced up in the same time,

$$S_2 = 2.4\,t + \frac{1}{2} \times 1.2\,t^2$$

The sum of the two must be 2.75 m.

Hence, $(4.9 + 0.6)t^2 = 2.75$; $t = 0.707$ s.

(b) Displacement of the bolt is given by

$$S_1 = -2.4 \times 0.707 + \frac{1}{2} \times 9.81 \times 0.707^2 = 0.75 \text{ m}$$

The distance travelled by the bolt is the sum of distances it goes up first and then comes down. It goes up until its velocity becomes zero,

$$V^2 - U^2 = -2\,gs$$

$$0 - 2.4^2 = -2 \times 9.81 \times S\,; \qquad S = 0.293 \text{ m}$$

It comes down such that the final downward displacement is 0.75 m, i.e., it traverses 0.293 m down and then 0.75 m down, making a total distance of 0.293 + 0.293 + 0.75 = 1.34 m.

***Example* 5.15** A bullet is projected so as to graze the tops of two walls each of height 20 m located at distances of 30 and 170 m in the same line from the point of projection as shown in Fig. Ex. 5.15. Find the angle and the speed of projection of the bullet.

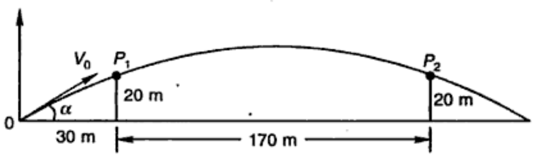


Fig. Ex. 5.15

Solution

From the equation,

$$z = x \tan \alpha - \frac{gx^2}{2V_0^2 \cos^2 \alpha}$$

for the two points P_1 and P_2,

$$20 = 30 \tan \alpha - \frac{9.81 \times 30^2}{2V_0^2 \cos^2 \alpha} = 30 \tan \alpha - \frac{4415}{V_0^2 \cos^2 \alpha} \qquad \text{(i)}$$

and

$$20 = 170 \tan \alpha - \frac{9.81 \times 170^2}{2V_0^2 \cos^2 \alpha} = 170 \tan \alpha - \frac{141\,755}{V_0^2 \cos^2 \alpha} \qquad \text{(ii)}$$

we can determine both α and V_0.

Calling '$V_0^2 \cos^2 \alpha$' as a parameter p

$$642 = 963 \tan \alpha - \frac{141\,755}{p}$$

$$20 = 170 \tan \alpha - \frac{141\,755}{p}$$

whence $\tan \alpha = 0.784$; $\alpha = 38.1°$

Substituting the value of α in (i),

$$V_0 = 45 \text{ m/s}$$

***Example* 5.16** A shell bursts on contact with the ground and pieces fly off in all directions with speeds up to 30 m/s. A person is standing 40 m away. What is the time duration over which he can be hit by a piece?

Solution With the maximum initial velocity of 30 m/s, the horizontal range of 40 m requires that

$$40 = \frac{30^2 \times \sin 2\alpha}{9.81}; \quad \sin 2\alpha = 0.435$$

whence $2\alpha = 25.8°$ or $154.2°$; $\alpha = 12.9°$ or $77.1°$

The first piece that can hit the person has an angle of projection of 12.9° and initial velocity 30 m/s takes time

$$t_1 = \frac{2 \times 30 \times \sin 12.9°}{9.81} = 1.36 \text{ seconds.}$$

The last piece that can hit the person has an angle of projection of 77.1° and initial velocity 30 m/s which takes time $t_2 = \dfrac{2 \times 30 \times \sin 77.1}{9.81} = 5.96$ seconds.

How about the maximum time taken by a piece at less than 30 m/s? Well, then the higher value of α would be less than 77.1° and hence t_2 would be less.

The duration over which the person can be hit by a piece is a period of 4.6 seconds, beginning 1.36 seconds and ending 5.96 seconds after the bursting of the shell.

***Example* 5.17** The maximum horizontal range of a gun is R_{max}. Determine the firing angle which should be used to hit a target located at a distance $R_{max}/2$ on the same level.

Solution The range of a bullet fired with a velocity V_0 at an angle α is given by

$$R = \frac{V_0^2 \sin 2\alpha}{g} \qquad \text{(i)}$$

This would be maximum for $\alpha = 45°$

$$R_{max} = V_0^2 / g$$

The target is located at

$$R = R_{max} / 2 = V_0^2 / 2g \tag{ii}$$

Equating it to (i),

$$\frac{V_0^2 \sin 2\alpha}{g} = \frac{V_0^2}{2g}$$

$$\sin 2\alpha = \frac{1}{2}$$

whence $\qquad 2\alpha = 30° \quad \text{and} \quad \alpha = 15°$

The gun should therefore, be fired at an inclination of 15° to hit the target.
One may check the result by considering the equation of trajectory,

$$z = x \tan \alpha - \frac{g x^2}{2V_0^2 \cos^2 \alpha}$$

For the target, $\qquad z = 0,\ x = V_0^2/2g$

$$0 = \frac{V_0^2}{2g} \tan \alpha - \frac{V_0^2}{8g \cos^2 \alpha}$$

whence, $\qquad \tan \alpha = \dfrac{1}{4 \cos^2 \alpha}; \sin 2\alpha = 1/2$

which is the same as determined earlier.

5.7 MOTION REFERRED TO MOVING FRAMES OF REFERENCE

In order to arrive at the description of the motion of a point with reference to an inertial frame for the application of Newton's laws it may be necessary to first ascertain the motion in relation to a moving frame and then refer it to an inertial frame for the sake of convenience. The moving frame may, in general, translate and rotate as well as accelerate linearly or angularly. It is the purpose of the following treatment to arrive at a systematic procedure for referring the space motion of a point with respect to a frame of reference if its motion is known in relation to another frame moving with respect to the former.

This task is achieved through a series of simple steps for the sake of clarity of understanding.

(a) Relative Motion of Two Points

Consider a pair of points, P_1 and P_2 moving with the velocities $\mathbf{V}_1$ and $\mathbf{V}_2$ and the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ with respect to a fixed frame of reference. The velocity of P_1 with respect to P_2 is given by

$$\mathbf{V}_{12} = \mathbf{V}_1 - \mathbf{V}_2 \tag{5.34}$$

and the acceleration of P_1 with respect to P_2 is given by

$$\mathbf{a}_{12} = \mathbf{a}_1 - \mathbf{a}_2 \tag{5.35}$$

This is demonstrated in Fig. 5.8. It can also be appreciated that

$$\mathbf{V}_{21} = \mathbf{V}_2 - \mathbf{V}_1 = -\mathbf{V}_{12}$$

and

$$\mathbf{a}_{21} = \mathbf{a}_2 - \mathbf{a}_1 = -\,\mathbf{a}_{12}$$

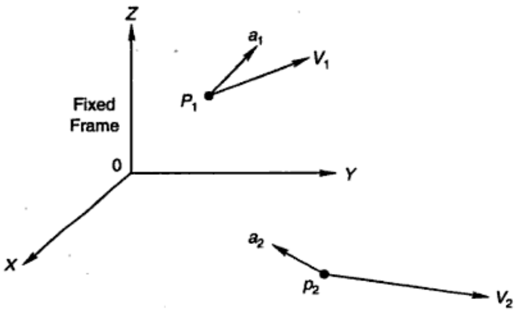


Fig. 5.8 ***Relative Velocity and Relative Acceleration***

***Example* 5.18** Two roads cross at 60° angle at an intersection at O. At an instant of time, a scooter 1 at 100 m east of the intersection moving at a velocity of 50 km/hr decelerates at 5 km/hr/s as it approaches the intersection O. At the same instant, a car 2 passes the intersection at a velocity of 20 km/hr and accelerates at 10 km/hr/s as shown in Fig. Ex. 5.18. Determine (a) the velocity of the scooter with respect to the car and (b) the acceleration of the car with respect to the scooter at the instant of observation as well as after a lapse of 3 s and, in general after t s.

Solution Choosing the x and y axes as shown,

At the instant of observation,

$$\mathbf{V}_1 = -50\ \mathbf{i}\ \text{km/h}$$

$$\mathbf{a}_1 = +\,5\ \mathbf{i}\ \text{km/h/s}$$

$$\mathbf{V}_2 = 20\ (\sin 60\ \mathbf{i} + \cos 60\ \mathbf{j})$$

$$= 17.32\ \mathbf{i} + 10\ \mathbf{j}\ \text{km/h}$$

$$\mathbf{a}_2 = 10(\sin 60\ \mathbf{i} + \cos 60\ \mathbf{j})$$

$$= 8.66\ \mathbf{i} = 5\ \mathbf{j}\ \text{km/h/s}$$

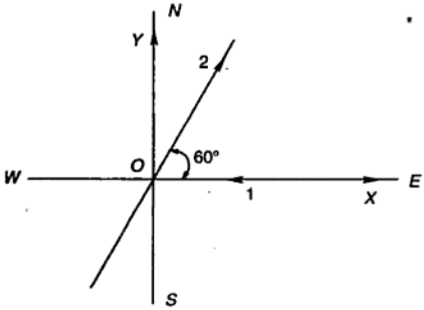


Fig. Ex. 5.18

(a) $\mathbf{V}_{12} = \mathbf{V}_1 - \mathbf{V}_2 = -50\,\mathbf{i} - 17.32\,\mathbf{i} - 10\,\mathbf{j}$

$= -67.32\,\mathbf{i} - 10\,\mathbf{j}$ km/h

(b) $\mathbf{a}_{21} = \mathbf{a}_2 - \mathbf{a}_1 = 8.66\,\mathbf{i} + 5\,\mathbf{j} - 5\,\mathbf{i}$

$= 3.66\,\mathbf{i} + 5\,\mathbf{j}$ km/h/s

At an instant after a lapse of 3 s

$$\mathbf{V}_1 = -50\,\mathbf{i} + 3 \times 5\,\mathbf{i} = -35\,\mathbf{i} \text{ km/h}$$

$$\mathbf{V}_2 = 17.321\,\mathbf{i} + 10\,\mathbf{j} + 3(8.66\,\mathbf{i} + 5\,\mathbf{j})$$

$$= 43.3\,\mathbf{i} + 25\,\mathbf{j} \text{ km/h}$$

(a) $\mathbf{V}_{12} = \mathbf{V}_1 - \mathbf{V}_2 = -35\,\mathbf{i} - 43.3\,\mathbf{i} - 25\,\mathbf{j}$

$= -78.3\,\mathbf{i} - 25\,\mathbf{j}$ km/h

(b) the relative acceleration remains the same;

$$\mathbf{a}_{21} = 3.66\,\mathbf{i} + 5\,\mathbf{j} \text{ km/h/s}$$

In general, after a lapse of t s

$$\mathbf{V}_1 = -50\,\mathbf{i} + 5t\mathbf{i} = (-50 + 5t)\,\mathbf{i} \text{ km/h}$$

$$\mathbf{V}_2 = 17.32\,\mathbf{i} + 10\,\mathbf{j} + (8.66\,\mathbf{i} + 5\,\mathbf{j})t$$

$$= (17.32 + 8.66t)\,\mathbf{i} + (10 + 5t)\,\mathbf{j} \text{ km/h}$$

The relative velocity of the scooter with respect to the car is

$$\mathbf{V}_{12} = (-50 + 5t - 17.32 - 8.66t)\,\mathbf{i} - (10 + 5t)\,\mathbf{j} \text{ km/h}$$

$$= (-67.32 - 3.66t)\,\mathbf{i} - (10 + 5\,t)\,\mathbf{j} \text{ km/h}$$

The relative acceleration remains the same as at the earlier instants because the acceleration of each vehicle is unaltered with time.

$$\mathbf{a}_{21} = 3.66\,\mathbf{i} + 5\,\mathbf{j}$$

***Example* 5.19** A rod follower *AB* is subjected to a vertical up-and-down movement while resting on the circular contour of radius 30 cm of a cam. The cam moves to the right with a velocity of 5 cm/s and an acceleration of 10 cm/s^2. Find the velocity and acceleration of the point *B* on the rod at the instant of interest as shown in Fig. Ex. 5.19.

Solution It is an interesting and a fairly simple problem which can be solved by a variety of methods.

Let us demonstrate the power of two different methods by solving this problem.

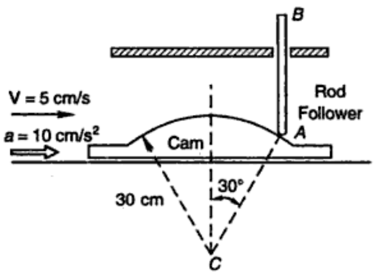


Fig. Ex. 5.19

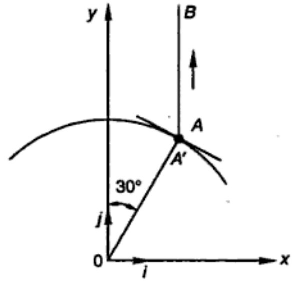


Fig. Ex. 5.19 *(Solution)*

Method I

The point B moves with the same velocity and acceleration as A. The motion of *A* may in turn be related to the motion of *A′*, a coincident point on the cam

$$\mathbf{V}_A = \mathbf{V}_{A'} + \mathbf{V}_{AA'}$$

$$\mathbf{V}_A \mathbf{j} = 0.05\ \mathbf{i} + \mathbf{V}_{AA'}\ (-0.866\ \mathbf{i} + 0.5\ \mathbf{j}) \qquad \text{(i)}$$

since the velocity of *A* must be along the *y*-axis, that of *A′* is given along the *x*-axis and the velocity of *A* with respect to *A′* is assumed up the tangent intuitively. From Eq. (i),

$$0.05 - 0.866\ V_{AA'} = 0$$

$$V_A - 0.5\ V_{AA'} = 0$$

whence

$$V_{AA'} = 0.0577 \text{ m/s}$$

$$V_A = 0.0289 \text{ m/s} = V_B$$

Similarly, for the acceleration,

$$\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{AA'} + \frac{V_{AA'}^2}{0.3}(-0.5\ \mathbf{i} - 0.866\ \mathbf{j})$$

$$= 0.1\ \mathbf{i} + a_{AA'}(-0.866\ \mathbf{i} + 0.5\ \mathbf{j}) + \frac{0.0577^2}{0.3}\ (-0.5\ \mathbf{i} - 0.866\ \mathbf{j}) \qquad \text{(ii)}$$

From this, the *x*-component of acceleration is equated to zero.

$$0.1 - 0.866\ a_{AA'} - 0.0055 = 0$$

Then, $a_A = 0.5a_{AA'} + 0.0096$

whence

$$a_{AA'} = 0.109 \text{ m/s}^2$$

$$a_A = -0.045 \text{ m/s}^2 = a_B$$

Hence, the point *B* moves up with a velocity of 0.0289 m/s and decelerates at 0.045 m/s² while moving upward.

Method 2

One may like to visualise the cam to be at rest and the point *A* of the rod follower to slide up the circular path with the *x*-component of velocity –5 cm/s and *x*-component of acceleration -10 cm/s^2, the equation of the path, then being

$$x^2 + y^2 = (0.3)^2$$

Differentiating it with respect to time *t*,

$$x\dot{x} + y\dot{y} = 0 \qquad \text{(i)}$$

and differentiating it again with respect to time *t*,

$$x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0 \qquad \text{(ii)}$$

The position of *A* is given by

$$x = 0.3 \sin 30° = 0.15 \text{ m}$$

$$y = 0.3 \cos 30° = 0.26 \text{ m}$$

From Eq. (i)

$$\dot{y} = -\frac{0.15 \times (-0.05)}{0.26} = 0.0288 \text{ m/s}$$

which must be the velocity of *A* and of *B*.

From Eq. (ii)

$$\ddot{y} = -\frac{0.15 \times (-0.1) + (-0.05)^2 + 0.0288^2}{0.26} = 0.045 \text{ m/s}^2$$

which must be tne acceleration of *A* and *B*.

(b) Translation of a Moving Frame

If a moving frame of reference *x-y-z* translates with respect to a fixed frame *X-Y-Z*, as shown in Fig. 5.9, then any pair of parallel lines in the two frames remain parallel to each other with the passage of time. It follows that the unit vectors **i**, **j** and **k** in the moving frame remain parallel to themselves and their magnitudes remaining unity, *the unit vectors remain invariant.*

If the velocity of the moving frame is $\mathbf{V}_0$ and the velocity of a point with respect to the moving frame is $\mathbf{V}_{Pm}$, the velocity of the point with respect to the fixed frame is given by

$$\mathbf{V}_{Pf} = \mathbf{V}_{Pm} + \mathbf{V}_0 \tag{5.36}$$

Similarly, the acceleration of a point with respect to the fixed frame is

$$\mathbf{a}_{Pf} = \mathbf{a}_{Pm} + \mathbf{a}_0 \tag{5.37}$$

where $\mathbf{a}_{Pm}$ is the acceleration of the point with respect to the moving frame and $\mathbf{a}_0$ is the linear acceleration of the moving frame with respect to the fixed frame.

It should be understood that a translating frame may translate either in space or in a plane or along a straight line. The velocity $\mathbf{V}_0$ and acceleration $\mathbf{a}_0$ of the moving origin may, in general, be in different directions at any instant of time. Only for the case of rectilinear translation, i.e., for motion along a straight line would $\mathbf{V}_0$ and $\mathbf{a}_0$ be collinear.

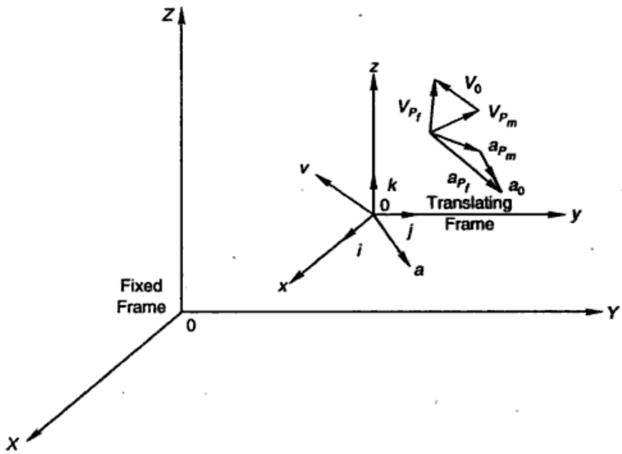


Fig. 5.9 ***Translation of a Moving Frame***

(c) Rotation of a Moving Frame

Consider a moving frame of reference *x-y-z* and a fixed or inertial reference frame *X-Y-Z*. Infinitesimal rotations $d\theta_x$, $d\theta_y$ and $d\theta_z$ of the moving frame are specified about the *X*, *Y* and *Z* axes respectively and are represented by vector components along the respective axes with **i**, **j**, **k** unit vectors as shown in Fig. 5.10.

Angular velocity components ω_x, ω_y and ω_z refer to the rates of rotation about *X*, *Y* and *Z* axes respectively.

$$\omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$

$$= \frac{d\theta_x}{dt}\mathbf{i} + \frac{d\theta_y}{dt}\mathbf{j} + \frac{d\theta_z}{dt}\mathbf{k}$$

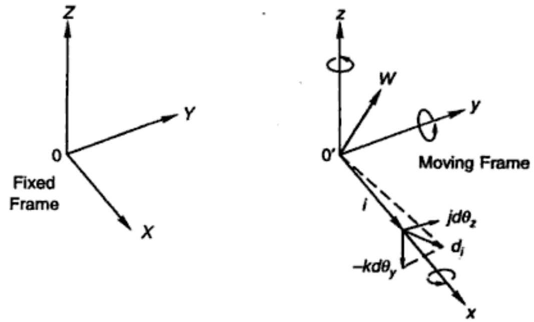


Fig. 5.10 ***Rotation of a Moving Frame***

(d) Derivatives of Moving Unit Vectors

Consider the change in the unit vector **i** observed from the fixed frame $(d\,\mathbf{i})_f$, due to the differential rotation $(d\theta_x, d\theta_y, d\theta_z)$ to the moving frame as shown in Fig. 5.10:

$d\theta_x$ produces no change in **i**

$d\theta_y$ produces a change $= -\mathbf{k}\,d\theta_y$ in **i**

$d\theta_z$ produces a change $= \mathbf{j}\,d\theta_z$ in **i**

Total change $(d\,\mathbf{i})_f = \mathbf{j}\,d\theta_z - \mathbf{k}\,d\theta_y$

and the rate of change of **i** with respect to t is given by

$$\left(\frac{d\,\mathbf{i}}{dt}\right)_f = \mathbf{j}\frac{d\theta_z}{dt} - \mathbf{k}\frac{d\theta_y}{dt} = \mathbf{j}\,\omega_z - \mathbf{k}\,\omega_y$$

which is the same as $\omega \times \mathbf{i}$

Hence $$\left(\frac{d\,\mathbf{i}}{dt}\right)_f = \omega \times \mathbf{i}$$

Similarly, expressions for the rates of change of **j** and **k** can be obtained:

$$\boxed{\begin{aligned}\left(\frac{d\,\mathbf{i}}{dt}\right)_f &= \omega \times \mathbf{i}\\ \left(\frac{d\mathbf{j}}{dt}\right)_f &= \omega \times \mathbf{j}\\ \left(\frac{d\mathbf{k}}{dt}\right)_f &= \omega \times \mathbf{k}\end{aligned}} \qquad (5.38)$$

(e) Derivative of a Constant Vector in a Moving Frame

Consider a fixed position vector **r** of a point P fixed with respect to a moving frame x-y-z which is rotating at an angular velocity ω as shown in Fig. 5.11.

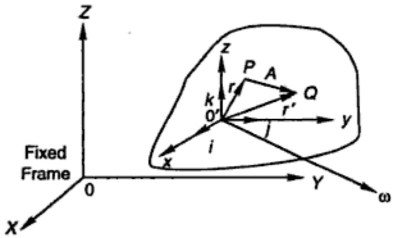


Fig. 5.11 ***Reference to Fixed and Moving Frames***

The axis of rotation must be along the ω vector. Writing the position vector of P and differentiating it with respect to t as observed from the moving frame,

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\left(\frac{dr}{dt}\right)_f = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} + x\left(\frac{d\,\mathbf{i}}{dt}\right)_f + y\left(\frac{d\,\mathbf{j}}{dt}\right)_f + z\left(\frac{d\,\mathbf{k}}{dt}\right)_f$$

Since the position vector is fixed with respect to the moving system,

$$\left(\frac{d\mathbf{r}}{dt}\right)_m = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = 0$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_f = x(\omega \times \mathbf{i}) + y(\omega \times \mathbf{j}) + z(\omega \times \mathbf{k})$$

$$= \omega \times (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

$$= \omega \times \mathbf{r}$$

Hence,

$$\left(\frac{d\mathbf{r}}{dt}\right)_f = \omega \times \mathbf{r}$$

Let there be a vector **A** fixed with respect to the x-y-z frame rotating at ω. if the position vectors of its end are **r** and **r′**, then **A** = **r′** – **r**.

$$\left(\frac{d\mathbf{r}}{dt}\right)_f = \omega \times \mathbf{r}$$

$$\left(\frac{d\mathbf{r}'}{dt}\right)_f = \omega \times \mathbf{r}'$$

$$\left(\frac{d}{dt}(\mathbf{r}' - \mathbf{r})\right)_f = \omega \times (\mathbf{r}' - \mathbf{r})$$

$$\left(\frac{d\,\mathbf{A}}{dt}\right)_f = \omega \times \mathbf{A} \qquad (5.39)$$

(f) Derivatives of a Position Vector for Different References

Let the position vector of a moving point P be $\mathbf{r}$ referred to the moving frame x-y-z and $\mathbf{R}$ referred to the fixed frame X-Y-Z.

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{r}$$

Differentiating $\mathbf{r}$ with respect to t observed from the moving frame and fixed frame respectively,

$$\left(\frac{d\mathbf{r}}{dt}\right)_m = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_f = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} + x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}$$

$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$+ x(\omega \times \mathbf{i}) + y(\omega \times \mathbf{j}) + z(\omega \times \mathbf{k})$$

$$= \left(\frac{d\mathbf{r}}{dt}\right)_m + \omega \times (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

$$= \left(\frac{d\mathbf{r}}{dt}\right)_m + \omega \times \mathbf{r} \tag{5.40}$$

(g) Velocity of a Point

The velocity of a point relative to a reference frame is the time derivative, as seen from the reference of the position vector with respect to that reference. Referring to Fig. 5.12.

$$\mathbf{V}_{Pf} = \left(\frac{d\mathbf{R}}{dt}\right)_f$$

$$\mathbf{V}_{Pm} = \left(\frac{d\mathbf{r}}{dt}\right)_m$$

Differentiating $\mathbf{R}$ with respect to time for the f reference

$$\left(\frac{d\mathbf{R}}{dt}\right)_f = \left(\frac{d\mathbf{R}_0}{dt}\right)_f + \left(\frac{d\mathbf{r}}{dt}\right)_f$$

or

$$\mathbf{V}_{Pf} = \mathbf{V}_0 + \left[\left(\frac{d\mathbf{r}}{dt}\right)_m + \omega \times \mathbf{r}\right]$$

$$\boxed{\mathbf{V}_{Pf} = \mathbf{V}_{Pm} + \mathbf{V}_0 + \omega \times \mathbf{r}} \tag{5.41}$$

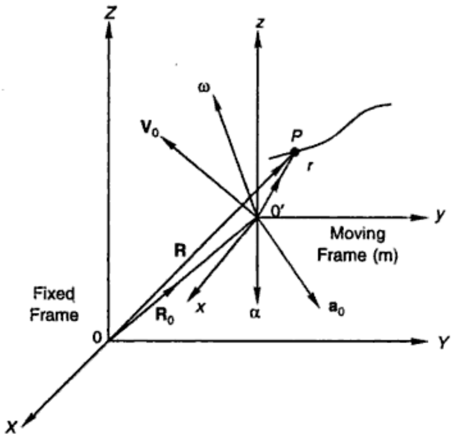


Fig. 5.12 ***Motion Observed from fixed and Moving Frames***

This is interpreted by stating that the velocity $\mathbf{V}_{Pf}$ of a particle as observed from a fixed reference frame must be the vector sum of the velocity $\mathbf{V}_{Pm}$ as observed from a moving reference frame, the velocity $\mathbf{V}_0$ of the moving origin and the term $\omega \times \mathbf{r}$ due to rotation of the moving frame at an angular velocity ω with respect to the fixed frame.

(h) Acceleration of a Point

The acceleration of a point relative to a reference frame is the time derivative, as seen from the reference frame of the velocity relative to that reference. Referring to Fig. 5.12 again,

$$\mathbf{a}_{Pf} = \left(\frac{d}{dt}\mathbf{V}_{Pf}\right)_f$$

$$\mathbf{a}_{Pm} = \left(\frac{d}{dt}\mathbf{V}_{Pm}\right)_m = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_m$$

$$\mathbf{a}_{Pf} = \left(\frac{d}{dt}\mathbf{V}_{Pf}\right)_f$$

$$= \mathbf{a}_0 + \left(\frac{d}{dt}\mathbf{V}_{Pm}\right)_f + \left(\frac{d}{dt}(\omega \times \mathbf{r})\right)_f$$

$$\left(\frac{d}{dt}\mathbf{V}_{Pm}\right)_f = \left(\frac{d\,\mathbf{V}_{Pm}}{dt}\right)_m + \omega \times \mathbf{V}_{Pm}$$

$$= \mathbf{a}_{Pm} + \omega \times \mathbf{V}_{Pm}$$

$$\left(\frac{d}{dt}(\omega \times \mathbf{r})\right)_f = \omega \times \left(\frac{d\mathbf{r}}{dt}\right)_f + \left(\frac{d\omega}{dt}\right)_f \times \mathbf{r}$$

$$= \omega \times \left[\left(\frac{d\mathbf{r}}{dt} \right)_m + \omega \times \mathbf{r} \right] + \alpha \times \mathbf{r}$$

$$= \omega \times [\mathbf{V}_{Pm} + \omega \times \mathbf{r}] + \alpha \times \mathbf{r}$$

$$= \omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r}) + \alpha \times \mathbf{r}$$

Finally,

$$\boxed{\mathbf{a}_{Pf} = \mathbf{a}_{Pm} + \mathbf{a}_0 + \alpha \times \mathbf{r} + 2\,\omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r})} \quad (5.42)$$

The interpretation of the terms is as follows:

- $\mathbf{a}_{Pf}$ — acceleration of the point *P* with respect to the fixed frame of reference.
- $\mathbf{a}_{Pm}$ — acceleration of the point *P* with respect to the moving frame of reference.
- $\mathbf{a}_0$ — translational acceleration of the origin of the moving frame with respect to the fixed origin.
- $\alpha \times \mathbf{r}$ — acceleration due to the moving frame accelerating with an angular acceleration α; the tangential component as seen from the fixed origin. It vanishes when α and $\mathbf{r}$ are parallel, collinear or when either is zero.
- $2\omega \times \mathbf{V}_{Pm}$ — Coriolis component of acceleration due to ω, the rotation of the moving frame and $\mathbf{V}_{Pm}$, the relative velocity of the point *P* in the moving frame. It is zero when ω is parallel to or collinear with $\mathbf{V}_{Pm}$ or either of them vanishes.
- $\omega \times (\omega \times \mathbf{r})$ — normal component of acceleration as seen from the fixed frame; also called centripetal acceleration, which for plane circular motion of the point becomes $\mathbf{r}\omega^2$. It vanishes where ω and $\mathbf{r}$ are collinear or when either ω or $\mathbf{r}$ is zero.

(i) A Note on Coriolis Acceleration

It is interesting to demonstrate the origin of Coriolis acceleration and the concept of its direction physically by a simple example.

Consider a slidable collar *P* made to slide at a constant velocity $\mathbf{V}_{Pm}$ with respect to a rod rotating at a constant angular velocity ω about **O** as shown in Fig. 5.13(a). Placing a frame of reference on the rod, we notice that the relative acceleration of *P* with respect to the rotating rod, angular acceleration of the rotating frame and the absolute acceleration of the origin on the rotating frame are specified as zero. For this simple case of plane motion only, the centripetal component $\omega \times (\omega \times \mathbf{r})$ and the Coriolis component $2\omega \times \mathbf{V}_{Pm}$ are non-zero. The former, $\mathbf{r}\,\omega^2$ in magnitude, is directed radially towards *O* whereas the latter, $2\omega \times \mathbf{V}_{Pm}$ is directed at right angles to $\mathbf{V}_{Pm}$ and ω.

In order to appreciate the origin of the Coriolis acceleration, consider the change in system configuration over a small time interval Δt as shown in Fig. 5.13(b). Initially, the velocity of *P* was made up of two components:

Radial: V_{Pm} shown by *OA*

Tangential: $r\omega$ shown by *OC*

Finally, after a lapse of time Δt, the components of the velocity of *P* are:

Radial: V_{Pm} shown by *OB*

Tangential: $(r + \Delta r)\omega$ shown by *OE*

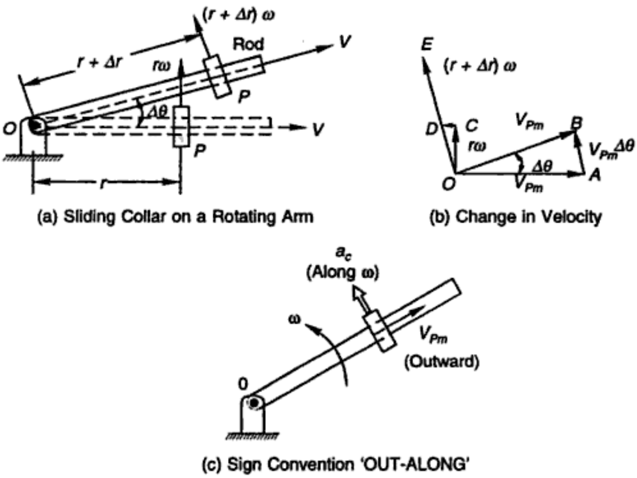


Fig. 5.13 ***Coriolis Acceleration***

The initial velocity vector P can be obtained by adding OA and OC and the final velocity vector by adding OB and OE, but these are purposely not shown to avoid unnecessary details. It is preferred to show the change of the velocity of P in three stages:

1. Due to the rotation of the relative sliding velocity
   $AB = V_{Pm}\,\Delta\theta$
2. Due to the rotation of the rod
   $CD = r\omega\,\Delta\theta$
3. Due to the sliding effect on the rotational velocity
   $DE = (r + \Delta r)\,\omega - r\omega = \Delta r\,\omega$

It is the sum of AB and DE, divided by Δt, in the limiting case which is called Coriolis acceleration:

$$\text{Coriolis acceleration } a_c == \lim_{\Delta t \to 0}\left(\frac{V_{Pm}\cdot\Delta\theta + \Delta r\cdot\omega}{\Delta t}\right)$$

$$= V_{Pm}\,\frac{d\theta}{dt} + \frac{dr}{dt}\,\omega = \omega\,V_{Pm} + \omega\,V_{Pm}$$

or

$$a_c = 2\omega\,V_{Pm}$$

directed perpendicular to the rod as indicated by the limiting case of the velocity diagram for $\Delta t \to 0$ and $\Delta\omega$ getting smaller.

Similarly, the rate of change of CD corresponds to the centripetal acceleration:

$$\text{Centripetal acceleration } a_r = \lim_{\Delta t \to 0}\left[\frac{r\omega\cdot\Delta\theta}{\Delta t}\right]$$

or

$$a_r = r\omega \frac{d\theta}{dt} = r\omega^2$$

directed towards the centre of rotation of the rod as also observed from the limiting case of the velocity diagram.

It is sometimes difficult to visualise the direction of the Coriolis acceleration. Actually, there is no difficulty about the direction of any component of acceleration when proceeding vectorially but a rule or sign convention (shown in Fig. 5.13(c)) may be stated for the graphical procedure applicable to the plane motion of rigid bodies:

If the slider collar moves radially **out** *with respect to the centre of rotation of the arm on which it slides, the Coriolis acceleration of the* **collar** *is* **along** *the direction of rotation. This is called the Out-Along conventoion*

***Example* 5.20** A platform as shown in Fig. Ex. 5.20 rotates about its axis with an angular speed of 2 rad/s counterclockwise and decelerates with an angular deceleration of 3 rad/s^2. A rod *OA* rotates about the hinge *O* with an angular velocity 4 rad/s and accelerates at a rate 5 rad/s^2 with respect to the platform. A collar *B* slides outward on the rod *OA* with a velocity of 2 m/s and an acceleration of 3 m/s^2 with respect to the rod. Compute the absolute velocity and acceleration of the collar if it is at 0.5 m from *O*.

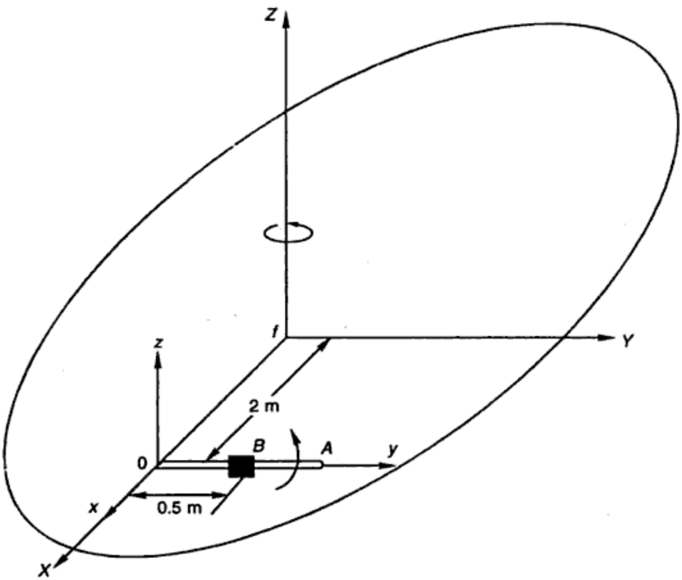


Fig. Ex. 5.20

Solution Let the fixed frame *XYZ* be fixed with the ground with its origin coincident with the centre of the platform and let the moving frame *xyz* be attached to the rod *OA* with its origin at *O*.

Next, it is necessary to identify the motion of the collar with respect to the moving frame and the motion of the moving frame with respect to the fixed frame. The collar moves with respect to the moving frame identified with the rod such that

$$\mathbf{r} = 0.5\,\mathbf{j},\ \mathbf{V}_{Pm} = 2\,\mathbf{j} \quad \text{and} \quad \mathbf{a}_{Pm} = 3\,\mathbf{j}$$

The moving frame moves with respect to the fixed frame such that

$$\omega = 2\,\mathbf{k} + 4\,\mathbf{k} = 6\,\mathbf{k}$$

$$\alpha = \dot{\omega} = 5\,\mathbf{k} - 3\,\mathbf{k} = 2\,\mathbf{k}$$

$$\mathbf{R}_0 = 2\,\mathbf{i}$$

$$\mathbf{V}_0 = \dot{\mathbf{R}}_0 = 2\,\mathbf{k} \times 2\,\mathbf{i} = 4\,\mathbf{j}$$

$$\mathbf{a}_0 = \ddot{\mathbf{R}}_0 = -3\,\mathbf{k} \times 2\,\mathbf{i} + 2\,\mathbf{k} \times 4\,\mathbf{j} = -6\,\mathbf{j} - 8\,\mathbf{i}$$

The absolute velocity and acceleration of the collar are determined by recalling the corresponding expressions:

$$\begin{aligned}\mathbf{V}_{Pf} &= \mathbf{V}_{Pm} + \mathbf{V}_0 + \omega \times \mathbf{r}\\ &= 2\,\mathbf{j} + 4\,\mathbf{j} + 6\,\mathbf{k} \times 0.5\,\mathbf{j}\\ &= -3\,\mathbf{i} + 6\,\mathbf{j}\ \text{m/s}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{Pf} &= \mathbf{a}_{Pm} + \mathbf{a}_0 + \alpha \times \mathbf{r} + 2\,\omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r})\\ &= 3\,\mathbf{j} - 6\,\mathbf{j} - 8\,\mathbf{i} + 2\,\mathbf{k} \times 0.5\,\mathbf{j} + 2(6\,\mathbf{k} \times 2\,\mathbf{j})\\ &\quad + 6\,\mathbf{k} \times (6\,\mathbf{k} \times 0.5\,\mathbf{j})\\ &= -33\,\mathbf{i} - 21\,\mathbf{j}\ \text{m/s}^2\end{aligned}$$

Alternatively, the moving frame *xyz* could have been attached to the platform with the origin either at *O* or at the centre of the platform. If the origin is located at *O*, the collar moves with respect to the moving frame such that

$$\mathbf{r} = 0.5\,\mathbf{j}$$

$$\mathbf{V}_{Pm} = 2\,\mathbf{j} + 4\,\mathbf{k} \times 0.5\,\mathbf{j} = -2\,\mathbf{i} + 2\,\mathbf{j}\ \text{m/s}$$

$$\begin{aligned}\mathbf{a}_{Pm} &= (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z\\ &= (3 - 0.5 \times 4^2)\,\mathbf{j} - (0.5 \times 5 + 2 \times 2 \times 4)\,\mathbf{i}\\ &= -18.5\,\mathbf{i} - 5\,\mathbf{j}\ \text{m/s}^2\end{aligned}$$

The motion of the moving frame with respect to the fixed frame is such that

$$\omega = 2\,\mathbf{k}$$

$$\alpha = \dot{\omega} = -3\,\mathbf{k}$$

$$\mathbf{R}_0 = 2\,\mathbf{i}$$

$$\mathbf{V}_0 = \dot{\mathbf{R}}_0 = 4\,\mathbf{j}$$

$$\mathbf{a}_0 = \ddot{\mathbf{R}}_0 = -6\,\mathbf{j} - 8\,\mathbf{i}$$

Again, from the expressions for the absolute velocity and acceleration,

$$\begin{aligned}\mathbf{V}_{Pf} &= \mathbf{V}_{Pm} + \mathbf{V}_0 + \omega \times \mathbf{r}\\ &= -2\,\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{j} + 2\,\mathbf{k} \times 0.5\,\mathbf{j}\\ &= -3\,\mathbf{i} + 6\,\mathbf{j}\text{ m/s}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{Pf} &= \mathbf{a}_{Pm} + \mathbf{a}_0 + \alpha \times \mathbf{r} + 2\,\omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r})\\ &= -18.5\,\mathbf{i} - 5\,\mathbf{j} - 6\,\mathbf{j} - 8\,\mathbf{i} - 3\,\mathbf{k} \times 0.5\,\mathbf{j}\\ &\quad + 2(2\,\mathbf{k} \times (-2\,\mathbf{i} + 2\,\mathbf{j})) + 2\,\mathbf{k} \times (2\,\mathbf{k} \times 0.5\,\mathbf{j})\\ &= -33\,\mathbf{i} - 21\,\mathbf{j}\text{ m/s}^2\end{aligned}$$

***Example* 5.21** A crane rotates about a vertical axis with a constant angular velocity of 0.4 rad/s while the boom is being raised with a constant angular velocity of 0.5 rad/s relative to the cab as shown in Fig. Ex. 5.21. If the length of the boom is 10 m, determine (a) the angular velocity of the boom, (b) the angular acceleration of the boom, (c) the velocity of the tip of the boom and (d) the acceleration of the tip of the boom.

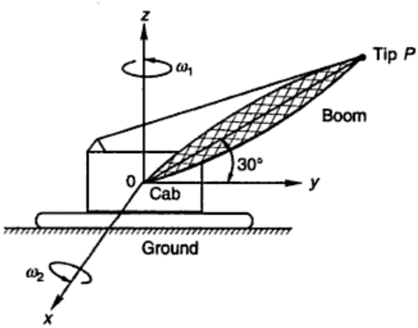


Fig. Ex. 5.21

Solution The problem may be tackled in a variety of ways by considering the moving frame attached to any moving component at any desired point. We shall consider some of the different possibilities with a view to gaining experience of analysing such motions.

Method 1

Moving frame x-y-z attached to the rotating cabin with its origin at O and the fixed frame X-Y-Z fixed in space but with its origin also at O.

The angular velocity ω of the moving frame is then 0.4 rad/s about the z axis, i.e.,

$$\omega = 0.4\ \mathbf{k}$$

From the given length $OP = 10$ m, the tip of the boom P is located by the position vector

$$\mathbf{r} = 10\ (\cos 30\ \mathbf{j} + \sin 30\ \mathbf{k})$$

$$= 8.65\ \mathbf{j} + 5\ \mathbf{k}$$

The velocity of P with respect to the fixed frame is given by

$$\mathbf{V}_{Pf} = \mathbf{V}_{Pm} + \mathbf{V}_0 + \omega \times \mathbf{r}$$

where $$\mathbf{V}_{Pm} = 0.5\ \mathbf{i} \times (8.66\ \mathbf{j} + 5\ \mathbf{k})$$

$$= 4.33\ \mathbf{k} - 2.5\ \mathbf{j}$$

$$\mathbf{V}_0 = 0\text{, the origin being coincident with } O$$

and $$\omega \times \mathbf{r} = 0.4\ \mathbf{k} \times (8.66\ \mathbf{j} + 5\ \mathbf{k}) = -3.465\ \mathbf{i}$$

Hence, $$\mathbf{V}_{Pf} = -3.465\ \mathbf{i} - 2.5\ \mathbf{j} + 4.33\ \mathbf{k}\ \text{m/s}$$

The angular velocity of the boom must be the sum of the angular velocity of the moving frame on the cab and the angular velocity of the boom with respect to the cab, i.e.,

$$\omega_{\text{boom}} = \omega_1 + \omega_2$$

$$= 0.4\ \mathbf{k} + 0.5\ \mathbf{i}\ \text{rad/s}$$

The angular acceleration of the boom can be likewise visualised as that due to the rotation of the boom on a rotating cab.

$$\alpha_{\text{boom}} = \omega_1 \times \omega_2$$

$$= 0.4\ \mathbf{k} \times 0.5\ \mathbf{i} = 0.2\ \mathbf{j}\ \text{rad/s}^2$$

The acceleration of P with respect to the fixed frame is given by

$$\mathbf{a}_{Pf} = \mathbf{a}_{Pm} + \mathbf{a}_0 + \alpha \times \mathbf{r} + 2\ \omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r})$$

where $$\mathbf{a}_{Pm} = 0.5\ \mathbf{i} \times (0.5\ \mathbf{i} \times (8.66\ \mathbf{j} + 5\ \mathbf{k}))$$

$$= -2.16\ \mathbf{j} - 1.25\ \mathbf{k}$$

$$\mathbf{a}_0 = 0$$

$$\alpha \times \mathbf{r} = 0$$

$$2\ \omega \times \mathbf{V}_{Pm} = 2 \times 0.4\ \mathbf{k} \times (4.33\ \mathbf{k} - 2.5\ \mathbf{j})$$

$$= 2\ \mathbf{i}$$

$$\omega \times (\omega \times \mathbf{r}) = 0.4\,\mathbf{k} \times (0.4\,\mathbf{k} \times (8.66\,\mathbf{j} + 5\,\mathbf{k}))$$

$$= -1.385\,\mathbf{j}$$

Hence,

$$\mathbf{a}_{Pf} = -2.165\,\mathbf{j} - 1.25\,\mathbf{k} + 2\,\mathbf{i} - 1.385\,\mathbf{j}$$

$$= 2\,\mathbf{i} - 3.55\,\mathbf{j} - 1.25\,\mathbf{k}\ \text{rad/s}^2$$

Method 2

Moving frame x-y-z attached to the boom with its origin at O and the fixed frame fixed in space but with its origin also at O.

In this case,

$$\omega = 0.4\,\mathbf{k} + 0.5\,\mathbf{i}$$

is the angular velocity of the moving frame attached to the boom.

$$\mathbf{r} = 8.66\,\mathbf{j} + 5\,\mathbf{k} \text{ as before}$$

$$\mathbf{V}_{Pf} = \mathbf{V}_{Pm} + \mathbf{V}_0 + \omega \times \mathbf{r}$$

where $\mathbf{V}_{Pm} = 0$, the point P being on the boom itself and

$$\mathbf{V}_0 = 0, \text{ the origin being coincident with } O.$$

$$\omega \times \mathbf{r} = (0.4\,\mathbf{k} + 0.5\,\mathbf{i}) \times (8.65\,\mathbf{j} + 5\,\mathbf{k})$$

$$= -3.465\,\mathbf{i} - 2.5\,\mathbf{j} + 4.33\,\mathbf{k}$$

Hence, $\mathbf{V}_{Pf} = -3.465\,\mathbf{i} - 2.5\,\mathbf{j} + 4.33\,\mathbf{k}$ m/s

The angular acceleration of the boom is obtained by $\omega_1 \times \omega_2$ by imagining the moving frame undergoing a relative rotation with respect to a rotating frame.

$$\alpha_{boom} = 0.4\,\mathbf{k} \times 0.5\,\mathbf{i} = 0.2\,\mathbf{j}\ \text{rad/s}^2$$

This is also the angular acceleration of the moving frame in this case.

$$\alpha = \alpha_{boom} = 0.2\,\mathbf{j}\ \text{rad/s}^2$$

The acceleration of P with respect to the fixed frame is

$$\mathbf{a}_{Pf} = \mathbf{a}_{Pm} + \mathbf{a}_o + \alpha \times \mathbf{r} + 2\omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r})$$

where $\mathbf{a}_{Pm} = 0$

Now, $\mathbf{a}_0 = 0$

$$\alpha \times \mathbf{r} = 0.2\,\mathbf{j} \times (8.66\,\mathbf{j} + 5\,\mathbf{k}) = 1\,\mathbf{i}$$

$$2\,\omega \times \mathbf{V}_{Pm} = 0$$

$$\omega \times (\omega \times \mathbf{r}) = (0.4\,\mathbf{k} + 0.5\,\mathbf{i}) \times ((0.4\,\mathbf{k} + 0.5\,\mathbf{i}) \times (8.66\,\mathbf{j} + 5\,\mathbf{k}))$$

$$= 1\,\mathbf{i} - 3.55\,\mathbf{j} - 1.25\,\mathbf{k}$$

Hence,

$$\mathbf{a}_{Pf} = 2\,\mathbf{i} - 3.55\,\mathbf{j} - 1.25\,\mathbf{k}\ \text{m/s}^2$$

Method 3

No moving frame at all X-Y-Z fixed with reference to frame fixed in space with its origin at O.

The angular velocity of the boom is due to ω_1 of the cab and ω_2 of the boom with respect to the cab, i.e.,

$$\omega_{boom} = \omega_1 + \omega_2$$
$$= 0.4\,\mathbf{k} + 0.5\,\mathbf{i}\text{ rad/s}$$

The angular acceleration of the boom is obtained by

$$\alpha_{boom} = \left(\frac{d\omega_{boom}}{dt}\right)_f = \left(\frac{d\omega_1}{dt}\right)_f + \left(\frac{d\omega_2}{dt}\right)_f$$

Now, $$\left(\frac{d\omega_1}{dt}\right)_f = 0$$

because the cab rotates at a constant angular velocity with respect to the fixed frame but the boom has a constant angular velocity with respect to the cab. Observing that ω_2 rotates with the cab at ω_1

$$\alpha_{boom} = \left(\frac{d\,\omega_2}{dt}\right)_f = \omega_1 \times \omega_2$$
$$= 0.4\,\mathbf{k} \times 0.5\,\mathbf{i} = 0.2\,\mathbf{j}\text{ rad/s}^2$$

The velocity of P is given by

$$\mathbf{V}_P = \omega \times \mathbf{r}$$
$$= (0.4\,\mathbf{k} + 0.5\,\mathbf{i}) \times (8.66\,\mathbf{j} + 5\,\mathbf{k})$$
$$= -3.465\,\mathbf{i} - 2.5\,\mathbf{j} + 4.33\,\mathbf{k}\text{ m/s}$$

The acceleration of P is likewise computed:

$$\mathbf{a}_P = \alpha_{boom} \times \mathbf{r} + \omega_{boom} \times (\omega_{boom} \times) + \mathbf{r})$$
$$= 0.2\,\mathbf{j} \times (8.66\,\mathbf{j} + 5\,\mathbf{k}) + ((0.4\,\mathbf{k} + 0.5\,\mathbf{i}) \times (0.4\,\mathbf{k} + 0.5\,\mathbf{i}) \times (8.66\,\mathbf{j} + 5\,\mathbf{k}))$$
$$= 2\,\mathbf{i} - 3.55\,\mathbf{j} - 1.25\,\mathbf{k}\text{ m/s}^2$$

Method 4

By multiple references. A moving frame X-Y-Z attached to the boom with its origin at O and another moving frame attached to the cab with its origin at O and a fixed frame X-Y-Z also with its origin at O.

The velocity and acceleration are now determined first with respect to the intermediate reference m_1 and then with respect to the fixed frame in the next step.

For the first step in respect of velocity.

$$\mathbf{V}_{Pm1} = \mathbf{V}_{Pm} + \mathbf{V}_0 + \omega \times \mathbf{r}$$

where

$$\mathbf{V}_{Pm} = 0$$

$$\mathbf{V}_0 = 0$$

$$\omega \times \mathbf{r} = 0.5\,\mathbf{i}\,(8.66\,\mathbf{j} + 5\,\mathbf{k}) = 4.33\,\mathbf{k} - 2.5\,\mathbf{j}$$

ω being the angular velocity of the boom with respect to the cab.

Hence,

$$\mathbf{V}_{Pm1} = -\,2.5\,\mathbf{j} + 4.33\,\mathbf{k}$$

and for the second step,

$$\mathbf{V}_{Pf} = \mathbf{V}_{Pm1} + \mathbf{V}_0 + \omega \times \mathbf{r}$$

where

$$\mathbf{V}_{Pm1} = -\,2.5\,\mathbf{j} + 4.33\,\mathbf{k}$$

$$\mathbf{V}_0 = 0$$

$$\omega \times \mathbf{r} = 0.4\,\mathbf{k}\,(8.66\,\mathbf{j} + 5\,\mathbf{k}) = -3.465\,\mathbf{i}$$

ω being the angular velocity of the cab with respect to the ground.

Hence,

$$\mathbf{V}_{Pf} = 3.465\,\mathbf{i} - 2.5\,\mathbf{j} + 4.33\,\mathbf{k}\ \text{m/s}$$

Similarly, for the first step in respect of acceleration,

$$\mathbf{a}_{Pm1} = \mathbf{a}_{Pm} + \mathbf{a}_0 + \alpha \times \mathbf{r} + 2\,\omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r})$$

where

$$\mathbf{a}_{Pm} = 0$$

$$\mathbf{a}_0 = 0$$

$$\alpha \times \mathbf{r} = 0$$

$$2\omega \times \mathbf{V}_{Pm} = 0, \text{ as } \mathbf{V}_{Pm} = 0$$

$$\omega \times (\omega \times \mathbf{r}) = 0.5\,\mathbf{i} \times (0.5\,\mathbf{i} \times (8.66\,\mathbf{j} + 5\,\mathbf{k}))$$

$$= -2.165\,\mathbf{j} - 1.25\,\mathbf{k}$$

Hence,

$$\mathbf{a}_{Pm1} = -2.165\,\mathbf{j} - 1.25\,\mathbf{k}$$

and for the second step,

$$\mathbf{a}_{Pf} = \mathbf{a}_{Pm1} + \mathbf{a}_0 + \alpha \times \mathbf{r} + 2\omega + \mathbf{V}_{Pm1} + \omega \times (\omega \times \mathbf{r})$$

$$= -2.165\,\mathbf{j} - 1.25\,\mathbf{k} + 0 + 0 + 2 \times 0.4\,\mathbf{k} \times (-2.5\,\mathbf{j} + 4.33\,\mathbf{k})$$

$$+ 0.4\,\mathbf{k} \times (0.4\,\mathbf{k} \times (8.66\,\mathbf{j} + 5\,\mathbf{k})$$

$$= 2\,\mathbf{i} - 3.55\,\mathbf{j} - 1.25\,\mathbf{k}\ \text{m/s}^2$$

Some comments can be made on the choice of a method. Methods 1 and 2 invoke a single moving frame whereas method 3 does not require any moving frame and method 4 requires more than one moving frame. Method 3 tends to be difficult because it requires thinking of the complete motion in one go. Method 4 offers simplicity of understanding and is indeed the choice if the number of moving components is large. There can be a number of intermediate moving frames for convenience. Methods 1 and 2 combine two steps into one and reduce the length of the procedure at the cost of clarity. However, in many problems, it is necessary to choose one moving frame for expressing the motion of every two rotating members. Methods 1 and 2 are, therefore, representative of a typical choice.

The origin of every moving frame was fixed at O only for convenience because then $\mathbf{V}_0 = 0$ and $\mathbf{a}_0 = 0$. There is no bar to fix the origin of the moving frame at any point. For example, it can be fixed at P, the tip of boom itself. Then, $\mathbf{r} = 0$ and $\mathbf{V}_0 \neq 0$ and $\mathbf{a}_0 \neq 0$.

Table 5.2 Expressions for Velocity and Acceleration in Different Coordinate Systems

Entity	*Cartesian Coordinates*	*Cylindrical Coordinates*
$\mathbf{r}$	$x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$	$r\,\mathbf{e}_r + z\,\mathbf{e}_z$
$\mathbf{V}$	$u\,\mathbf{i} + v\,\mathbf{j} + \omega\,\mathbf{k}$	$V_r\,\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\,\mathbf{e}_z$
a_x	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z}$	$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$
a_y	$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial z}$	$\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$
a_z	$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + \omega\frac{\partial \omega}{\partial z}$	$\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}$

Concept Review Questions

1. Comment on the truth of the following statements:
   (a) The displacement of a particle during an interval of time may not be the same as the distance moved by it.
   (b) Directions of velocity and acceleration of a point at any instant may not be the same whereas the directions of displacement and velocity at any instant must be the same.
   (c) The acceleration of a particle undergoing simple harmonic motion must be directed towards the centre of oscillations whereas the velocity may either be towards or away from it.
   (d) A vector may be constant in a moving frame of reference but it may not be constant as observed from a fixed frame of reference.
2. State the assumptions made in the derivations of the relations

$$V = U + at$$

$$S = Ut + \frac{1}{2}at^2$$

and $$V^2 - U^2 = 2as$$

and hence state when these relations are not applicable.

3. Explain the meaning of the terms: centripetal acceleration, Coriolis acceleration and normal acceleration as applied to the motion of a particle.
4. Show that

$$\frac{d\mathbf{e}_r}{dt} = \omega\,\mathbf{e}_\theta$$

$$\frac{d\mathbf{e}_\theta}{dt} = -\omega\,\mathbf{e}_r$$

$$\frac{d\mathbf{e}_n}{dt} = \omega\,\mathbf{e}_t$$

$$\frac{d\mathbf{e}_t}{dt} = -\omega\,\mathbf{e}_n$$

5. Under what circumstances are the cylindrical coordinates preferred to the rectangular coordinates and under what conditions are the path coordinates preferred to both?
6. Would the unit vectors **i**, **j** and **k** of a moving frame appear to change with time as observed from a fixed frame, if the moving frame
   (a) translates at a constant velocity?
   (b) translates rectilinearly with a variable velocity?
   (c) rotates at a constant rotational velocity?
   (d) rotates at a variable rotational velocity?

Tutorial Problems

5.1 The acceleration of a particle is given by

$$a = t^3 - 3t^2 + 5 \text{ m/s}^2$$

where the time t is in s. If the velocity of the particle at $t = 1$ s is 6.25 m/s and the displacement is 8.80 m, calculate the velocity and the displacement at $t = 2$ s.

(**Ans.** 8 m/s and 16.1 m)

5.2 A particle, starting from rest, moves in a straight line and its acceleration is given by

$$a = 50 - 36t^2 \text{ m/s}^2$$

where t is in s. Determine (a) the velocity of the particle when it has travelled 52 m, and (b) the time taken by it before it comes to rest again. (**Ans.** 4 m/s, 2.04 m)

5.3 A particle passes through a point (3, 4, 5) with a velocity of

$$V = 10\,\mathbf{i} + 11\,\mathbf{j} + 12\,\mathbf{k}$$

at a time $t = 1$ s. A constant acceleration

$$\mathbf{a} = 2\,\mathbf{i} - 3\,\mathbf{j}$$

is impressed upon it for 10 s. Compute the position and velocity of the particle at the final instant. (**Ans.** 203, –36, 125; 30 **i** – 19 **j** + 12 **k**)

5.4 A particle is observed at $t = 0$, and 2 s when it passes through positions (0, 3, 0) and (20, 3, – 4) respectively. If the velocity vector has the form

$$\mathbf{V} = At\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$$

determine its position and velocity at $t = 5$ s. (**Ans.** (125, 3, –10), 25 **i** – 2 **k** m/s)

5.5 A point moving with simple harmonic motion has an amplitude of 1 m and the period of one complete oscillation is 2 s. Determine its displacement, velocity and acceleration at an instant 0.4 s after passing an extremity.

(**Ans.** 0.309 m, 2.99 m/s, 3.05 m/s^2)

5.6 A particle moving with simple harmonic motion performs 10 complete oscillations per minute and its speed, when at a distance of 8 cm from the centre of oscillation is 60% of the maximum speed. Find the speed of the particle when it is 6 cm from the centre of oscillation. (**Ans.** x_0 – 10 cm, $V = 8.38$ cm/s)

5.7 A particle , moving with simple harmonic motion, has a time period of 0.6 s. Its speed at its mean position is 1.5 m/s. Determine its speed when it is half way between its mean position and an extremity. (**Ans.** 1.3 m/s at $x = 0.0715$ m)

5.8 Determine from first principles, the angle at which a bullet must be fired over a horizontal plane such that the greatest height attained by it equals the range on the plane. (**Ans.** $\alpha = 76°$)

5.9 Two guns are projected at each other, one upward at an angle of 30° and the other at the same angle of depression, the muzzles being 30 m apart as shown in Fig. Prob. 5.9. If the guns are shot with velocities of 350 m/s upward and 300 m/s downward respectively, find when and where the bullets may meet.

(**Ans.** 0.0462 s, (14 m, 8.07 m))

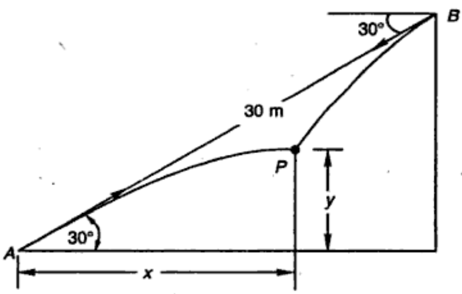


Fig. Prob. 5.9

5.10 Show that there are two directions in which a particle may be projected at the same velocity so that it passes through a given target. Establish the minimum velocity-of-projection requirement such that the particle does reach the target.

(**Ans.** $\theta_1 = 1/2 \sin^{-1} (gx/v^2), \pi/2 - \theta_1$;

$V \min \geq [gx^2/2(x \tan\theta - y)]^{1/2}$ for θ_1 and θ_2)

5.11 The horizontal distance of a target to be hit by a projectile is 10,000 m. The shell leaves the gun with a velocity of 600 m/s as shown in Fig. Prob. 5.11. What must be the angle of elevation α of the gun if a mountain 2000 m high intervening midway between the gun and the target is to be cleared? (**Ans.** $\alpha = 82.1°$)

5.12 A projectile is fired from a cliff 120 m above sea level with an initial velocity of 500 m/s directed at an angle of elevation of 30° to the horizontal. Estimate the time of flight and the horizontal range if the target is at the sea level.

(**Ans.** 51.4 s, 22 276 m)

5.13 Two particles are projected simultaneously from two points A and B such that h is the horizontal distance and k the vertical distance between them. They are projected at

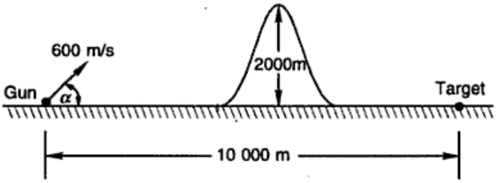


Fig. Prob. 5.11

the same inclination α to the horizon with the same velocity V as shown in Fig. Prob. 5.13. Show that their distance from each other will be minimum after a time.

$$t = \frac{h}{2\,V \cos \alpha}$$

and that the minimum distance will be k.

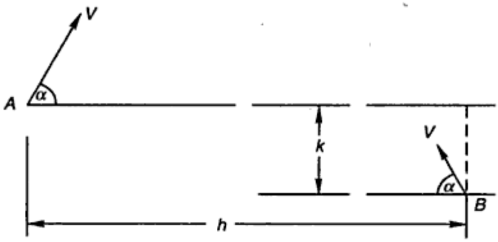


Fig. Prob. 5.13

5.14 If a body travels half its total path in the last second of its free fall, starting from rest, find the total time and height of its fall. (**Ans.** 3.414 seconds; 57.2 metres)

5.15 A ball rolls off the top of a stairway with a horizontal velocity of 1.5 m/s. The steps are 20 cm wide and 20 cm high. Which step will the ball hit first?

(**Ans.** The path $y = -2.18\,x^2$ intersects the line $y = -x$ at $x = 0.45$ m; hence, 3rd step).

5.16 Determine the minimum speed with which the motorcycle must leave the 30° ramp at A to reach the point B, clearing the pond in between. (**Ans.** 37 km/hour)

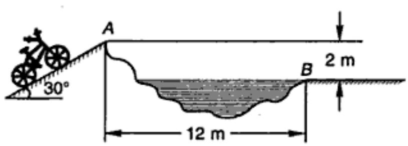


Fig. Prob 5.16

5.17 A helicopter is descending vertically downward with a uniform velocity. At a certain instant, a food packet is dropped from it which takes 5 seconds to reach the ground. As this packet strikes the ground, another food packet is dropped from it, which takes 4 seconds to reach the ground. Find the velocity with which the helicopter is descending and its height, when second packet is dropped. Also find the distance travelled by the helicopter during the interval of dropping the packets.

(**Ans.** 11.04 m/s downwards at 177.8 m height; 55.2 m downwards)

5.18 A point moves on a curve $xy = 16$ according to the law $x = 4\ t^2$ where x and y are expressed in meters and t is in seconds. Find the magnitude and direction of the velocity of the point (a) when $t = 1$ second and (b) when $x = 2$ metres.

(**Ans.** $V_x = 8\ t$, $V_y = -8/t^3$ in m/s; (a) $t = 1$, $x = y = 4$; $V_x = 8$, $V_y = -8$ m/s
(b) $x = 2$, $y = 8$, $t = 0.707$s; $V_x = 5.66$, $V_y = -22.6$ m/s)

5.19 A gun fires a bullet with such an initial velocity and such an angle of elevation that the maximum height to which it rises is h. Find the maximum range that can be obtained with the same initial velocity.

(**Ans.** $h = z_{max} = \dfrac{V_0^2}{2g}$ at $\alpha = 90°$: $x_{max} = V_0^2 \sin 2\alpha/g = 2h$ at $\alpha = 45°$)

5.20 A body travels a distance s in a duration of t seconds. It starts from rest and ends at rest. In the first part of its journey it moves at a constant acceleration a and in the second part with a constant retardation r. Show that

$$t = \sqrt{2s\,(1/a + 1/r)}$$

(Hint: Set up the three equations; $t = t_1 + t_2$, $s = 1/2\ (a\ t_1^2 + r\ t_2^2)$ and $a\ t_1 = r\ t_2$ and eliminate t_1 and t_2)

5 21 The rotor of a motor has an angular acceleration which is directly proportional to the time t. The motor starts from rest at time $t = 0$. After 3 seconds, the rotor has completed 5 revolutions. Obtain the equation of motion of the rotor and estimate its angular velocity at $t = 2$ seconds (**Ans.** $\alpha = d\omega/dt = 20/9\ \pi t$; 14 rad/s)

5.22 A 250 m long railway train is travelling along a curved track of 1 km radius at a speed of 60 km/hour and decelerates at 0.2 g. Calculate the velocity and acceleration of the engine as seen by the guard at the tail end of the train.

(**Ans.** 16.67 (**i** – **j**) m/s; – 1.68 **i** – 2.24 **j** m/s^2)

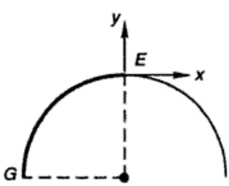


Fig. Prob. 5.22

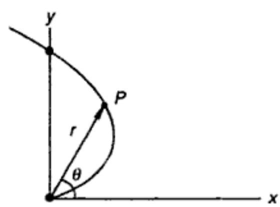


Fig. Prob. 5.23

5.23 The path of a particle P is an Archimedean spiral. The motion of the particle is defined by the relations,

$$r = 10\ t \qquad \text{and} \qquad \theta = 2\ t$$

where r is in metres, t is in seconds and θ is in radians. Determine the velocity and acceleration of the particle (a) when $t = 0$ and (b) when $t = 0.25$ seconds.

(**Ans.** (a) $10\ i$, $40\pi j$; (b) $10\ j - 5\ \pi\ i$, $-40\ \pi\ i - 10\pi^2 j$)

5.24 An aeroplane is flying with a constant velocity v at a constant height h. Show that, if a gun is fired point blank at the aeroplane as it passes directly over the gun with an angle of elevation α, the shell will hit the aeroplane provided

$$2(V\cos\alpha - v)\ v \tan^2\alpha = gh$$

where V is the initial velocity of the shell.

5.25 A bomber is flying horizontally at a speed of 500 km/h at an altitude of 3 km such that a ship lies in a vertical plane through the line of sight as shown in Fig. Prob 5.25. Determine the angle of the line of sight of the bomber with the ship at the instant a bomb is released so as to hit the ship. Where would the bomber be at the instant the ship is wrecked? (**Ans.** $\theta = 48.9°$; Over the ship)

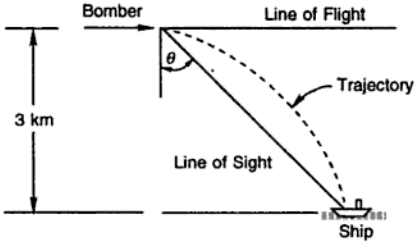


Fig. Prob. 5.25

5.26 The motion of a point in the vertical plane is given by

$$r = 3\,t^2,\ \theta = 0.5 \sin \pi t/4$$

where r is in cm and θ in radian and t in s. Determine the velocity and acceleration of the point when $t = 3$ s. (**Ans.** 19.5 cm/s, 16.4 cm/s^2)

5.27 A wheel rotates at an angular speed 10 rad/s and the rotational speed increases at 2 rad/s^2. A collar C moves out on a horizontal spoke such that its speed and acceleration with respect to the spoke are 3 m/s and 2 m/s^2 respectively as shown in Fig. Prob. 5.27. Compute the absolute velocity and acceleration of the collar if it is at 0.5 m from the centre of rotation. (**Ans.** 5.83 m/s; 77.62 m/s^2)

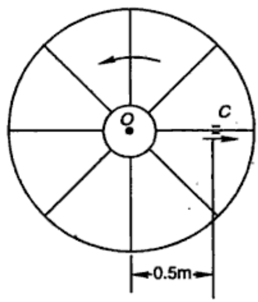


Fig. Prob. 5.27

5.28 A tracking device stationed at the launching point of a missile records the r and θ coordinates of the missile with the passage of time. It is noticed that if r is in km, θ in degrees and t in s, the following expressions represent the motion closely

$$r = 2\,t - t^2/20$$
$$\theta^2 = 1300 - t^2$$

for the plane trajectory of the missile. Estimate the position , velocity and acceleration of the missile at $t = 20$ s. (**Ans.** 20 km, – 0.2333 km/s. 0.104 km/s^2)

5.29 A particle moves on a frictionless wire bent into a cubic $y = 2x^3$. At a point (1, 2), the speed of the particle is 3 m/s and it decreases at a rate of 2 m/s. Compute its velocity and acceleration in terms of the rectangular coordinates.

(**Ans.** 0.493, 2.96 m/s; –0.79, –1.84 m/s^2)

5.30 A particle P slides down an incline of 30° frictionlessly and then moves up a circular arc of 1 m radius as shown in Fig. Prob. 5.30. Compute the velocity and acceleration of the particle at A just before the start of the arc and at B midway on the arc.

(**Ans.** 5.18 m/s; 8.5, 27.3 m/s^2)

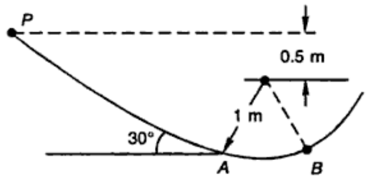


Fig. Prob. 5.30

5.31 A load P is being raised by means of an assemblage of two links AB and BP as shown in Fig. Prob. 5.31. At the instant of interest link AB rotates and accelerates at 3 rad/s and 4 rad/s^2 respectively with respect to the ground whereas link BP with an angle of 90° AB rotates and accelerates at 5 rad/ and 2 rad/s^2 with respect to link AB. If AB is 5 m long and BP is 2 m long with an angle of 60° at the instant of interest, determine the acceleration of the load with respect to the ground reference.

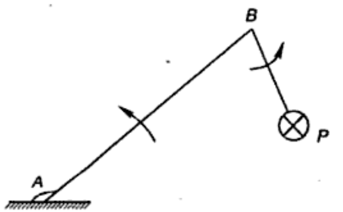


Fig. Prob. 5.31

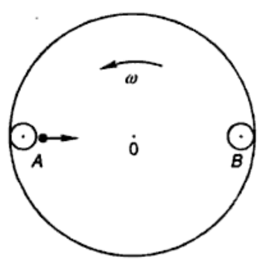


Fig. Prob. 5.32

5.32 Two boys A and B stand diametrically opposite on a plane horizontal table of diameter 5 m rotating anticlockwise at 10 radians per second as shown in Fig. Prob. 5.32. If A throws a ball towards B at a speed of 5 m/s, how will it tend to move on the table and why? (**Ans.** 50 m/s tangential, 5 m/s radius)

5.33 A rotating spotlight is at a perpendicular distance l from a horizontal floor. The light revolves at constant N revolutions per minute about a horizontal axis perpendicular to the plane representing it in Fig. Prob. 5.33. Drive expressions for the velocity and acceleration of the light spot travelling along the floor. Let θ be the angle between the vertical line l and the light beam at time t.

(**Ans.** $\omega = 0.1051\ N \sec^2 \theta$, $\alpha = 0.0221\ N^2 \sec^2 \theta \tan \theta$)

5.34 A flexible chain of length l rests on a smooth table with length c overhanging the edge as shown in Fig. Prob 5.34. The system originally at rest is released. Describe the motion. The chain weighs w N/m.

(**Ans.** $\ddot{x} - gx/l = gc/l$; $x = \frac{1}{2}c \exp(\sqrt{g/l\,t}) + \frac{1}{2}c \exp(-\sqrt{g/l}t)$)

5.35 A straight tube is attached to a vertical shaft at a fixed angle α as shown in Fig. Prob. 5.35. The shaft rotates with a constant angular velocity ω. A particle moves along the tube with a constant velocity V relative to the tube. Find the magnitude of the acceleration of the particle when it is at a distance l along the tube from the centre.

$$\left(\textbf{Ans. } a = l\omega^2 \sin \alpha \sqrt{1 + \left(\frac{2V}{l\,\omega}\right)^2}\right)$$

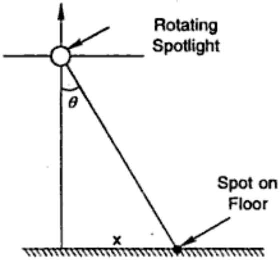


Fig. Prob. 5.33

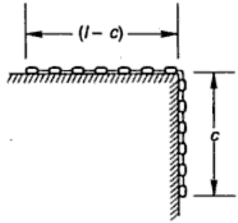


Fig. Prob. 5.34

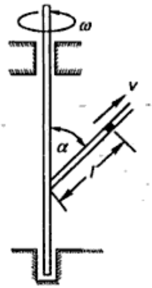


Fig. Prob. 5.35

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. The displacement of a point
   (a) implies the distance moved by the point
   (b) is a vector, from the initial to the final position of the point
   (c) is always less than the distance traversed by the point
   (d) is independent of the distance and the direction of movement of the point
2. The relationship $V^2 - U^2 = 2as$, with conventional notation, is applicable for
   (a) all possible motions of a point
   (b) constant velocity of a point
   (c) constant acceleration of a point
   (d) variable acceleration of a point
3. One of the following assumptions is *not* necessary in obtaining the equation for parabolic trajectory of a particle:
   (a) Air resistance is negligible
   (b) The gravitational acceleration g is constant
   (c) The body can be represented by a particle
   (d) The body must not change its mass during the motion

4. The unit vector 'normal' to a curve
   (a) is directed towards the local centre of curvature
   (b) is directed outward along the join of the centre of curvature and the point
   (c) is the same as the radial unit vector
   (d) must only be perpendicular to the path of the point
5. The Coriolis acceleration may *not* vanish if the
   (a) relative velocity of the moving point becomes zero
   (b) rotational velocity of the moving frame becomes zero
   (c) rotational velocity of the moving frame and the relative velocity become collinear
   (d) angular acceleration of the point becomes zero

Answers to Multiple-Choice Questions

1 (b), 2 (c), 3 (d), 4 (a), 5 (d).

6 Dynamics of a Particle

and of the Mass Centre of Any System

6.1 INTRODUCTION

The study of dynamics refers to the motion of bodies under the application of action, i.e., external forces or moments. The same laws of dynamics are applicable to the motion of a particle and of the centre of mass of any system undergoing translation under the application of forces. This is because the mass of a particle is assumed to be concentrated at a point which is also its centre of mass. This fact is supported by kinematic considerations; the general motion of a rigid body may be considered to comprise the translation of the mass centre and a rotation superimposed upon it. The laws and principles studied in this chapter apply to the translation of any body or a system of bodies. The terms 'centre of mass' and 'particle' are at times used interchangeably in this chapter and should cause no confusion on the scope of application of the equations.

The dynamics of a mass centre or of a particle is governed by the Newton's law

$$\boxed{\mathbf{F} = \frac{d}{dt}(m\,\mathbf{V})}$$

This is the fundamental equation of motion which governs the interaction of the applied force **F** with the motion of a particle or a mass centre. Problems in dynamics may be concerned with the determination of the motion, i.e., acceleration, velocity and positions for a prescribed force or vice versa.

It may be stated at the outset that the work-energy principle, impulse-momentum principle and the moment of momentum principle are alternative forms of Newton's law. One or the other may be preferred under different circumstances. A comparative study of the equivalent dynamical equations is given in Table 6.1. This is indeed a summary of the principles derived and discussed in this chapter.

6.2 EQUATION OF MOTION

The equation of motion due to Newton for the centre of mass of any system or a particle of *constant mass m* may be written as

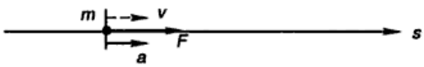
$$\boxed{\mathbf{F} = m\frac{d\mathbf{V}}{dt} = m\,\mathbf{a} = m\frac{d^2\mathbf{r}}{dt^2}} \qquad (6.1)$$

It may be noted that for a particle, the net force, velocity and acceleration refer to the point representation of the particle, while for a rigid body, the net force may be applied anywhere on it but the velocity and acceleration are referred to the mass centre only as shown in Fig. 6.1.

$$\mathbf{F} = m\frac{d\mathbf{V}}{dt} = m\mathbf{a} \qquad \mathbf{F} = m\frac{d\mathbf{V}_c}{dt} = m\mathbf{a}_c$$

Fig. 6.1 ***Implication of Newton's Law***

The motion can be determined from a knowledge of the applied force **F**. Let us first consider some simple cases of rectilinear translation as visualised in Fig. 6.2.



(a) F = Const.
(b) F = F(t)
(c) F = F(v)
(d) F = F(s)

Fig. 6.2 ***Rectilinear Translation***

Case (a): Constant Force F along *s*-direction

Then, $$a = \frac{d^2 s}{dt^2} = F/m$$

On integration,

$$v = \frac{ds}{dt} = F/m\, t + C_1$$

and $$s = \frac{1}{2} F/m\, t^2 + C_1 t + C_2$$

The constants of integration are determined from the given conditions.

Case (b): Force *F*(*t*) is Function of Time along *s*-direction

Then, $$a = \frac{d^2 s}{dt^2} = F(t)/m$$

On integration,

$$v = \frac{ds}{dt} = \int F(t)/m\, dt + C_1$$

and $$s = \int \left(\int F(t)/m\, dt + C_1 \right) dt + C_2$$

where the constants C_1 and C_2 are again determined from the given conditions.

Table 6.1 Comparative Study of the Equivalent Dynamical Equations for a Particle or a Mass Centre

	Newton's Law	*Work Energy Equation*	*Impulse Momentum Principle*	*Moment of Momentum Principle*
General form for a particle	$\mathbf{F} = \frac{d}{dt}(m\mathbf{V}) = m\mathbf{a}$	$W_{12} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = KE_2 - KE_1$	$\int_{t_1}^{t_2} \mathbf{F} \cdot dt = m(\mathbf{V}_2 - \mathbf{V}_1)$	$\mathbf{M}_0 = \mathbf{r}_0 \times \mathbf{F} = \frac{d}{dt}(\mathbf{r}_0 \times m\mathbf{V}) = \frac{d\mathbf{H}_0}{dt}$
Statement for a particle	Net force acting on a particle equals the rate of change of momentum or the mass times acceleration of the particle	Net work done by the external forces acting on a particle equals the change in kinetic energy possessed by the particle.	Impulse of the net force acting on a particle over a period of time equals the change of momentum of the particle over the same period of time	Moment of the resultant force on a particle about a fixed point O equals the time rate of change of the moment of momentum referred to the point
General form for the mass centre of a body	$\mathbf{F} = \frac{d}{dt}(m\mathbf{V}_c) = m\,\mathbf{a}_c$	$W'_{12} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}_c = KE_{2c} - KE_{1c}$	$\int_{t_1}^{t_2} \mathbf{F} \cdot dt = m(\mathbf{V}_{2c} - \mathbf{V}_{1c})$	$\mathbf{M}_0 = \mathbf{r} \times \mathbf{F} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{V}_c) = \frac{d\mathbf{H}_0}{dt}$
Statement for the mass centre of a body	Net force acting on a body equals the rate of change of momentum associated with the mass centre or the mass times acceleration of the centre of mass of the body	Net work done by the external forces, as if acting at the mass centre of a body, equals the change in linear kinetic energy associated with the mass centre of the body	Impulse of the net force acting on a body over a period of time equals the change of momentum associated with the mass centre of the body over the same period of time	Moment of the resultant force at the mass centre of a system about a fixed point O equals the time rate of change of moment of momentum about the same point.
Special cases; Associated Principles	When $\mathbf{F} = 0$; in the absence of net external force, a particle or the mass centre of body must continue to move with constant velocity: $\mathbf{V} = \text{Const.}$	When $\mathbf{F} = -\nabla PE$; in a conservative force field, the mechanical energy of a particle or that associated with the mass centre is conserved; $KE + PE = \text{Const.}$	When $\mathbf{F} = 0$; in the absence of an external force, the linear momentum of a particle or that associated with the mass centre is conserved: $\mathbf{p} = \Sigma m\mathbf{V} = \text{Const.}$	When $\mathbf{M}_0 = 0$; in the absence of the moment of the net external force taken about a fixed point, the moment of momentum relative to the same point is conserved; $\mathbf{H} = \Sigma(\mathbf{r} \times m\,\mathbf{V}) = \text{Const.}$

Table 6.1 (*Contd.*) **Comparative Study of the Equivalent Dynamical Equations for a Particle or a Mass Centre**

	Newton's Law	*Work Energy Equation*	*Impulse Momentum Principle*	*Moment of Momentum Principle*
Circumstances under which preferred	When the external force or acceleration is desired to be evaluated	When the work done may be determined and/or when one of the velocities is to be evaluated; particularly useful for motion in a conservative force field.	When the impulses or application of forces over a period of time is involved and also when the external force or impulse vanishes	When the force on a particle is directed towards or away from a fixed point

Case (c): Force $F(v)$ is a Function of Speed along s-direction

Then, $$a = \frac{dv}{dt} = F(v)/m$$

or, $$\frac{dv}{F(v)} = \frac{1}{m}\,dt$$

On integration,

$$\int \frac{dv}{F(v)} = \frac{1}{m}t + C_1$$

which provides $$v = f(t)$$

and on further integration, yields an expression for the displacement s, the constant being determined from the given conditions.

Case (d): Force $F(s)$ is Function of the Rectilinear Displacement s

Then, $$a = \frac{dv}{dt} = F(s)/m$$

or, $$\frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = F(s)/m$$

or $$v\,dv = \frac{1}{m}\,F(s)\,ds$$

On integration,

$$\frac{v^2}{2} = \frac{1}{m}\int F(s)\,ds + C_1$$

and $$v = \frac{ds}{dt} = \left[\frac{2}{m}\int F(s)\,ds + C_1\right]^{1/2}$$

Separating the variables and integrating again provides s as a function of t, the constants of integration being determined from a knowledge of the given conditions.

If the motion of a particle is prescribed, the force required to accomplish it may be determined by employing the equation of motion

$$\mathbf{F} = m\,\mathbf{a}$$

and substituting the value of **a** in it.

***Example* 6.1** A particle of mass 1 kg moves in a straight line under the influence of a force which increases linearly with time at the rate of 60 N/s, it being 40 N initially. Determine the position, velocity and acceleration of the particle after a lapse of 5 s if it started from rest at the origin.

Solution From the statement of the problem,

$$F = 40 + 60t$$

which, by Newton's law should equal mass times acceleration of the particle. Since the mass is 1 kg,

$$a = \frac{d^2x}{dt^2} = 40 + 60t$$

Integrating the terms with respect to time t.

$$v = \frac{dx}{dt} = 40t + 30t^2 + C_1$$

and integrating again,

$$x = 20t^2 + 10t^3 + C_1t + C_2$$

From the initial conditions,

$$v = 0 \quad \text{and} \quad x = 0 \quad \text{at } t = 0$$

the constant C_1 and C_2 vanish.

Hence, $v = 40t + 30t^2$

and $x = 20t^2 + 10t^3$

At the instant, $t = 5$ s

$$a = 40 + 60 \times 5 = 340 \text{ m/s}^2$$

$$v = 40 \times 5 + 30 \times 5^2 = 950 \text{ m/s}$$

$$x = 20 \times 5^2 + 10 \times 5^3 = 1750 \text{ m from the origin.}$$

***Example* 6.2** A particle moving with a velocity v along a straight line is retarded such that the retardation is (a) proportional to velocity and (b) proportional to square of velocity. Determine the expressions for velocity as a function of time and the distance traversed before it comes to rest for both cases.

Solution For case (a), let the retardation be kv

i.e., $$a = \frac{dv}{dt} = -k\,v$$

or $$\frac{dv}{dt} + k\,v = 0;$$

On integration $$\int_u^v \frac{dv}{v} + k\int_0^t dt = 0$$

or $$\log_e \frac{v}{u} + k\,t = 0$$

$$v = u^{-kt}$$

Also, $$a = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds} = -k\,v$$

or $$\int_u^v dv + k \int_0^s ds = 0$$

$$v - u + k\,s = 0$$

$$v = u - k\,s$$

$$s = \frac{u - v}{k}$$

For case (b), let the retardation be $\mu\, v^2$

i.e., $$a = \frac{dv}{dt} = -\mu\, v^2$$

or $$\frac{dv}{dt} + \mu v^2 = 0$$

On integration, $$\int_u^v \frac{dv}{v^2} + \mu \int_0^t dt = 0$$

$$-\frac{1}{v} + \frac{1}{u} + \mu\, t = 0$$

or $$v = \frac{u}{1 + \mu\, u\, t}$$

Also, $$a = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds} = -\mu v^2$$

or $$v\frac{dv}{ds} + \mu v^2 = 0$$

$$\int_u^v \frac{dv}{v} + \mu \int_0^s ds = 0$$

$$\log_e \frac{v}{u} + \mu\, s = 0$$

$$v = u\, e^{-\mu s}$$

$$s = \frac{1}{\mu} \log_e \frac{u}{v}$$

It may be noticed that the distance traversed before coming to rest in case (a) is finite whereas that in case (b) is infinite!

***Example* 6.3** If a body of mass m moves through a liquid at low velocity, the force of resistance due to viscosity is given by

$$F = k\,v$$

where k is the resisting force at unit velocity. Show that the velocity would decrease exponentially with time and linearly with displacement.

Solution The motion of the centre of mass of the body is given by

$$F = m\,a = m\frac{dv}{dt} = -\,k\,v$$

whence $$\frac{dv}{dt} = -\,k/m\;v$$

or $$\frac{dv}{v} = -\frac{k}{m}\,dt$$

On integration,

$$\log_e v = -\frac{k}{m}t + C$$

Using the condition, $v = v_0$ at $t = 0$, $C = \log_e v_0$

$$\log_e\left(\frac{v}{v_0}\right) = -\frac{k}{m}t$$

or $$\frac{v}{v_0} = e^{-k/mt}$$

which shows that the velocity decreases exponentially with respect to time.

Writing $$v = \frac{dx}{dt}$$

$$\frac{dx}{dt} = v_0 \cdot e^{-k/mt}$$

or $$dx = v_0 \cdot e^{-k/mt}\,dt$$

which, upon integration gives

$$x = \frac{v_0}{k}\left(1 - e^{-k/mt}\right)$$

Substituting for v_1

$$v = v_0 - k\,x$$

which shows that the velocity of the body decreases linearly with displacement.

***Example* 6.4** Identify the correct and incorrect response(s) in the following:

A particle is acted upon by a force of constant magnitude which is always perpendicular to the velocity of the particle. The motion of the particle takes place in a plane. It follows that:

(A) its velocity is constant.
(B) its acceleration is constant.
(C) its kinetic energy is constant.
(D) it moves in a circular path.

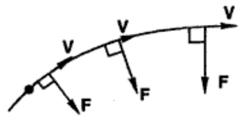


Fig. Ex. 6.4

In this case,

$$\mathbf{V} \cdot \mathbf{F} = 0$$

Since **F** has no component along the velocity vector, it must be constant in magnitude.

Since **F** is constant in magnitude, the particle must have an acceleration also constant in magnitude only. Directions of **V**, **F** and **a** vary.

(A) and (B) are incorrect because only the magnitudes of velocity and acceleration must be constant, not their directions.

The kinetic energy $1/2\ mV^2$ is not constant as V is not constant. (C) is incorrect. Also (D) is incorrect because the given conditions may bring about motion in a non-circular path.

***Example* 6.5** The angular velocity of a flywheel is observed to decrease by 10% in the first minute. Calculate the decrease in the second minute if the retardation is proportional to the angular velocity.

Solution
Given that

$$\frac{d\omega}{dt} = -k\,\omega$$

Hence,

$$\frac{d\omega}{dt} + k\,\omega = 0$$

and

$$\int_{\omega_0}^{\omega} \frac{d\omega}{\omega} + k\int_0^t dt = 0$$

or

$$\log\frac{\omega}{\omega_0} + k\,t = 0$$

and

$$\frac{\omega}{\omega_0} = e^{-kt}$$

In the first one second, $\frac{\omega}{\omega_0}$ becomes 90%.

In the second second, it will become 90% of that, i.e., 81% of the original velocity.

***Example* 6.6** An object of mass m falls vertically down in a medium with the resistance R proportional to the velocity. Obtain an expression for the velocity at time t if it starts from rest at time $t = 0$. What is the terminal velocity?

Solution The motion of the centre of mass C of the object may be studied by employing the Newton's law,

$$F = m\,a = m\frac{dv}{dt}$$

but the net external force on the object is given by

$$F = m\,g - k\,v$$

as shown in Fig. Ex. 6.6(a) (Solution).

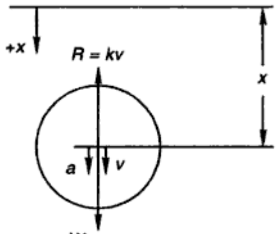


Fig. Ex. 6.6(a) *(Solution)*

Therefore $$m\frac{dv}{dt} = m\,g - k\,v$$

or $$\frac{dv}{mg/k - v} = \frac{k}{m}\,dt$$

On integrating,

$$-\ln(mg/k - v) = \frac{k}{m}t + C_1$$

Recognising that $v = 0$ at $t = 0$

$$C_1 = -\ln(mg/k)$$

Hence $$-\ln(mg/k - v) = \frac{k}{m}t - \ln(mg/k)$$

or $$\frac{mg/k - v}{mg/k} = e^{(-k/m)\,t}$$

from which $$v = mg/k(1 - e^{(-k/m)\,t}) \tag{i}$$

The terminal velocity of the centre of mass occurs at t tending to infinity;

$$V = mg/k \tag{ii}$$

If it is desired to obtain an expression for the displacement of the centre of mass of the object, the fact that $v = \frac{dx}{dt}$ together with Eq. (i) provides

$$dx = mg/k(1 - e^{(-km)\,t})\,dt$$

whence, by integration with the prescribed condition,

$$x = [mg/k]t + m^2g/k^2e^{(-k/m)\,t} - m^2g/k^2$$

or $$x = V\,t - \frac{V^2}{g}(1 - e^{(-k/m)t})$$

It may also be added that the case of the resistance R being proportional to the velocity of an object is a factual situation for low-speed movements through viscous fluids. At higher speeds, however, the resistance is proportional to the square of velocity of the object.

Another very important comment must be noted. The object was assumed to have a symmetrical shape such that the weight and resistance act along the same line. This assumption was not necessary. In fact, the object chosen could have been unsymmetrical and the line of action of the resistance to motion could have been displaced with respect to that of the weight *mg*, although parallel as shown in Fig. 6.6(b) (Solution). The equations still apply as far as the motion of the centre of mass is concerned and the results obtained are correct because the net external force is still given by

$$F = m\,g - k\,v$$

and the displacement, velocity and acceleration of the centre of mass are implied. The centre of mass drops vertically down. It is quite a different matter whether the object rotates or not. The rotation aspect is not within the purview of the Newton's law of motion.

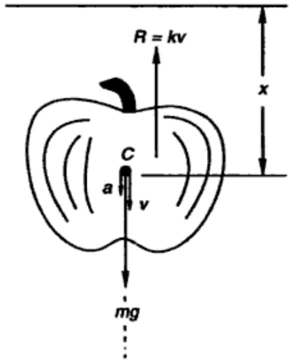


Fig. Ex. 6.6(b) ***(Solution)***

***Example* 6.7** Two blocks of mass 2.9 kg and 1.9 kg are suspended from a rigid support *S* by two inextensible wires each of length 1 metre, see Fig. Ex. 6.7. The upper wire has negligible mass and the lower wire has a uniform mass of 0.2 kg/m. The whole system of blocks, wires and support have an upward acceleration of 0.2 m/s^2. Acceleration due to gravity is 9.8 m/s^2.

(i) Find the tension at the mid-point of the lower wire.
(ii) Find the tension at the mid-point of the upper wire.

Solution For the free-body diagram of the lower mass together with 50% of the

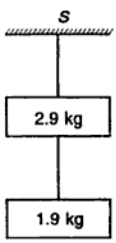


Fig. Ex. 6.7

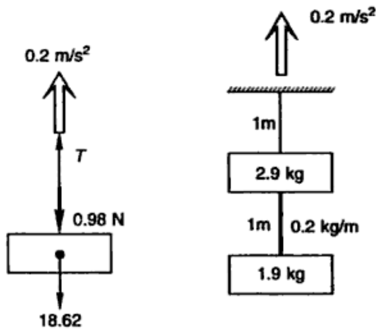


Fig. Ex. 6.7 ***(Solution)***

lower wire, the forces acting are as shown in Fig. Ex. 6.7 (Solution). Applying the Newton's law of motion.

$$T - 18.62 - 0.98 = (1.9 + 0.1) \times 0.2$$

whence $T = 0.4 + 19.6 = 20$ N

In order to find the tension at the mid point of the upper wire, consider the free body diagram of the entire part below it. Then,

$$T - 18.62 - 1.96 - 2.9 \times 9.8 = (1.9 + 0.2 + 2.9) \times 0.2$$

whence $T = 50$ N

***Example* 6.8** A block of mass $m = 5$ kg rests on a smooth inclined surface of a wedge of mass $M = 10$ kg. The wedge is resting on a smooth horizontal surface. Assuming the pulley to be weightless, smooth and frictionless and the string to be light and inextensible, find the acceleration of mass M.

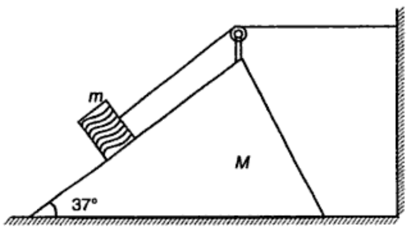


Fig. Ex. 6.8

Solution From the free body diagram of the mass m, as shown in Figs. Ex. 6.8(a) and (b) (Solution),

$$R = mg \cos 37° = 5 \times 9.81 \times 0.8 = 39.2 \text{ N}.$$

$$T = mg \sin 37° = 5 \times 9.81 \times 0.6 = 29.4 \text{ N}.$$

Now, let us consider the f.b.d. the wedge.
The horizontal force acting on it is $R \sin 37°$

$$= 39.2 \times 0.6 = 23.6 \text{ N}$$

whence $10 \times a = 23.6$

$$a = 2.36 \text{ m/s}^2$$

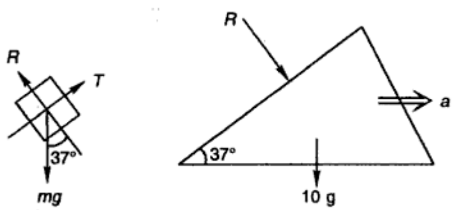


Fig. Ex. 6.8 (a and b) *(Solution)*

***Example* 6.9** Two blocks $m_1 = 2$ kg and $m_2 = 5$ kg are initially resting on the floor. They are connected by a light and inextensible cord running over a weightless and frictionless pulley as shown in Fig. Ex. 6.9. Find the acceleration of each block and the pulley if an upward force F applied to the pulley is (i) 14 N, (ii) 70 N. Find the force required to lift both the blocks.

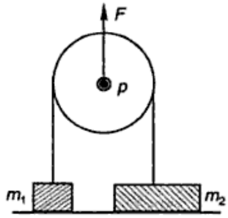


Fig. Ex. 6.9

Solution Weights of the two blocks are $2 \times 9.81 = 19.62$ and $5 \times 9.81 = 49.05$ N respectively.

(i) With $F = 14$ N, neither of the weights is overcome; none will be lifted. $a_1 = 0 = a_2 = a_p$

(ii) With $F = 70$ N, tension in each cord = 35 N.

For the smaller block,

$$35 - 19.62 = 2\,a_1;\ a_1 = 7.69 \text{ m/s}^2$$

but the bigger block cannot move, $a_2 = 0$

The pulley moves up with the average velocity and acceleration,

$$a_p = (7.69 + 0)/2 = 3.85 \text{ m/s}^2$$

Both the blocks will move up if tension in each string exceeds 49.05 N, the weight of the bigger block. Then $F = 2 \times 49.05 = 98.10$ N.

***Example* 6.10** Two blocks A and B are held stationary 10 m apart on a 20° incline as shown in Fig. Ex. 6.10. The coefficient of dynamic friction between the plane and A is 0.3 whereas between the plane and B is 0.1. If the blocks are released simultaneously, calculate the time taken and distance travelled by each block before they are at the verge of collision.

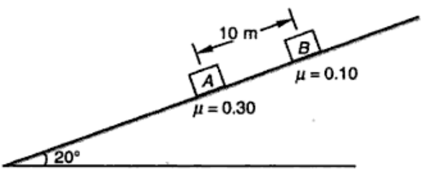


Fig. Ex. 6.10

Solution From the free-body diagrams of the blocks, as shown in Fig. Ex. 6.10 (Solution) for block A,

$$m_A g \sin 20° - 0.3 R_A = m_A a_A$$

$$R_A = m_A g \cos 20°$$

or
$$m_A g \sin 20° - 0.3\, m_A g \cos 20° = m_A a_A$$

or
$$9.81 \times 0.342 - 0.3 \times 9.81 \times 0.94 = a_A$$

$$a_A = 0.59 \text{ m/s}^2$$

Similarly, for block B,

$$m_B g \sin 20° - 0.1 R_B = m_B a_B$$

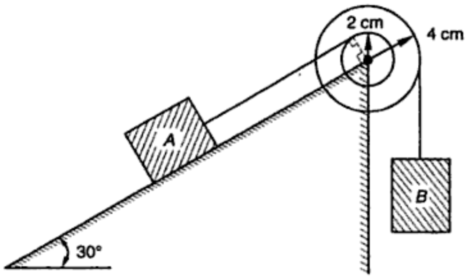
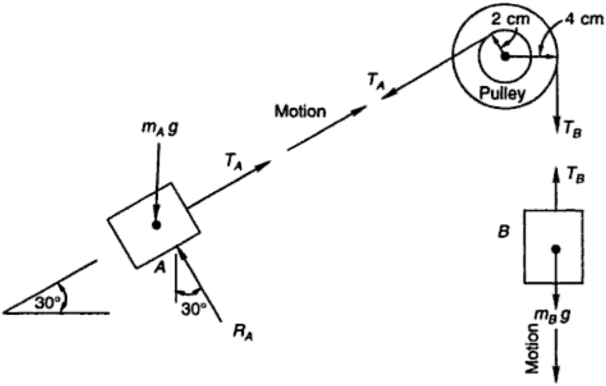


Fig. Ex. 6.11



Figs. Ex. 6.11 (a) and (b) ***(Solution)***

(Solution), the forces along the direction of motion are evaluated and substituted in the equations of motion:

(*a*) *For frictionless incline*

For block *A*,

$$T_A - m_A g \sin 30° = m_A a_A \tag{ii}$$

For block *B*,

$$m_B g - T_B = m_B a_B \tag{iii}$$

From kinematic considerations for the step-pulley,

$$a_B = 2a_A = 2 \times 2 = 4 \text{ m/s}^2 \tag{iv}$$

Substituting Eqs. (i) and (iv) into Eqs. (ii) and (iii),

$$2T_B - 3 \times 9.81 \times 0.5 = 3 \times 2$$

$$9.81\ m_B - T_B = 4\ m_B$$

From the former,

$$T_B = \frac{6 + 14.715}{2} = 10.36 \text{ N}$$

and from the latter,

$$m_B = \frac{10.36}{5.81} = 1.78 \text{ kg}$$

(*b*) *For frictional incline*

For block *A*.

$$T_A - m_A g \sin 30° - 0.3\, R_A = m_A\, a_A$$

where $R_A = m_A\, g \times \cos 30°$

$= 3 \times 9.81 \times 0.866 = 25.49 \text{ N}$

Then, $T_A - 3 \times 9.81 \times 0.5 - 0.3 \times 25.49 = 3 \times 2$

or $T_A - 22.36 = 6$

whence $T_A = 28.36$

For block *B*,

$$m_B\, g - T_B = m_B\, a_B$$

Employing the facts that $T_A = 2T_B$ and $a_B = 2a_A = 4 \text{ m/s}^2$

it becomes $9.81 m_B - 28.36/2 = 4\, m_B$

whence $m_B = 2.44 \text{ kg}$

It may be noted that the presence of friction on the incline results in considerably higher tension in the cords and requires a larger mass of the hanging block to cause the same acceleration of the block on the incline. A little reflection will show that if a single pulley was employed instead of a step-pulley, the problem would be lot easier but the mass of block *B* required for the same purpose would be considerably more!

***Example* 6.12** A painter of mass 100 kg standing in a jhoola, i.e., suspended cage of mass 25 kg has arranged to pass the rope over a fixed pulley. He pulls the rope with an acceleration in order to rise. At an instant, he exerts an effective weight of 450 N on the jhoola, find the

(i) acceleration of the painter and
(ii) tension in the string.

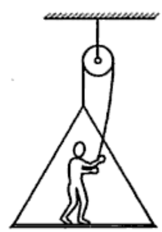


Fig. Ex. 6.12

Solution Let us draw the free body diagram of the jhoola as also of the painter alone. (Ref. Ex. 6.12 (Solution))
For the two free-bodies respectively,

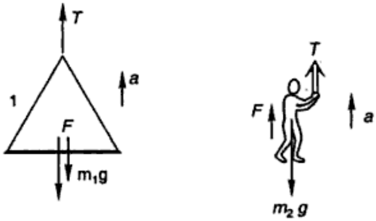


Fig. Ex 6.16 ***(Solution)***

$$T - F - m_1 g = m_1 a \qquad T + F - m_2 g = m_2 a$$

$$T - 450 - 25 \times 9.81 = 25\,a \qquad T + 450 - 100 \times 9.81 = 100\,a$$

$$T - 25\,a = 695.3 \qquad T - 100\,a = 531$$

From these two equations,

$$a = 2.19 \text{ m/s}^2, \quad T = 750 \text{ N}.$$

6.3 D'ALEMBERT PRINCIPLE: INERTIA FORCES

If the mass centre of a body or a particle of mass m is subjected to a net force **F** and it acquires an acceleration **a**, then,

$$\mathbf{F} + (-m\,\mathbf{a}) = 0 \tag{6.2a}$$

This is indeed a restatement of Newton's law but it suggests that the term $(-\,ma)$ may be considered as a fictitious force, often called *D'Alembert force* or the inertia force as depicted in Fig. 6.3.

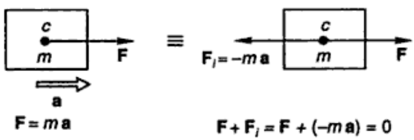


Fig. 6.3 ***Newton's Law = D'Alembert's Principle***

According to the D'Alembert principle, the net external force **F** actually acting on the body and the inertia force $\mathbf{F}_i$ together keep the body in a state of 'fictitious equilibrium'

$$\mathbf{F} + \mathbf{F}_i = 0 \tag{6.2b}$$

The principle tends to give the solution procedure of a dynamic problem an appearance akin to that of a static problem. The rule of equilibrium for statics, i.e.,

$$\Sigma\,\mathbf{F} = 0$$

may, therefore, be employed for a dynamic problem with the introduction of the concept of *fictitious dynamic equilibrium*.

The significance of D'Alembert principle does not end with the extension of our ability of using the methods of statics in dynamics but goes beyond to be coupled with the principle of virtual work and to lead to an alternative formulation of mechanics on the basis of energy considerations. A glimpse of this formulation is given in Chapter 10 under the heading 'Variational Principles'.

Let us consider the motion of a particle along a circular path on a smooth plane with reference to Newton's law and the D'Alembert principle. At any instant of time, the velocity must be tangential to the circular path and the acceleration may consist of a tangential component and a radially inward or centripetal component. If the speed of the body is constant, it experiences only the centripetal acceleration, equal in magnitude to v^2/r. The force **F** that must act on the body to enable the body to move in a circular path must be radially inwards at all times such that

$$\mathbf{F} = m\,\mathbf{a}$$

with a magnitude mv^2/r.

This is so shown for two positions of the body in Fig. 6.4(a). The force **F** actually required to be acted on the body is called the centripetal force.

According to the D'Alembert principle, the equation of motion may be written as

$$\mathbf{F} + (-m\,\mathbf{a}) = 0$$

and the fictitious force ($-m$ **a**) is called the centrifugal force. The equation is interpreted by saying that the body may be considered in a state of 'equilibrium' under the application of two forces: the actual or centripetal force **F** and a fictitious or centrifugal forces *CF* equal to ($-m$ **a**).

The two forces must be equal and opposite. The centripetal force being radially inward, the centrifugal force must be radially outward at any instant. This is so shown for two positions of the body in Fig. 6.4(b).

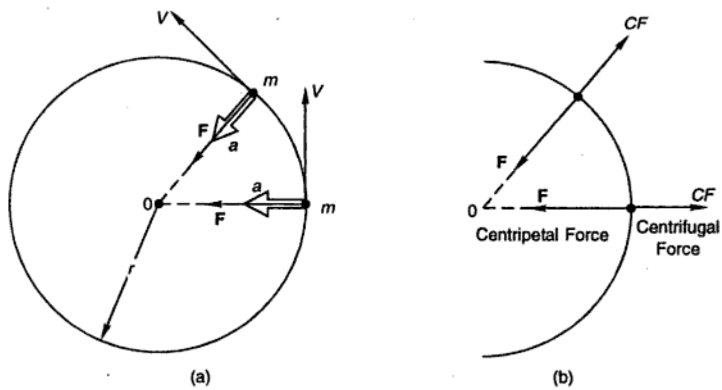


Fig. 6.4 ***Motion Along a Circular Path***

Since the concept of centrifugal force replaces that of the 'acceleration' of the body, the body is considered 'nonaccelerating' or in 'equilibrium' once the centrifugal force is imagined to be acting upon it.

Let us consider the traditional example of a small stone of mass *m* tied at the end of a string of length *l* and whirled at a constant speed *v*. If whirled in a horizontal plane, the stone is subjected to a vertically downward force equal to its weight *mg* and a radially outward horizontal centrifugal force equal to mv^2/r. The force due to the string, also in a horizontal plane, can only balance the centrifugal force leaving the vertical force *mg* unbalanced. The conclusion is that a stone cannot be whirled, by means of a string, keeping the string in a horizontal plane. Instead, the string must be inclined downward, going outward, to provide a vertical component in the string force to balance the weight as shown in Fig. 6.5. The centrifugal force is still horizontal. It may be observed that the angle of inclination θ is given by

$$\tan\theta = \frac{mg}{CF} = \frac{mg}{mv^2/r} = \frac{gr}{v^2} \tag{6.3}$$

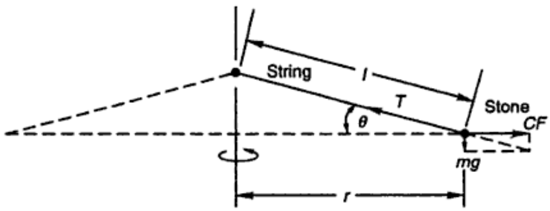


Fig. 6.5 ***Whirling of a Stone in a Horizontal Plane***

The angle is independent of the mass of the stone but increases as the velocity decreases or the radius of the circular path increases.

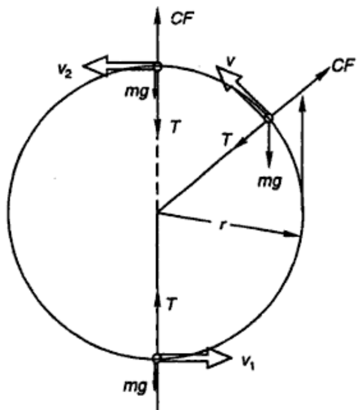


Fig. 6.6 ***Whirling of a Stone in a Vertical Plane***

If the stone is whirled in a vertical plane, the three forces, i.e., its weight *mg* acting downward, the centrifugal force *CF* equal to mv^2/r acting radially outward and the string force T radially inward may be considered to keep it in 'equilibrium' at any instant as shown in Fig. 6.6. At the lowest position,

$$T = CF + mg = \frac{mv^2}{r} + mg$$

whereas at the uppermost position,

$$CF = T + mg$$

whence

$$T = CF - mg = \frac{mv^2}{r} - mg$$

The tension in the string must be positive at the lowest position for all values of v but it can drop to zero at the uppermost position if

$$\frac{mv^2}{r} - mg \le 0$$

or

$$\boxed{v \le \sqrt{gr}} \tag{6.4}$$

Since a string cannot remain straight without being in tension, the stone will not reach the uppermost position on the circle if the velocity of whirling drops below $\sqrt{gr}$.

6.4 WORK, POWER AND ENERGY

If a body is subjected to a force **F** and the point of application of the force is displaced by an infinitesimal displacement $d\mathbf{r}$ as shown in Fig. 6.7(a), the work done by the force is defined as the scalar or dot product of the force and the infinitesimal displacement

$$dW = \mathbf{F} \cdot d\mathbf{r} \tag{6.5}$$

$$= F\, dr \cos\theta$$

where θ is the angle between the force and displacement vectors. In other words, the work done by a force is the product of the magnitude F of force and the distance $dr \cos\theta$ moved by the point of application of the force in the direction of the force. Alternatively, the work done may be considered to be the product of the magnitude dr of the displacement and the component $F \cos\theta$ of the force acting in the direction of the displacement of the point of application of the force. The work done by a force may be positive or negative depending upon whether the force component is directed along or opposite to the direction of displacement, i.e., whether the angle between the force and displacement is acute or obtuse.

For a particle, the force acts at the same point for which the displacement is considered. For a rigid body, on the other hand, the displacement of the point of application of the force must be considered.

For a finite displacement of a particle, therefore, the work done due to a force **F** is given by

$$\boxed{W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}} \tag{6.6}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ refer to the initial and final positions 1 and 2 as shown in Fig. 6.7(a).

The force $\mathbf{F}$ acting on the particle may be a function of the space coordinates and the displacement $d\mathbf{r}$ may also be in space. The elementary work done can, therefore, be written as

$$dW = (F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= F_x\,dx + F_y\,dy + F_z\,dz$$

where F_x, F_y and F_z may vary with the space coordinates. The total work done to displace the particle from state 1 to state 2 is given by

$$W = \int_1^2 dW = \int_1^2 F_x\,dx + \int_1^2 F_y\,dy + \int_1^2 F_z\,dz$$

For a finite displacement of a rigid body, the work done by a force $\mathbf{F}$ acting at the centre of mass C is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}_c$$

where $d\mathbf{r}_c$ refers to the displacement of C as in Fig. 6.7(b). If the force $\mathbf{F}$ acted at some other point P such that C did not fall on the line of action of $\mathbf{F}$ as in Fig. 6.7(c), then the work done by the force $\mathbf{F}$ on the body would be

$$W = \int_{\mathbf{r}_{1p}}^{\mathbf{r}_{2p}} \mathbf{F} \cdot d\mathbf{r}_p$$

where $d\mathbf{r}_p$ refers to the displacement of P. It must be noted that the expression.

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}_c$$

does not mean the work done by the force as it is acting but, instead, the work done by the force $\mathbf{F}$ *as if it acted through the centre of mass C.*

It is interesting to observe the cases where no work is done by a force: (a) when the displacement is zero, e.g., by the force acting on a stationary structure and (b) when the displacement is perpendicular to the force applied, e.g., work by the gravitational force on an object moving horizontally on the surface of the earth.

The rate at which work is done is called *power*, given by

$$\text{Power} = \frac{dW}{dt} \tag{6.7}$$

Since $\quad dW = \mathbf{F} \cdot d\mathbf{r}$

for a force $\mathbf{F}$ whose point of application is displaced by an infinitesimal displacement $d\mathbf{r}$, the power dissipated in a process is referred to the power associated with the dissipative forces.

The kinetic energy possessed by a particle of mass m moving with a velocity $\mathbf{v}$ is defined as

$$KE = \frac{1}{2} mv^2 \tag{6.9}$$

where v is the speed or magnitude of $\mathbf{v}$. The kinetic energy of a rigid body of mass m in translation at a velocity $\mathbf{v}$ is also defined by the same expression but the kinetic energy of rigid body in general translational and rotational motion must be determined from a consideration of the velocity of the individual elements dm.

$$KE = \int_{\text{body}} \frac{1}{2} v^2 \, dm \tag{6.10}$$

where the integration is taken over the entire mass of the body. Similarly, the total kinetic energy for a system of n particles is obtained by the addition of the kinetic energies possessed by the individual particles,

$$KE = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 \tag{6.11}$$

It is also possible to demonstrate that the total kinetic energy of a system can be considered to be composed of two parts: the kinetic energy of the total mass moving with the velocity v_c of the mass centre and the kinetic energy of motion of the elements of the system relative to the mass centre:

$$KE = \frac{1}{2} mv_c^2 + \sum \frac{1}{2} m_i v_{ic}^2$$

It may be remarked that the units of work and energy are the same. In SI units, the unit of work is joule, J ≡ N m and the unit of energy is kg m^2/s^2 ≡ Nm ≡ J.

A force field is said to be conservative if the force $\mathbf{F}$ in the field is continuous in space and is expressible as a gradient of a scalar function ϕ, i.e.,

$$\mathbf{F} = \mathbf{F}(x, y, z)$$

and

$$\mathbf{F} = \mathbf{grad}\, \phi = \nabla \phi$$

or

$$F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

whence

$$F_x = \frac{\partial \phi}{\partial x}, F_y = \frac{\partial \phi}{\partial y}, F_z = \frac{\partial \phi}{\partial z} \tag{6.12}$$

The scalar function or force potential ϕ must be such that its partial derivative with respect to a coordinate results in the force component along that direction.

Alternatively, the condition for $\mathbf{F}$ to be equal to the gradient of ϕ requires that

$$\nabla \times \mathbf{F} \text{ or } \mathbf{curl}\, \mathbf{F} = 0$$

instead of the radial lines. The expression for a change in potential energy is then given by

$$\Delta PE = mgh$$

as is commonly used for earth-bound objects.

***Example* 6.13** A vertical lift of total mass 500 kg acquires an upward velocity of 2 m/s over a distance of 3 m of motion with constant acceleration, starting from rest. Calculate the tension in the cable supporting the lift.

If the lift, while stopping, moves with a constant deceleration and comes to rest in 2 s, calculate the force transmitted by a man of mass 75 kg on the floor of the lift during that interval.

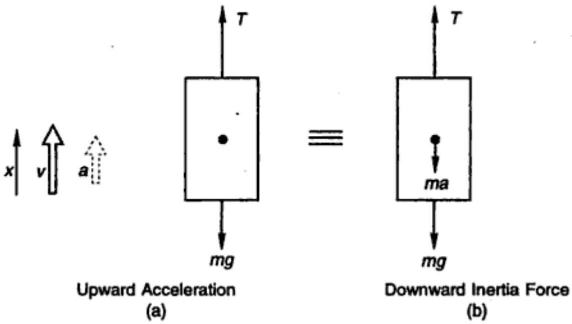


Fig. Ex. 6.13 (a) and (b) (*Solution*)

Solution The upward acceleration a is obtained by using

$$V_2^2 - V_1^2 = 2as$$

$$a = \frac{2^2}{2 \times 3} = 0.67 \text{ m/s}^2$$

Examining the forces acting on the lift as shown in Fig. Ex. 6.13(a) (Solution), the net force must equal mass times acceleration in accordance with the Newton's law,

$$T - mg = ma$$

whence

$$T = m(g + a)$$

$$= 500(9.81 + 0.67)$$

$$= 5240 \text{ N}$$

Alternately the D'Alembert principle suggests that an inertia force equal to $(-ma)$, i.e., ma downward be imagined acting on the lift together with the external force T and mg as shown in Fig. Ex. 6.13(b) (Solution) and the problem be solved as an equilibrium problem in statics. By this principle,

$$\Sigma \mathbf{F} = 0$$

$$T - mg + (-ma) = 0$$

or $$T - 500 \times 9.81 - 500 \times 0.67 = 0$$

whence $$T = 5240 \text{ N}$$

While stopping, the acceleration of the lift is

$$a = \frac{0-2}{2} = -1 \text{ m/s}^2$$

The negative sign of acceleration implies that the acceleration is opposed to the direction of velocity, as shown in Fig. Ex. 6.13(c) (Solution).

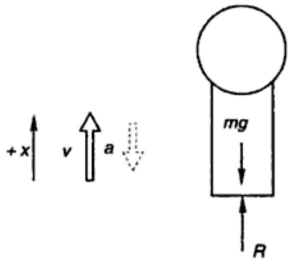
The force transmitted by the man on the floor on the lift equals the reaction R exerted by the lift on the man. With reference to the free-body diagram,

$$R - m\,g = m\,a$$

$$R = m\,a + m\,g$$

$$= 75\,(-1 + 9.81)$$

$$= 661 \text{ N}$$



(c) Man being decelerated

Fig. Ex. 6.13 (c) *(Solution)*

The force transmitted by the man on the floor of the lift is, therefore, 661 *N* downwards.

It is interesting to understand that the man in the lift experiences the acceleration imposed upon him and, consequently, experiences a change in his weight *W*. The reaction *R* exerted by the lift on the man is equal and opposite to the weight *W* felt by him. The weight felt by a man is less than *mg* while accelerating downwards and more than *mg* while accelerating upwards. In particular, if the lift was to be accelerated downwards at an acceleration *g*, the man would feel weightless and the reaction by the floor of the lift on the man would become zero. The weight felt by a man in a lift moving at a constant velocity would be *mg*. The pictorial representation of the weight felt by him in a lift during a round trip is shown in Fig. Ex. 6.13(d) (Solution).

***Example* 6.14** A segment of a smooth circular curved road of radius 30 m is located in a zone of 40 km/h speed limit. What should be the angle of banking so that a vehicle may travel on it without any outward side thrust?

If the coefficient of static friction between the road and tyres of a vehicle was 0.3, calculate the maximum possible speed before the vehicle experiences a side-slip.

Solution When a vehicle negotiates a circular curve, it may be considered in 'equilibrium' under the action of three forces, namely the weight *mg*, centrifugal force *CF* equal to mv^2/r and reaction *R* which is normal to the road in the absence of friction.

Resolving the forces along the incline and writing for equilibrium,

$$mg \sin\theta = F\cos\theta = \frac{mv^2}{r}\cos\theta$$

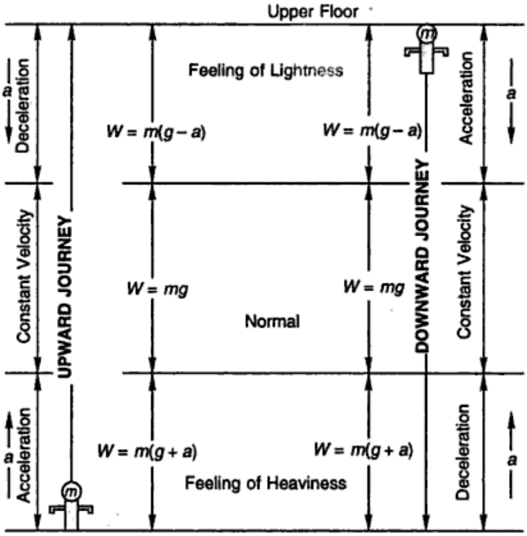


Fig. Ex. 6.13(d) ***(Solution) Weight Felt in a Lift***

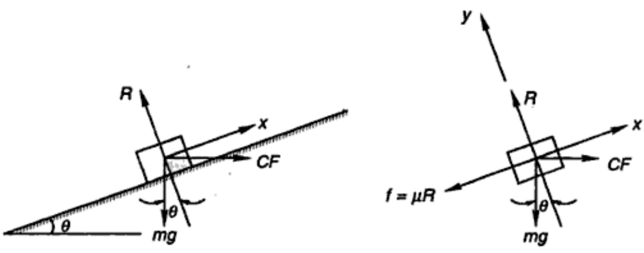


Fig. Ex. 6.14 (a and b) ***(Solution)***

whence $\tan\theta = \dfrac{v^2}{gr}$

For the case in hand,

$$\tan\theta = \frac{(40 \times 1000/3600)^2}{9.81 \times 30} = 0.42$$

and $\theta = 22.76°$

In the presence of frictional force which would act down the plane if the velocity exceeds 40 km/h, the limiting velocity may be worked out with reference to the free-body diagram drawn in Fig. Ex. 6.14(b) (Solution).

Along the x-axis,

$$CF \cos\theta - \mu R - mg \sin\theta = 0$$

or
$$\frac{mv^2}{r}\cos\theta - \mu R - mg\sin\theta = 0 \tag{i}$$

and along the y-axis,

$$R - mg\cos\theta - CF\sin\theta = 0$$

or
$$R - mg \text{ co } \theta - \frac{mv^2}{r}\sin\theta = 0 \tag{ii}$$

From Eq. (ii)

$$R = mg\cos\theta + \frac{mv^2}{r}\sin\theta$$

Substituting in Eq. (i) and simplifying it,

$$v^2 = \frac{\mu + \tan\theta}{1 - \mu\tan\theta}\cdot gr \tag{iii}$$

For the case in hand,

$$v^2 = \frac{0.3 + \tan 22.76^\circ}{1 - 0.3\tan 22.76^\circ}\times 9.81 \times 30$$

whence
$$v = 15.565 \text{ m/s}$$

$$= 15.565 \times \frac{3600}{1000} = 56 \text{ km/h}$$

It is interesting to note that the angle of banking for a circular road calculated on the basis of no-friction is safer because the presence of friction allows even higher maximum velocity without any danger of side-slip. In other words the angle of banking calculated by taking friction into account would have been less than 22.76° as may be seen by substituting the given values in Eq. (iii).

***Example* 6.15** A small spherical object comes rolling down a ramp and leaves the edge horizontally with a velocity V_0. Describe its trajectory.

Solution The spherical object would continue to have its rolling motion at constant angular velocity, in the absence of air resistance, on leaving the ramp.

At any instant of time t,

$$x = V_0 t \qquad z = -1/2\, gt^2$$

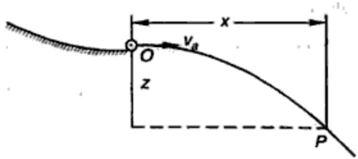


Fig. Ex. 6.15

***Example* 6.17** A particle of mass 0.05 kg slides down the circular periphery of a horizontal smooth cylinder of radius 12 cm when let go from the top position, as shown in Fig. Ex. 6.17. Determine the reaction of the cylinder on it when it reaches 30° degree position.

Will it leave the surface? If so, where and with what velocity?

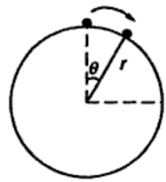


Fig. Ex. 6.17

Solution From the freebody diagram of the mass as shown in Fig. Ex. 6.17 (Solution) at an angle θ,

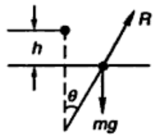


Fig. Ex. 6.17 (*Solution*)

$$mg\cos\theta - R = m\frac{v^2}{r} \qquad \text{(i)}$$

Also, by conservation of energy,

$$mgh = mg\,(r - r\cos\theta) = \frac{1}{2}mv^2$$

or $$g(1-\cos\theta) = \frac{1}{2}v^2/r \qquad \text{(ii)}$$

Eliminating v^2/r between (i) and (ii)

$$\begin{aligned} R &= mg\cos\theta - 2\,mg\,(1-\cos\theta) \\ &= 0.05\times 9.81\times 0.866 - 2\times 0.05\times 9.81\,(1-0.866) \\ &= 0.29\text{ N.} \end{aligned}$$

It must leave the circular periphery, latest when it reaches the extremity of the horizontal diameter when the velocity is zero or the mass is zero. With its finite mass and velocity it will lose contact with the cylinder where $R = 0$, i.e..
Equating $mg\cos\theta$ with $m\cdot 2g\cdot(1-\cos\theta)$

$$\begin{aligned} \cos\theta &= 2\,(1-\cos\theta) \\ \cos\theta &= 2 - 2\cos\theta \\ 3\cos\theta &= 2 \\ \theta &= \cos^{-1} 2/3 = 48.2° \end{aligned}$$

At this location,

$$\begin{aligned} v^2 &= 2\,gr\,(1-\cos\theta) \\ &= 2\times 9.81\times 0.12\,(1 - 2/3) = 0.78 \end{aligned}$$

whence, $$v = 0.886\text{ m/s}$$

The direction of the velocity must be tangent to the circle at that point, i.e., at (– 48.2°) with the horizontal direction.

***Example* 6.18** Two blocks m_1 and m_2, of masses 2 and 4 kg respectively are placed one on the other on a horizontal table and connected to a suspended weight

M through a frictionless pulley as shown in Fig. Ex. 6.18. The coefficient of static friction between m_1 and m_2 is 0.4 and the coefficient of kinetic friction between m_2 and the table is 0.2. Find the maximum mass of the block M in order that m_2 accelerates over the table without m_1 slipping over m_2.

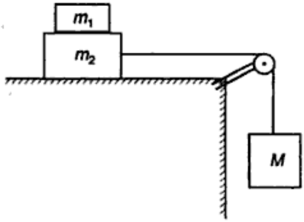


Fig. Ex. 6.18

Solution For the free body diagram of mass m_1 as shown in Fig. Ex. 6.18(a) (Solution), the frictional force is 0.4 × 2 × 9.81, i.e., 7.85 N. The maximum acceleration of m_1 should be limited to a, such that

$$m_1 \times a = 7.85; \; a = \frac{7.85}{2} = 3.925 \text{ m/s}^2$$

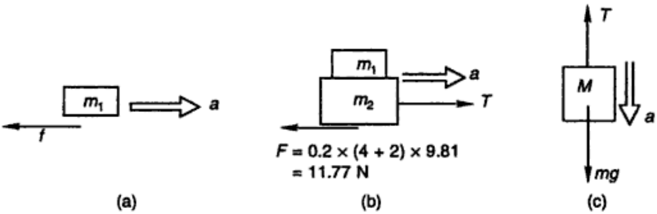


Fig. Ex. 6.18 (a, b and c) *(Solution)*

Now, let us draw the free body diagram of the two masses m_1 and m_2 together as also for the suspended mass M (ref. Fig. Ex. 6.18(b) (Solution)).

By dynamical equations,

$$T - 11.77 = (2 + 4) \times 4; \qquad T = 35.77 \text{ N}$$

$$Mg - T = M \times 4; \; M = 35.77/(9.81 - 4) = 6.15 \text{ kg}$$

***Example* 6.19** A bob of mass m = 20 grams is attached to a 20 cm long string tied to the apex of a cone with rough surface as shown in Fig. Ex. 6.19. At time t = 0, the cone is imparted a constant angular acceleration $\alpha = 0.5$ rad/s^2 about its vertical axis. Assuming that the bob has no relative motion w.r.t. the surface of the cone,

(i) at what time will the bob leave contact with the surface?

(ii) what would be tension in the string at that instant?

Solution Consider the free body diagram of the bob at any instant in rotation as shown in Fig. Ex. 6.19 (Solution).

$$T \cos 30 - R \cos 60° = mr\,\omega^2$$

$$T \sin 30° + R \sin 60° = mg$$

whence, $$\frac{\sqrt{3}}{2} T - \frac{R}{2} = m \times 0.173\,\omega^2$$

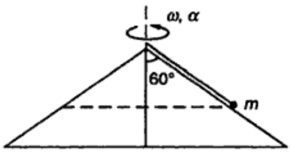


Fig. Ex. 6.19

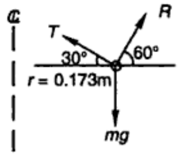


Fig. Ex. 6.19 *(Solution)*

and $$\frac{T}{2} - \frac{\sqrt{3}}{2}R = m \times 9.81$$

which leads to $$R = (0.173\ \omega^2 - \sqrt{3} \times 9.81)$$

It leaves contact when $R = 0$; $\omega = 9.9$ rad/s

(i) From the relationship,

$$\omega = \omega_0 + \alpha t$$

$$t = 9.9/0.5 = 19.8 \text{ seconds}$$

(ii) Solving for T from the above,

$$T = 0.39 \text{ N}$$

***Example* 6.20** Two masses m_1 and m_2 are connected by a massless inextensible string which passes over a massless and frictionless pulley Fig. Ex. 6.20. This constitutes an *Atwood's machine*. Find the acceleration of mass m_1 and the tension in the string as the system moves under gravity.

Solution Considering the free-body diagram of each of the masses (Fig. Ex. 6.20),

$$m_1 g - T = m_1 a$$

$$T - m_2 g = m_2 a$$

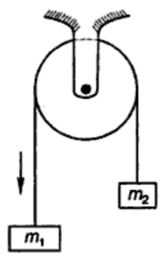


Fig. Ex. 6.20

From these equations,

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

and $$T = \frac{(2m_1\ m_2)}{m_1 + m_2} g$$

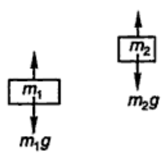


Fig. Ex. 6.20 *(Solution)*

Atwood used this system to determine the value of g, the acceleration due to gravity. If $m_1 = 2m_2$, which meant $a = g/3$, i.e., the system accelerates at one third the value of g. He used different values of m_1 and m_2 and measured the resulting acceleration whence he computed the value of g;

$$g = \frac{m_1 + m_2}{m_1 - m_2} a.$$

One can get the same result by employing the work-energy principle or the conservation-of-energy principle in this case. Let us apply the latter.

If the mass m_1 moves down by x the resulting velocity would be given by considering

$$PE + KE = \text{Const.}$$

$$(m_1 - m_2)g\,x = \frac{1}{2}(m_1 + m_2)\,v^2$$

or

$$(m_1 - m_2)g\,x = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2$$

Differentiating it w.r.t. time,

$$(m_1 - m_2)g\dot{x} = (m_1 + m_2)\,\ddot{x}\,\dot{x}$$

and dividing by $\dot{x}$, which in general is non-zero,

$$(m_1 - m_2)g = (m_1 + m_2)\,a$$

whence

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

***Example* 6.21** Establish that the gravitational force field close to the surface of the earth and the force field due to a linear spring are conservative force fields.

Solution

(a) Gravitational force field

Consider a particle of mass m placed in a parallel gravitational force field close to the surface of the earth is shown in Fig. Ex. 6.21(a).

$$F_x = 0,\ F_y = 0$$

$$F_z = -mg, \text{ the weight of the particle}$$

A potential function ϕ should exist for a conservative force field. For this case,

$$F_x = \frac{\partial\phi}{\partial x} = 0 \qquad \text{(i)}$$

$$F_y = \frac{\partial\phi}{\partial y} = 0 \qquad \text{(ii)}$$

$$F_z = \frac{\partial\phi}{\partial z} = -mg \qquad \text{(iii)}$$

From Eqs. (i) and (ii) it may be seen that the potential function is not a function of x or y. Therefore integrating Eq. (iii),

$$\phi = -mgz + \text{Constant}$$

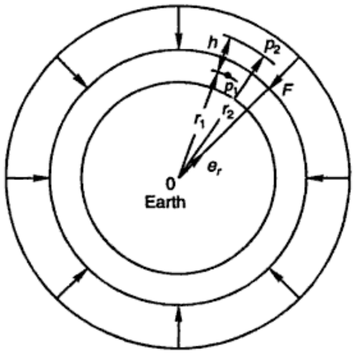


Fig. Ex. 6.21(a)

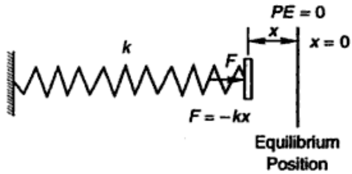


Fig. Ex. 6.21(b)

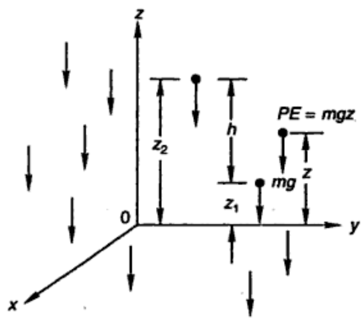


Fig. Ex. 6.21 *(Solution)*

The existence of ϕ is a guarantee for the field to be conservative. The potential energy for the gravitational force field is

$$PE = -\phi = mgz$$

it being reckoned zero at the origin of the coordinates arbitrarily.

(b) Force field of a linear spring

For a linear spring compressed by x, the force field is such that

$$F_x = -kx$$

$$F_y = 0, F_z = 0$$

In an effort to determine the potential function, let

$$F_x = \frac{\partial \phi}{\partial x} = -kx \tag{i}$$

3. In certain situations, where the work done may be evaluated readily, the velocities may be computed by employing the work-energy equation.

6.6 CONSERVATION OF MECHANICAL ENERGY

Let us now examine the case where the force is conservative. In a conservative force field, the work done in moving a particle or a mass centre from position 1 to 2 depends upon the potential energy *PE* of the end states only,

$$W = PE_1 - PE_2$$

Equating this expression for the work done with that provided by the work-energy equation,

$$PE_1 - PE_2 = KE_2 - KE_1$$

whence $$PE_1 + KE_1 = PE_2 + KE_2$$

or $$\boxed{(PE + KE)_1 = (PE + KE)_2} \qquad (6.16)$$

which implies that in a conservative force field, the sum of the potential energy and kinetic energy remains constant for all positions of a particle or a mass centre. The sum of the potential energy and kinetic energy is called mechanical energy. Equation (6.16) is referred to as the *Principle of conservation of mechanical energy.* In other words, *the mechanical energy of a mass centre or a particle is conserved when it moves in a conservative force field*, as also visualised in Fig. 6.9.

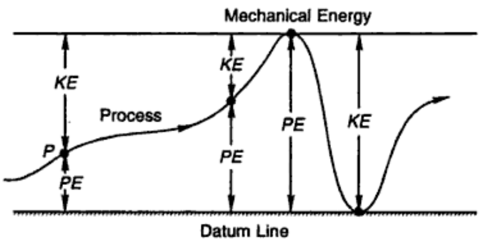


Fig. 6.9 ***Visualisation of the Motion with Conservation of Mechanical Energy (PE + KE = Const.)***

If a particle is, instead, made to move in a force field, a part of which is conservative and another part non-conservative, then

$$\mathbf{F} = \mathbf{F}_{\text{cons.}} + \mathbf{F}_{\text{non-cons.}}$$

and a potential energy function *PE* may be determined for the conservative part of the force $\mathbf{F}_{\text{cons}}$. The work-energy principle provides that

$$W = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = KE_2 - KE_1$$

The kinetic energy at each point is expressed as follows:

$$KE_A = 0$$

$$KE_B = \frac{1}{2}mv_B^2 = \frac{1}{2} \times 0.5 v_B^2 = 0.25 v_B^2$$

$$KE_C = \frac{1}{2}mv_C^2 = \frac{1}{2} \times 0.5 v_C^2 = 0.25 v_C^2$$

Employing the principle of conservation of mechanical energy,

$$(PE + KE) = \text{Constant}$$

$$50 + 0 = -1.24 + 0.25v_B{}^2 = -9.81 + 0.25v_C{}^2$$

whence $v_B = 14.32$ m/s

and $v_C = 15.47$ m/s

Let us consider a point D where the slider would come to rest again

$$PE_D = -2 \times 9.81 \times 0.5 + \frac{1}{2} \times 400 \times x^2$$

$$KE_D = 0$$

Again, by conserving mechanical energy,

$$50 + 0 = -2 \times 9.81 \times 0.5 + \frac{1}{2} \times 400 \times x^2$$

$$x = 0.55 \text{ m}$$

The stretch of the spring being 0.55 m, its stretched length should be 0.55 + 0.5 = 1.05 m which suggests that

$$CD = (1.05^2 - 0.5^2)^{1/2} = 0.92 \text{ m}$$

***Example* 6.24** A block of mass 5 kg is released from rest from a position A on a 30° incline as shown in Fig. Ex. 6.24. Determine the maximum compression of the spring if the spring constant is 8 N/cm and the coefficient of friction between the block and the incline is 0.2.

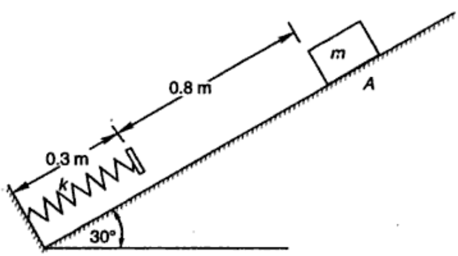


Fig. Ex. 6.24

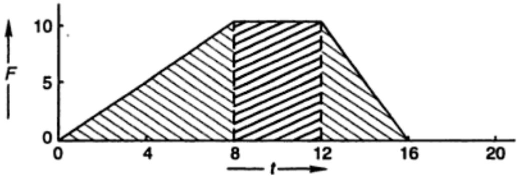


Fig. 6.10 ***Force vs Time Curve***

$$\text{Magnitude of } \mathbf{I} = \frac{10 \times 8}{2} = 10 \times (12-8) + \frac{10 \times (16-12)}{2} + 0$$

$$= 40 + 40 + 20 + 0 = 100 \text{ Ns}$$

computed as the area under the F-t curve.

The concept of impulse is particularly useful if large forces act over short intervals of time. The action may then be classified as impulsive and expressed in terms of an impulse **I** in its own right without regard to the force associated with it.

The impulse of a force is often called linear impulse in order to differentiate it from angular impulse which may arise due to a moment acting over a period of time.

6.8 IMPULSE-MOMENTUM PRINCIPLE

Newton's law for a particle or a mass centre provides that

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

which may be rewritten as

$$\mathbf{F}\,dt = m\,d\mathbf{v}$$

Integrating each side from an initial position at t_1 when the velocity is $\mathbf{v}_1$ to final position at t_2 when the velocity is $\mathbf{v}_2$

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = \int_{v_1}^{v_2} m\,d\mathbf{v} = m(\mathbf{v}_2 - \mathbf{v}_1) \tag{6.19}$$

In words,

Impulse = Change in momentum

or Initial momentum + Impulse = Final momentum

This is a vector equation which applies to a particle or mass centre of a rigid body or of a system of bodies. It is the velocity $\mathbf{v}_c$ of the mass centre that must be considered to evaluate the linear momentum of a rigid body at any instant, *no matter where the force actually acts on the body.*

The left hand side of Eq. (6.19) can be rewritten as

$$\int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{i}\int_{t_1}^{t_2} F_x\, dt + \mathbf{j}\int_{t_1}^{t_2} F_y\, dt + \mathbf{k}\int_{t_1}^{t_2} F_z\, dt$$

and similarly the right hand side can also be written as

$$= m(v_{2x} - v_{1x})\,\mathbf{i} + m(v_{2y} - v_{1y})\,\mathbf{j} + m(v_{2x} - v_{1z})\,\mathbf{k}$$

Equating the left hand and right hand sides of Eq. (6.19), we get

$$\int_{t_1}^{t_2} F_x\, dt = m(V_{2x} - V_{1x})$$

$$\int_{t_1}^{t_2} F_y\, dt = m(V_{2y} - V_{1y})$$

$$\int_{t_1}^{t_2} F_z\, dt = m(V_{2z} - V_{1z})$$

6.9 CONSERVATION OF MOMENTUM

It may be observed from the impulse-momentum principle that the momentum of a particle or of the centre of mass of a system is conserved in the absence of an external impulse or an external force acting on it. Then,

Initial momentum = Final momentum

***Example* 6.25** A force given by

$$\mathbf{F} = 3t^2\,\mathbf{i} + 5t\,\mathbf{j} - (8t^3 + 400)\,\mathbf{k}\ \text{N}$$

acts from $t = 0$ to $t = 10$ s. Determine the impulse of the force. If this impulse acted at the centre of mass of a body of mass 500 kg and brought it to rest, estimate the velocity of the body before it acted.

Solution

$$\mathbf{I} = \int \mathbf{F}\, dt$$

$$= \int_0^{10} 3t^2\, dt\,\mathbf{i} + \int_0^{10} 5t\, dt\,\mathbf{j} - \int_0^{10} (8t^3 + 400)\, dt\,\mathbf{k}$$

$$= \left| t^3 \right|_0^{10} \mathbf{i} + \left| 2.5t^2 \right|_0^{10} \mathbf{j} - \left| 2t^4 + 400t \right|_0^{10} \mathbf{k}$$

$$= 1000\,\mathbf{i} + 250\,\mathbf{j} - 24{,}000\,\mathbf{k}\ \text{Ns}$$

Applying the impulse-momentum principle in the vertical direction,

$$(10 \times 600 \times \sin 30 - 0) = F \times 10/1000$$

whence $$F = 300{,}000 \text{ N} = 300 \text{ kN}.$$

This must also be reaction by the ground on the frame in the vertically upward direction.

***Example* 6.28** A 0.001 kg bullet has a velocity of 1000 m/s as it enters a fixed block of wood. It comes to rest 0.002 s after entering the block. Determine the average force that acted on the bullet and the distance penetrated by it.

Solution From the impulse-momentum principle,

$$\mathbf{I} = \int_1^2 \mathbf{F}\, dt = F_{av} \cdot \Delta t = 0.002\, F_{av}$$

$$= m(\mathbf{v}_2 - \mathbf{v}_1) = 0.001(0 - 1000) = -1 \text{ N}$$

whence, $$F_{av} = -1/0.002 = -500 \text{ N}$$

which implies that a resistive force of 500 N acts opposite to the direction of motion of the bullet.

The distance penetrated by the bullet into the block may be worked out quite simply by employing the work energy principle or by using Newton's law of motion.

By the work-energy principle,

$$\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} m (v_2^2 - v_1^2)$$

or $$500 \times \text{distance penetrated} = \frac{1}{2} \times 0.001 \times (0^2 - 1000^2)$$

whence the distance penetrated = 1.0 m

Alternatively, the acceleration of the bullet should be

$$\mathbf{a} = \mathbf{F}/m = (\mathbf{v}_2 - \mathbf{v}_1)/\Delta t$$

$$= \frac{0 - 1000}{0.002} = -500{,}000 \text{ m/s}^2$$

and the distance penetrated is given by

$$v_2^2 - v_1^2 = 2a\, s$$

whence $$s = \frac{0 - 1000^2}{2 \times 500\,000} = 1.0 \text{ m}$$

***Example* 6.29** A person of mass 60 kg stands at one end of a 6 m long floating boat of mass 240 kg. If the person walks across to the other end at a steady rate of

1.2 m/s, determine (a) the velocity of the boat as observed by an observer on the ground during the process, (b) the distance by which the boat is shifted, (c) the velocity of the boat if the person stops at the other end and (d) the velocity of the boat if the person, while walking, falls out of the boat at the other end.

Solution Considering the person (1) and the boat (2) as a single system, there is no external force on it. The momentum of the person and the boat taken together must, therefore be conserved. Distances and velocities are referred positive to the right as shown in Fig. Ex. 6.29 (Solution).

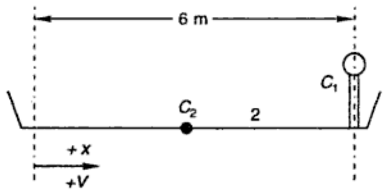


Fig. Ex. 6.29 *(Solution)*

(a) Initially, the person and the boat are at rest; their total momentum is zero. As the person moves with a velocity of 1.2 m/s to the right with respect to the boat, the boat may be moving with a velocity V_2 m/s. The absolute velocity of the person is, therefore,

$$V_1 = -1.2 + V_2 \text{ m/s}$$

Momentum of the system in the process is

$$m_1V_1 + m_2V_2 = 60(-1.2 + V_2) + 240\ V_2$$
$$= 300\ V_2 - 72$$

which must be equal to the initial momentum, i.e., zero

$$300\ V_2 - 72 = 0$$

whence $$V_2 = 0.24 \text{ m/s}$$

and $$V_1 = -1.2 + 0.24 = -0.96 \text{ m/s}$$

An observer on the ground will, therefore, observe the boat to be drifting to the right with a velocity of 0.24 m/s and the person moving to the left with a velocity of 0.96 m/s.

(b) The time taken by the person to walk across the other end is

$$t = 6/1.2 = 5 \text{ s}$$

and the distance the boat travels in 5 seconds is

$$x_2 = 5 \times 0.24 = 1.2 \text{ m}$$

(c) If the person stops at the other end, the boat should also stop because their total momentum must remain zero.

(d) If the person, while walking falls out of the boat at the other end, he would take with him a momentum

$$m_1V_1 = 60(-1.2 + 0.24) = -57.6 \text{ Ns}$$

The momentum possessed by the boat is, therefore, +57.6 Ns because the sum of the momenta equals zero all the time. The boat must have a velocity V_2 after the person fell out

$$V_2 = \frac{57.6}{240} = 0.24 \text{ m/s}$$

In the absence of any external force, the boat will thus continue to move to the right at 0.24 m/s.

An alternative approach to the solution of the first two parts of the problem would be as follows:

The person and the boat taken together do not experience any external force. Therefore, the centre of mass of the system must remain unaltered.

Initially $$x_c = \frac{60 \times 6 + 240 \times 3}{60 + 240} = 3.6 \text{ m}$$

Finally, after the person has moved to be other end,

$$3.6 = \frac{60 \times (0 + x_2) + 240(3 + x_2)}{60 + 240}$$

$$= \frac{300x_2 + 720}{300}$$

whence, $$x_2 = 1.2 \text{ m}$$

which shows that the boat must have moved by 1.2 m to the right.

The velocity of the boat during the process is determined from the knowledge of the time taken by the person and the displacement of the boat

$$V_2 = \frac{x_2}{t} = \frac{1.2}{6/1.2} = 0.24 \text{ m/s}$$

***Example* 6.30** A military truck of mass 4000 kg while developing a tractive force of 12 kN tows a jeep of mass 2000 kg with the help of an inextensible cable up an incline of 1 in 10. A winch mounted on the jeep is operated to approach the truck with a constant acceleration of 0.5 m/s^2. Before the winch is operated, the truck and jeep were travelling at 12 m/s each. Determine their speed after a lapse of 10 s (refer Fig. Ex. 6.30).

What would the final velocities be if the winch did not operate?

Solution This problem is best solved by employing the impulse-momentum equation. This preference is in view of the obvious requirement to deal with the action of forces over a prescribed time and to determine the velocities as a result of change of momentum.

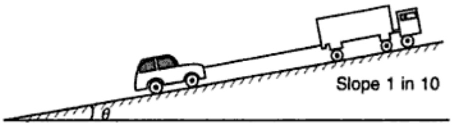


Fig. Ex. 6.30

Taking the truck and jeep together as a single system, the magnitude of the external force acting on it is

$$F = 12 - (4000 + 2000) \times 9.81 \sin \theta/1000$$
$$= 12 - (4 + 2) \times 9.81 \times 1/10 = 12 - 3.93 - 1.96$$
$$= 6.11 \text{ kN}$$

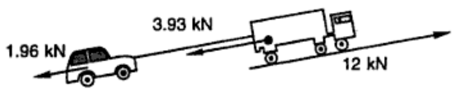


Fig. Ex. 6.30 *(Solution)*

The magnitude of the impulse acting on the system is

$$I = F \times 10 = 6.11 \times 10 = 61.1 \text{ kNs}$$

Initial momentum of the system is

$$p_1 = (4000 \times 12 + 2000 \times 12)/1000$$
$$= 72 \text{ kNs}$$

Denoting the final velocity of the truck by V_t and that of the jeep by V_j where

$$V_j = V_t + 0.5t$$

the final momentum of the system is

$$p_2 = (4000 \times V_t + 2000 \times V_j)/1000$$
$$= (4V_t + 2(V_t + 0.5 \times 10))$$
$$= 6V_t + 10 \text{ kNs}$$

Since

$$\text{Impulse} = \text{Change in momentum}$$
$$61.1 = 6V_t + 10 - 72$$

whence $V_t = 20.52$ m/s

and $V_j = 20.52 + 0.5 \times 10 = 25.52$ m/s

In the absence of a winch, the final momentum of the system would have been given by

$$p_2 = (4000 \times V_t + 2000 \times V_j)/1000$$
$$= 4V_t + 2V_j = 6V$$

and $$v_2 - v_1 = \frac{F}{m}(t_2 - t_1)$$

Taking time $t_1 = 0$ at the beginning when $v_1 = 0$

$$v_2 = \frac{168}{20} \times 5 = 42 \text{ m/s}$$

Alternatively, from

$$v = u + at$$

$$v = 0 + 8.4 \times 5 = 42 \text{ m/s}$$

(c) From the Newton's law,

$$F = m\frac{d^2 s}{dt^2}$$

$$s = \frac{F}{m} t^2 /2 + C_1 t + C_2$$

Noting that both C_1 and C_2 are zero for the given conditions,

$$s = \frac{168}{20} \times 5^2 /2 = 105 \text{ m}$$

Alternatively, from

$$s = ut + \frac{1}{2} at^2$$

$$s = 0 + \frac{1}{2} \times 8.4 \times 5^2 = 105 \text{ m}$$

(d) The kinetic energy is given by

$$\frac{1}{2}mv^2 = \frac{1}{2} \times 20 \times 42^2 = 17{,}640 \text{ J}$$

(e) The work done on the body is

$$\mathbf{F} \cdot \mathbf{s} = 168 \times 105 = 17{,}640 \text{ J}$$

which is indeed equal to the change in kinetic energy of the body.
(f) The momentum possessed is

$$m\,\mathbf{v} = 20 \times 42 = 840 \text{ kg m/s or Ns}$$

(g) The impulse imparted to the body is

$$\mathbf{I} = \mathbf{F}dt = 168 \times 5 = 840 \text{ Ns}$$

which is indeed equal to the change in momentum of the body.

For the second phase of the motion, i.e., along a horizontal surface, the force along the direction of motion is

$$-f = -\mu N = -0.2 \times 196.2 = -39.2 \text{ N}$$

The work done on the body is

$$-39.2 \times \text{distance travelled}$$

Equating it to the change in kinetic energy, −17,640 J

the distance travelled is 17,640/39.2 = 450 m

The impulse on the body is

$$-39.2 \times \text{time of action}$$

Equating it to the change in momentum,

$$0 - 840 = -840 \text{ Ns}$$

the time of travel is 840/39.2 = 21.43 s

Alternatively, the acceleration of the body can be determined as

$$a = -39.2/20 = -1.96 \text{ m/s}^2$$

and from the Newton's law in the form

$$F = m\frac{dv}{dt}$$

the time can be determined and from the form

$$F = m\frac{d^2 s}{dt^2}$$

the distance travelled can be calculated.

Alternatively, employing the relations

$$v = u + at \qquad \text{and} \qquad s = ut + 1/2\, at^2$$

$$t = \frac{0-42}{-1.96} = 21.43 \text{ s}$$

and $$s = 42 \times 21.43 + 1/2 \times (-1.96) \times 21.43^2 = 450 \text{ m}$$

6.10 MOMENT OF MOMENTUM

The term 'moment of momentum' is defined analogous to the moment of a force. The momentum possessed by a body of mass m moving with a linear velocity **v** being $m\mathbf{v}$, the definition for the moment of momentum **H** is that

$$\mathbf{H} = r \times m\mathbf{v}$$

where **r** is the position vector of the centre of mass of the body at a given instant with respect to a reference point.

The term 'angular momentum' is used interchangeably with moment of momentum. This term is possibly designed to denote its difference from the linear momentum $m\,\mathbf{v}$, also written simply as momentum. The physical interpretation of Moment of Momentum is not readily clear from its definition but the meaning can be appreciated in the following article, where the rate of change of moment of momentum is related to the external moment about the same reference point for the body.

6.11 MOMENT OF MOMENTUM EQUATION

Starting with the Newton's law in the form

$$\mathbf{F} = \frac{d}{dt}(m\,\mathbf{v})$$

and taking moment of each term of the equation about a point O in space as shown in Fig. 6.11.

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d}{dt}(m\,\mathbf{v})$$

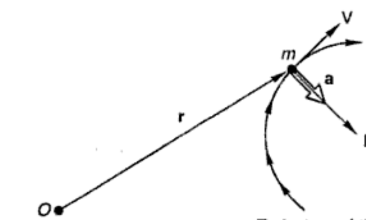


Fig. 6.11 ***Trajectory of the Particle***

The addition of a zero term, namely

$$\frac{d\,\mathbf{r}}{dt} \times m\,\mathbf{v} = 0$$

on the right hand side of the above equation yields

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d}{dt}(m\,\mathbf{v}) + \frac{d\,\mathbf{r}}{dt} \times m\,\mathbf{v}$$

or
$$\mathbf{M}_0 = \mathbf{r} \times \mathbf{F} = \frac{d}{dt}(\mathbf{r} \times m\,\mathbf{v}) \tag{6.20}$$

or
$$\boxed{\mathbf{M}_0 = \frac{d\,\mathbf{H}_0}{dt} = \dot{\mathbf{H}}_0}$$

This is the moment of momentum equation which states that the moment of the resultant force on a particle or at the mass centre of a system about a fixed point O equals the time rate of change of moment of momentum about the same point.

6.12 CONSERVATION OF MOMENT OF MOMENTUM

If the moment of the resultant force acting on a particle or at the centre of mass of a system about a fixed point in space is zero, then

$$\frac{d}{dt}(\mathbf{r} \times m\,\mathbf{v}) = 0$$

or

$$\mathbf{r} \times m\,\mathbf{v} = \text{Const.} = \mathbf{H} \tag{6.21}$$

i.e., **H**, the moment of momentum is conserved. The moment of the resultant force would be zero if either the force is zero or the force is directed towards or away from the origin *O*.

***Example* 6.32** A particle of mass 2 kg tied at the end of an inextensible string is rotated at 20 rad/s along a circle of 1 m radius over a smooth horizontal table top. The string is pulled down through a slot at the centre of the table top at a speed of 5 m/s as shown in Fig. Ex. 6.32. Calculate the speed of the particle when it reaches 0.5 m from the centre. Comment on the variation of tension in the string with time.

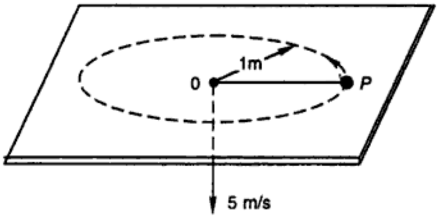


Fig. Ex. 6.32

Solution As the particle is rotated over the table and the string is pulled in, the only force that acts on the particle is the tension in the string which is always radial. Consequently, the fact that

$$\mathbf{r} \times \mathbf{F} = 0$$

prompts the utilisation of the principle of conservation of moment of momentum, i.e.,

$$\mathbf{r} \times m\mathbf{v} = \text{Const.} = \mathbf{H}$$

Since the motion is confined to a plane, the velocity at any instant of time must be given by

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta$$

and

$$\mathbf{r} \times m\mathbf{v} = \mathbf{r} \times m(v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta)$$

$$= \mathbf{r} \times mv_\theta\mathbf{e}_\theta = \mathbf{H}$$

or

$$h = H/m = r\,v_\theta$$

$$= r^2\omega = \text{Constant} \tag{i}$$

Initially, $h_1 = 1^2 \cdot 20 = 20 \text{ m}^2/\text{s}$

and finally, $h_2 = 0.5^2 \cdot \omega_2 = \omega/4 \text{ m}^2/\text{s}$

From the conservation principle,

$$h_1 = h_2$$

$$20 = \omega/4, \; \omega_2 = 80 \text{ rad/s}$$

and $v_{\theta 2} = 0.5 \times 80 = 40 \text{ m/s}$

Since $v_{r2} = 5 \text{ m/s}$

the speed of the particle is

$$v = \sqrt{40^2 + 5^2} = 40.3 \text{ m/s}$$

The tension in the string may be estimated by applying the Newton's law to the particle. There is no force and no acceleration in the tangential direction.

In the radial direction,

$$-F = 2(-r\omega^2)$$

or $$F = 2r\omega^2 \tag{ii}$$

It is prescribed that $r = 1 - 5t$

and from the moment of momentum conservation Eq. (i)

$$r^2\omega = 1^2 \cdot 20 = 20$$

$$\omega = \frac{20}{(1-5t)^2}$$

which, when substituted in Eq. (ii), gives

$$F = 2(1-5t)\frac{20^2}{(1-5t)^4}$$

or $$F = \frac{800}{(1-5t)^3} \text{ N}$$

It is interesting to note that the force required to pull the string at a constant speed is not constant; it increases as the particle approaches the centre-slot. The following table should supplement the understanding of the phenomenon.

Table 6.2

t (s)	r (m)	ω (rad/s)	F(N)
0	1	20	800
0.05	0.75	35.56	1896
0.10	0.50	80	6400
0.15	0.25	320	51,200
0.20	0	∞	∞

It is noticed that the particle turns increasingly faster as it approaches the centre and it may, therefore, be expected that the string must break at some position. However, in reality friction and drag come into play and neither the rotational speed nor tension is allowed to increase to such an extent. The fact that *the rotational velocity of a mass increases as it is drawn closer to the axis of rotation* is used by expert figure skaters on the ice. They start into a whirl with both their arms and one leg extended and then, upon drawing the arms and the leg in, they obtain a greatly increased angular velocity, which is both amusing and amazing!

***Example* 6.33** A thin circular ring of mass 100 kg and radius 2 m resting on a smooth surface is subjected to a sudden application of a force of 300 N at a point on its periphery. Calculate (a) the angular acceleration of the ring and (b) the acceleration of the mass centre.

Solution (a) Applying the moment of momentum equation with the origin chosen at the mass centre of the ring and realising that **r**, **v** and **M** are mutually perpendicular,

$$M = \frac{d}{dt}(rm\,v) = \frac{d}{dt}(r^2 m\omega)$$

$$= r^2 m\dot{\omega}$$

whence

$$\dot{\omega} = \frac{M}{r^2 m}$$

$$= \frac{300 \times 2}{2^2 \times 100} = 1.5 \text{ rad/s}^2$$

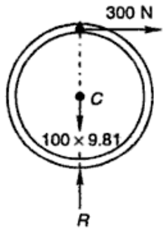


Fig. Ex. 6.33 *(Solution)*

(b) The acceleration of the mass centre may be determined by applying Newton's law,

$$\mathbf{F} = m\,\mathbf{a}_c$$

Then

$$\mathbf{a}_c = \frac{300}{100} = 3 \text{ m/s}^2$$

The Newton's law is indeed applicable although the force is acting at a point other than the centre of mass C.

Concept Review Questions

1. Recognise and explain the truth in the following statements:
   (a) The acceleration or deceleration for a rectilinear motion is ascertained on the basis of like or unlike signs for the acceleration and velocity vectors rather than on the basis of the acceleration vector alone.
   (b) An object may be travelling at a constant speed but it may have variable acceleration.
   (c) The laws of motion applicable to a particle must also be applicable to the centre of mass of any system whether or not the force acts at the centre of mass.

(d) The work-energy principle, impulse-momentum principle, moment of momentum principle and D'Alembert principle are alternative manifestations of the Newton's law of motion.

2. Under what circumstances is Newton's law of motion applicable? When would you prefer the work-energy, impulse-momentum or moment of momentum principle formulations? How does the D'Alembert principle differ from the Newton's law?
3. Which of the following are conserved in a central-force motion and why?
   (a) Force
   (b) Linear momentum
   (c) Moment of momentum
   (d) Mechanical energy.
4. Match the following definitions correctly with the features of the plots indicated.

Net applied force	Slope of velocity vs time
Change in kinetic energy	Area of acceleration vs time
Change in velocity	Area of velocity vs time
Change in the applied force	Slope of position vs time
Change in displacement	Area of force vs position
Instantaneous acceleration	Slope of kinetic energy vs position
Instantaneous velocity	Slope of linear momentum vs time
Change in linear momentum	Area of force vs time

Tutorial Problems

6.1 A stone is dropped into a well. If the splash is heard 2.50 seconds later determine the depth of the water surface assuming that the velocity of sound is 330 m/s.

(*Hint*: $d/330 + \sqrt{2d/9.81} = 2.50$, $d = 28.6$ m)

6.2 A particle accelerates as $a = 9\sqrt{x}$ where a is in m/s^2 and x is rectilinear displacement in metres. At $t = 3$ seconds, the displacement is 16 m and the velocity is 27.7 m/s. Determine the displacement, velocity and acceleration of the particle one second later.

(**Ans.** 67.5 m, 81.6 m/s, 73.9 m/s^2)

6.3 A block of mass 2 kg slides down the face of a smooth 45° wedge of mass 10 kg as shown in Fig. Prob. 6.3. The wedge is placed on a frictionless horizontal surface. Determine the acceleration of the wedge. (**Ans.** 0.89 m/s^2)

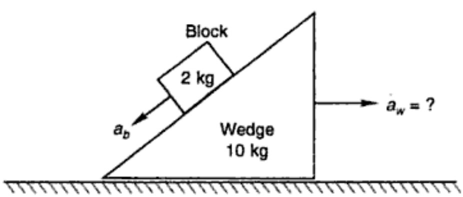


Fig. Prob. 6.3

6.4 A vehicle is uniformly accelerated upon an incline of 1 in 20 from rest and attains a velocity of 5 m/s in 15 s. Determine the tractive force if the vehicle has a mass of 2000 kg and the resistance to motion is 200 N. (**Ans.** 0.33 m/s^2, 1848 N)

6.5 Three blocks of masses m_1, m_2 and m_3 are connected by two cords as shown in Fig.

Prob. 6.5. Obtain an expression for the acceleration a of the system and determine the tension in the cords.

$$\left(\textbf{Ans. } a = \frac{(m_3 - \mu_1 m_1 - \mu_2 m_2)g}{m_1 + m_2 + m_3} \right.$$

$$T_1 = \mu_1 m_1 g + \frac{m_1 g(m_3 - \mu_1 m_1 - \mu_2 m_2)}{m_1 + m_2 + m_3}$$

$$\left. T_2 = m_3 g - \frac{m_3 g(m_3 - \mu_1 m_1 - \mu_2 m_2)}{m_1 + m_2 + m_3} \right)$$

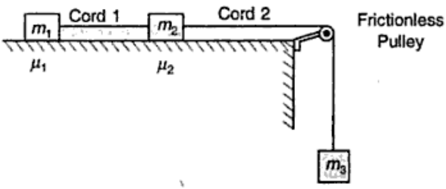


Fig. Prob. 6.5

6.6 A block A resting on a smooth floor and carrying a block B upon it is pulled by a horizontal force as shown in Fig. Prob. 6.6. Determine the acceleration of A to cause a slip between A and B if the coefficient of friction between them is μ.

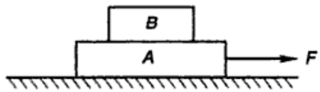


Fig. Prob. 6.6

(**Ans.** $a = \mu g$)

6.7 An object of mass m falls vertically down in a medium with resistance R proportional to the square of the velocity. Obtain an expression for the velocity at time t if it starts from rest at time $t = 0$. What is the terminal velocity?

$$\left(\textbf{Ans. } V = \sqrt{\frac{mg}{k}} \left[\frac{1 - e^{-2t\sqrt{kg/m}}}{1 + e^{-2t\sqrt{kg/m}}} \right]; \sqrt{\frac{mg}{k}} \right)$$

6.8 A particle of mass m is projected vertically upward with a velocity v_0 in a medium whose resistance is kv. Determine the time for the particle to come to rest. What would be the time if the resistance was kv^2 instead of kv?

$$\left(\textbf{Ans. } m/k \ln(1 + kv_0/mg) \text{ and } \sqrt{m/kg} \tan^{-1} v_0 \sqrt{k/mg} \right)$$

6.9 A horizontal force of 100 N is exerted on block A of mass 20 kg which is tied by an inclined string to block B of mass 5 kg as shown in Fig. Prob. 6.9. The coefficients of

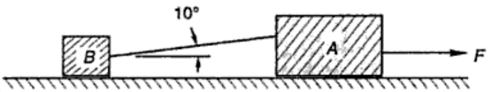


Fig. Prob. 6.9

friction between the plane and block surfaces A and B are 0.25 and 0.5 respectively. Calculate the tension in the string and the acceleration of the system.

(**Ans.** 28 N, 1.11 m/s^2)

6.10 A particle of mass m rests on the top of a smooth sphere of radius r as shown in Fig. Prob. 6.10. Assuming that the particle starts moving from rest, at what point will it leave the sphere?

(*Hint*: $-N + mg\cos\theta = mr\omega^2$ and $mg\sin\theta = mr\alpha$,

$$\omega = \frac{d\theta}{dt},\ \alpha = \frac{d\omega}{dt},\ \int_0^{\theta} \alpha\, d\theta = \int_0^{\omega} \omega\, d\omega$$

Substituting $N = 0$, $\cos\theta = 2/3$, $\theta = 48.2°$)

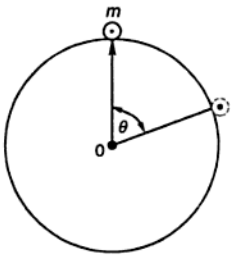


Fig. Prob. 6.10

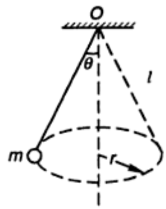


Fig. Prob. 6.11

6.11 A conical pendulum consists of a particle of mass m tied to a cord of length l such that it traces a circular path of radius r at an angular velocity ω when the cord makes an angle θ with the axis of rotation as shown in Fig. Prob. 6.11. If its velocity increases, the particle rises and the radius r of the circular path and angle θ increase. Derive a relationship between θ and ω for constant angular velocity and express the frequency in terms of θ.

$\left(\textbf{Ans. } \omega = \sqrt{g/l\cos\theta},\ f = \frac{1}{2\pi}\sqrt{g/l\cos\theta}\right)$

6.12 A small block of mass 2 kg, held by a cord rests on a smooth inclined plane which can turn about the vertical axis zz as shown in Fig. Prob. 6.12. The cord is 0.6 m long. Determine the tension in the cord when the angular velocity of the plane and block is 10 revolutions per minute.

Also calculate the angular velocity and tension in the cord when the block is at the verge of loosing contact with the incline plane.

(**Ans.** 10.75 N; 5.72 rad/s, 39.24 N)

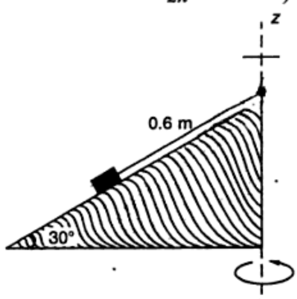


Fig. Prob. 6.12

6.13 A wagon of mass 5000 kg is loose shunted to acquire a speed of 10 m/s before it goes and hits a bumper. If the spring constant of the stationary bumper is 200 N/cm and the spring constant of the wagon pumper is 300 N/cm, calculate (a) the maximum compression of the stationary bumper and (b) the time taken to reach that state.

(**Ans.** 3.87 m; 0.72s)

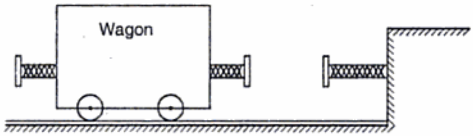


Fig. Prob. 6.13

6.14 A photograph of the 7.25 kg shot-put champion in action shows that the initial angle of projection was 45°. The toss on a level ground was 22 m. Calculate the velocity of projection and the time of flight (a) assuming that the height of the champion is negligible in comparison with the toss and (b) assuming that the height of the point of projection is 2 m. (**Ans.** 14.7 m/s, 2.15 s; 14.1 m/s, 2.2 s)

6.15 A 2 kg ball is suspended by an inextensible string from a ceiling to comprise a pendulum of length 3 m. The ball is released from a position where the string makes an angle of 45° with the vertical. Determine the velocity of the ball when the string makes an angle of 30° with the vertical and when it is at the bottom position.

(**Ans.** 3.06 m/s, 4.15 m/s)

6.16 Determine the minimum velocity a body must have at the top of a vertical circular cylinder of radius r if, when moving circularly, it is to remain in contact with the circular cylinder.

(**Ans.** $v_{\min} = \sqrt{gr}$)

6.17 A bead of mass m moves around a vertical circle of radius r. If the tangential velocity is v_t at the top of the circle, find the tangential velocity v_b at the base of the circle. Assume that the friction is negligible.

(**Ans.** $v_b = \sqrt{(v_t^2 + 4gr)}$)

6.18 A 1 kg stone is whirled at 60 revolutions per minute in a plane vertical circle of radius 1.5 m by means of a string. Determine the tensions in the string at the top and bottom positions. How would the tensions alter if

(a) the mass of the stone were halved?

(b) the whirling speed were halved?

(c) the radius of the circle were halved?

(**Ans.** 49.4, 69 N; 24.7, 34.5 N; 5.0, 24.6 N; 19.8, 39.4 N)

6.19 In a circus, a motor cyclist is moving inside a spherical cage of radius 3 m. The motor cycle and the man together have a mass of 725 kg. Find the least velocity with which the motor cyclist must pass the highest point on the cage without losing contact with the cage.

(*Hint*: The reaction $\left(mg - \dfrac{mV^2}{r}\right) \geq 0$

and $V = \sqrt{gr} = 5.42$ m/s)

6.20 A cyclist, riding at 5 m/s, wishes to turn as fast as he can without skidding. If the coefficient of static friction between the tyres and the road is 0.25, estimate the minimum safe radius of his turn.

(*Hint*: To avoid sliding, $\dfrac{mV^2}{r} \leq \mu mg$

and $V \leq \sqrt{\mu gr}; r = 10.2$ m)

6.21 A particle of mass m slides down a frictionless track and enters a vertical loop of radius r at A to 'loop the loop' as shown in Fig. Prob. 6.21. What should be the minimum height h at the starting point of the particle in order that it may make a complete circuit in the loop?

(*Hint*: $v_A = \sqrt{2gh}$ and the top position is the critical point were the reaction $N = 0$ and the velocity v is given by $v^2 = v_A^2 - 4gr$. Employing $mg = mv^2/r$, $h = 2.5r$. If the particle is started from a height less than $2.5r$ it will not reach the top position and if it is started from a height more than $2.5r$, it should loop the loop successfully.)

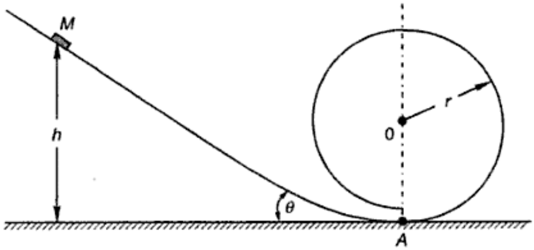


Fig. Prob. 6.21

6.22 A vertical shaft rotating at 5 rad/s has a light horizontal arm fixed with it. Two identical collars A and B each of mass 3 kg, slide out with respect to the arm at 2 m/s as shown in Fig. Prob. 6.22. Determine the angular acceleration of the arm.

(*Hint*: $d/dt\,(\mathbf{r} \times m\,\mathbf{V}) = 0 = d/dt(r^2 m\omega)$

$r^2 d\omega/dt + 2r\omega\, dr/dt = 0$

Hence, $d\omega/dt = -10$ rad/s^2)

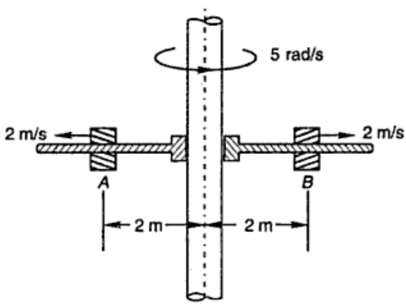


Fig. Prob. 6.22

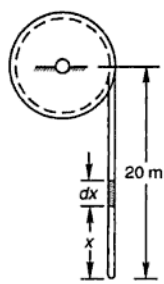


Fig. Prob. 6.23

6.23 Determine the work done in winding up a homogeneous cable which hangs from a horizontal drum if its free length is 20 m and weighs 1 kN as shown in Fig. Prob. 6.23. **(Ans.** 98.1 kJ)

6.24 A bullet enters a 5 cm thick plank with a speed of 600 m/s and leaves with a speed of 240 m/s. Determine the greatest thickness of the plank that can be penetrated by the bullet. **(Ans.** 5.95 cm)

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions.

1. The motion of a particle, in general, is described by
   (a) the Newton's law and not the work-energy equation
   (b) the impulse-momentum principle alone if there is no external force
   (c) the Newton's law, the work-energy equation, impulse momentum principle or the moment-of-momentum principle
   (d) the principles of conservation of energy and momentum
2. The D'Alembert principle
   (a) is a hypothetical principle
   (b) provides no special advantage over Newton's law
   (c) is based upon the existence of inertia forces
   (d) allows a dynamical problem to be treated akin to a statical problem
3. The centrifugal force
   (a) is not an inertia force
   (b) tends to overturn a body outwards on a curved path
   (c) is a fictitious force
   (d) is the real force experienced by a body negotiating a bend
4. A particle of mass m is projected with a velocity V making an angle of 45° with the horizontal. The magnitude of the angular momentum of the particle about the point of projection when the particle is at its maximum height h is
   (a) zero, (b) $m\,V^3/(4\sqrt{2}\,g)$, (c) $m\,V^3/(\sqrt{2}\,g)$, (d) $m\sqrt{2gh^2}$
5. The momentum of a system of two bodies is conserved
   (a) if either body does not exert a force on the other
   (b) when there is no external force acting on either body
   (c) when the external forces act only on one body
   (d) when the external moment on the system is zero.
6. The impulse-momentum principle is applicable
   (a) if there is no external force acting on the body
   (b) when the impulse is conserved
   (c) when the momentum is conserved
   (d) wherever Newton's law is applicable.
7. The principle of conservation of mechanical energy requires that
   (a) the acceleration should be zero
   (b) there should be no external forces
   (c) the motion should be restricted to the gravitational field only
   (d) the force-field should be conservative
8. A simple pendulum mounted on a lorry moving on a horizontal track with a constant acceleration a will be deflected away from the vertical
   (a) towards the direction of acceleration, $\theta = \sin^{-1}(a/g)$
   (b) against the direction of acceleration, $\theta = \sin^{-1}(a/g)$

(c) against the direction of acceleration, $\theta = \tan^{-1}\left(\frac{a}{g}\right)$

(d) towards the direction of acceleration, $\theta = \tan^{-1}\left(\frac{a}{g}\right)$

Answers to Multiple-Choice Questions

1 (c), 2 (d), 3 (c), 4 (b), 5 (b), 6 (d), 7 (d), 8 (c)

7 KINEMATICS OF A RIGID BODY

7.1 INTRODUCTION

A *rigid body* is an idealisation of the behaviour of a body. A body can be idealised as rigid when its dimensions and the relative positions of points within it do not change during the course of observation. Mathematically, distances between any pairs of points within the body remain constant. For example, two arbitrary points P_1 and P_2 shown in a rigid body in Fig. 7.1(a) remain a constant distance apart no matter what happens to the body. Mathematically, $P_1P_2 = k$. In other words, the body is undeformed under the static and dynamic actions. The idealisation of a rigid body allows the distribution of matter, uniformly or non-uniformly over the extent of the body.

Kinematics of a rigid body refers to the relationship of position, velocity and acceleration with time for a rigid body. We first visualise the types of motion a rigid body can undergo and classify them as *pure translation, pure rotation, plane motion* and *space motion*. We then proceed to study the kinematics in the same order for two reasons: one, to go from simple to not-so-simple and two, because the more general motions can be thought of as superpositions of the simpler motions namely translation and rotation.

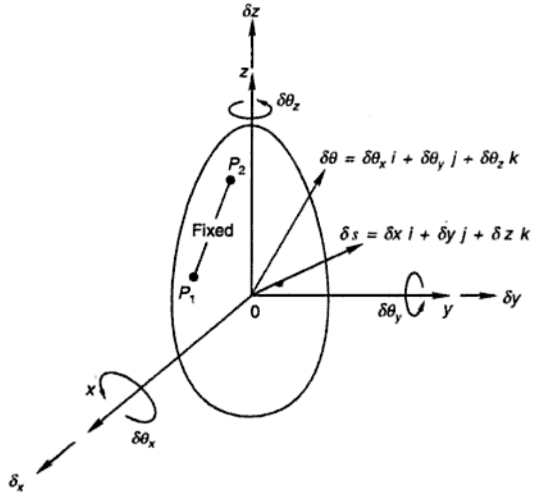
A great deal of attention is paid to the relative motion of rigid bodies with reference to the moving and fixed frames of reference. A number of examples on the analysis of motion, i.e., on the determination of velocity and acceleration, both analytically and graphically, are given.

7.2 TYPES OF MOTION

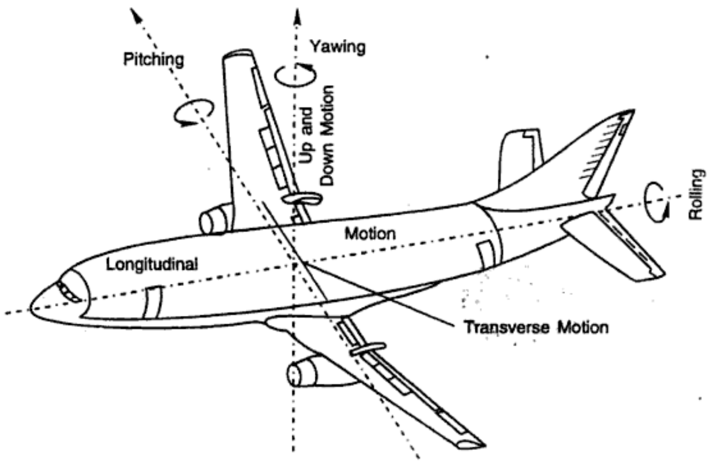
A rigid body may be displaced from its initial position *rectilinearly* along one or more of the three axes or *angularly* about one or more of the three axes, the set of axes being chosen arbitrarily.

There are altogether six modes of displacement and hence six coordinates are needed to specify the motion; a rigid body is said to have six degrees of freedom as shown in Figs. 7.1(a) and (b). Correspondingly, a rigid body may have six components each of velocity and of acceleration.

The degrees of freedom of a rigid body can be thought of in another way. Suppose that a rigid body is to be located in space during its motion. The number of independent coordinates required to specify its location in space is equal to the number of degrees of freedom it enjoys. A little reflection will show that if a body is held at one point, it can rotate about that point; if it is held at two points, it can rotate about the axis passing through the two points and if it is held at three non-collinear points, it is held fully constrained. It follows that the position of the body is fully specified if the coordinates of any three non-collinear points on the body are specified such as that shown in Fig. 7.1(c). Each of the three points requires three

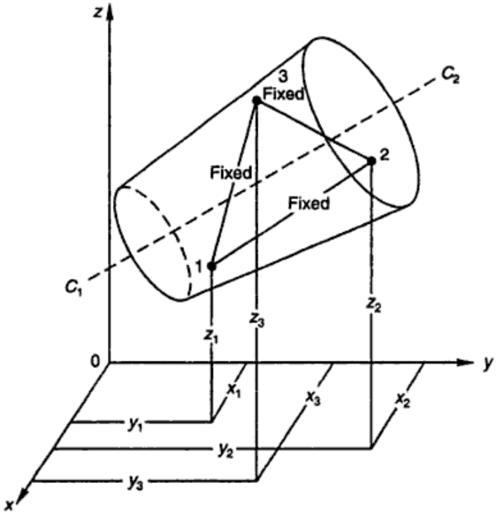


(a) Six Modes of Displacement of a Rigid Body



(b) Six Degrees of Freedom of an Aeroplane

Fig. 1 (*Contd.*) *Degrees of Freedom*



(c) Degrees of Freedom of a Rigid Body

Fig. 7.1 ***Degrees of Freedom***

	Cartesian Components			Numbers
	x	y	z	
Linear displacement	δ_x	δ_y	δ_z	6
Angular displacement	$\delta\theta_x$	$\delta\theta_y$	$\delta\theta_z$	
Linear velocity	V_x	V_y	V_z	6
Angular velocity	ω_x	ω_y	ω_z	
Linear acceleration	a_x	a_y	a_z	6
Angular acceleration	α_x	α_y	α_z	

coordinates to be specified which makes a total of nine coordinates for the entire body. Of these nine coordinates, only six are independent because the distance between the points remains fixed by the definition of the rigid body. In other words, the nine coordinates are

$$x_1, y_1, z_1 \qquad x_2, y_2, z_2 \qquad \text{and} \qquad x_3, y_3, z_3$$

and the three constraints are

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = C_1$$

$$(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = C_2$$

$$(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 = C_3$$

The number of independent coordinates is thus reduced to six.

The motion of a rigid body is usually subjected to certain kinematic constraints, thus bringing down the number of degrees of freedom.

A rigid body allowed to rotate about a fixed axis has only one degree of freedom, i.e., it can only undergo angular displacement about that axis. A rigid body allowed to translate in general has three degrees of freedom, i.e., it can acquire displacements along the x, y and z axes. If the translation is restricted in a plane, it has only two degrees of freedom and if it is required to go along a specified curve, it is left with only one degree of freedom.

The concept of the degrees of freedom of a rigid body can be extended to a number of connected rigid bodies. Each rigid body has six degrees of freedom. From the total, the number of constraints must be subtracted to find the degrees of freedom of the system of bodies.

A *mechanism* is an assemblage of rigid linkages which have relative motion between them but the motion of every linkage is uniquely determined if one of them is given a prescribed motion. A mechanism has, therefore only one degree of freedom.

A *structure* is an assemblage of members capable of withstanding loads without any change in their dimensions. No relative motion is allowed between the structural members. A structure can, therefore, be thought of as a rigid body with zero degree of freedom.

The definition of a rigid body has far-reaching implications. There can be no component of velocity of a point relative to another point along the line joining the two points in the rigid body because the distance between them cannot change. The velocity of a point relative to another point in the rigid body must, therefore, be wholly perpendicular to the join of the two points. For example, with reference to Fig. 7.1(c), point 1 can have any arbitrary velocity $\mathbf{v}_1$ but points 2 and 3 cannot have arbitrary velocities. The component of $\mathbf{v}_1$ along 1-2 must be the same as that of $\mathbf{v}_2$ along 1-2. Similarly, the component of $\mathbf{v}_3$ along 1-3 must equal the component of $\mathbf{v}_1$ along 1-3 and of $\mathbf{v}_3$ along 3-2 must equal the component of $\mathbf{v}_2$ along 3-2. In other words,

$\mathbf{v}_{21}$ is perpendicular to line 1-2

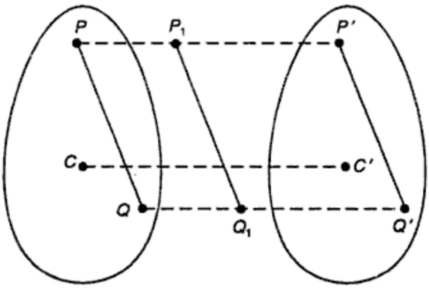
$\mathbf{v}_{12}$ is perpendicular to line 1-2

$\mathbf{v}_{13}$ is perpendicular to line 1-3, etc.

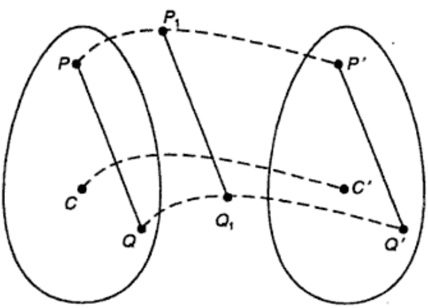
It implies that a point on a rigid body can only undergo an angular displacement with respect to another point on the body at any instant. The absence of relative velocity along the join of two points on a rigid body does not imply the absence of relative acceleration along the join of the points. In fact, whenever there is relative velocity between two points in a rigid body, there must be a component of acceleration along the join. In addition, there may be a component of acceleration normal to the join of the points resulting from the rate of change of the relative velocity of the points.

Translation

A rigid body is said to be in *translation* if the linear displacement of every point in the rigid body is the same. Translatory motion is characterised by the movement of a typical line element PQ parallel to itself. In other words, the translational motion of a rigid body is characterised by each point on the body to have the same velocity and also the same acceleration at an instant. It should be understood that a rigid body can undergo a change in velocity both in magnitude and direction during translation. In *rectilinear translation*, a typical point P translates along a straight line $P\,P_1\,P'$ and an element PQ moves to P_1Q_1, $P'Q'$ as shown in Fig. 7.2(a). In *curvilinear translation*, a typical point P may trace a plane or a space curve PP_1P' and a line $PQ \parallel P_1Q_1 \parallel P'Q'$ as shown in Fig. 7.2(b). The curve traced by each point must be identical on a rigid body in a translation. It follows that all the points of the body have the same linear displacement, same velocity and same acceleration at a given instant.



(a) Rectilinear Translation

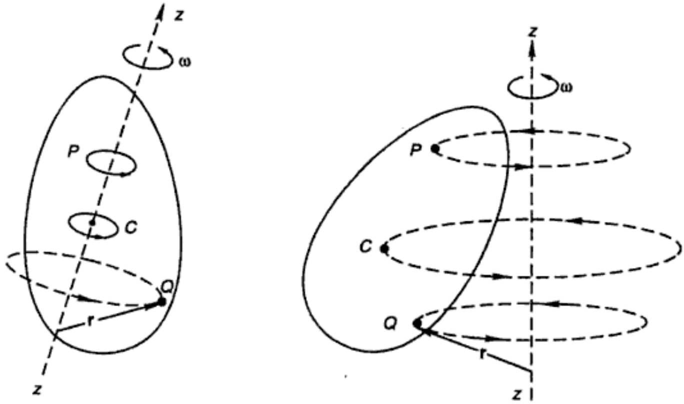


(b) Curvilinear Translation

Fig. 7.2 ***Translation of a Rigid Body***

Rotation

Rotational motion is characterised by the same angular displacement of all the points in the rigid body. It follows that the angular displacement, angular velocity and angular acceleration at a given instant are the same for all the points in the body with reference to the axis of rotation. The axis of rotation, chosen as z-z, may lie within the body or outside it as shown in Figs. 7.3(a) and (b). The trajectory of each point on the rigid body in rotation must be a circle with its centre on the axis of



(a) Axis within the Body (b) Axis outside the Body

Fig. 7.3 *Rotational Motions*

rotation. It may be noted that the velocity of a point with a position vector **r** on a body in rotation at angular velocity ω is given by

$$\mathbf{V}_{\theta} = \boldsymbol{\omega} \times \mathbf{r} \tag{7.1}$$

The acceleration of a point is likewise given by

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{7.2}$$

The words *pure translation* and *pure rotation* are often used to stress the absence of rotation and translation respectively.

In general, a rigid body may *translate* and *rotate* simultaneously. Conversely, a general rigid body motion may be thought of as a superposition of a pure rotation and a pure translation, viz., (a) pure rotation about an axis through a chosen point and (b) pure translation along the join of the initial and final positions of the chosen point.

A rigid body is said to be in fixed-axis rotation if there exists a fixed straight line within or outside the body such that the points identified with the body but on that line have zero velocity and zero acceleration. The straight line thus qualified is called *axis of rotation* of the body. Rotation may also be specified to be about a point if there exists only a fixed point identified with the body where both the velocity and acceleration vanish. An example of fixed-axis rotation is a shaft rotating in a fixed journal bearing and an example of fixed-point rotation is a spinning top rotating about the tip in steady or unsteady states.

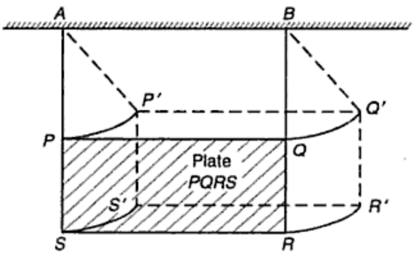
Plane Motion

The motion of a rigid body is said to be *plane motion* if all the points in the body stay in the same parallel planes. A plane motion may be composed of translation and rotation. Examples of plane rotation are given as follows:

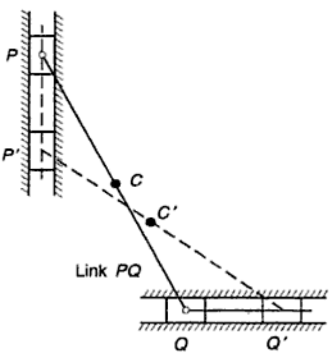
1. Curvilinear translation is a plane motion if the curve traced by any point on it is a plane curve. A thin plate hanging by two equal inextensible strings is an

example of plane curvilinear translation, e.g., *AP* and *BQ* oscillating in its own plane as shown in Fig. 7.4(a).

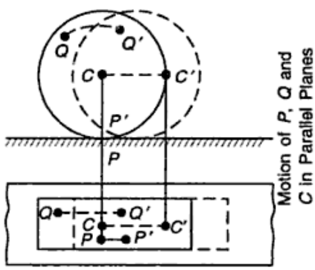
2. Linear translation of a rigid body must be a plane motion.
3. Rotation of a rigid body about a fixed axis must be a plane motion.
4. Translation and rotation can result in a plane motion if the rotation takes place about an axis perpendicular to a plane of motion as shown in Fig. 7.4(b).
5. Rolling without slipping of a cylindrical object on a flat or curved surface must be a plane motion as shown in Fig. 7.4(c).



(a) Curvilinear Translation



(b) Translation and Rotation



(c) Rolling of a Cylinder

Fig. 7.4 ***Examples of Plane Motion***

Space Motion

Space motion of a rigid body is a general type of motion, with 6 degrees of freedom, not constrained to be categorised in any of the restricted motions. Examples of such motions are the flying objects, vehicles on land, etc. The rolling of a cone on a flat or curved surface and the final stage of the motion of a spinning top are space motion. The space motion of a body in which one point remains fixed in space is called motion about a fixed point. An example of such a motion is the entire span of spinning motion of a top on a rough floor.

The equivalent combinations are described as follows:

Fig. 7.8(a)	Translation from PQ to $P'Q_1$ and rotation about P'
Fig. 7.8(b)	Translation from PQ to P_1Q' and rotation about Q'
Fig. 7.8(c)	Translation from PQ to P_1Q_1 and rotation about C'
Fig. 7.8(d)	Translation from PQ to P_1Q_1 and rotation about O'
Fig. 7.8(e)	Rotation from PQ to PQ_1 about P and translation to $P'Q'$
Fig. 7.8(f)	Rotation from PQ to P_1Q about Q and translation to $P'Q'$
Fig. 7.8(g)	Rotation from PQ to P_1Q_1 about C and translation to $P'Q'$
Fig. 7.8(h)	Rotation from PQ to P_1Q_1 about an arbitrary point O and translation to $P'Q'$

Of course, *translation and rotation for a plane motion are commutative*; the order of rotation and translation in the summation is immaterial as can be verified geometrically. For example, Figs. 7.8(e) and (f) show equal amount of translation and rotation but in Fig. 7.8(h) translation is done first and rotation later and in Fig. 7.8(f) it is the other way round. Similarly, Figs. 7.8(a) and (e), (c) and (f); (d) and (h) also commutative.

It is also important to note that, whatever be the mode of combination, the amount of rotation is the same. The angular velocity ω of every point on the element is therefore the same. It is, therefore, in order to use the term *angular velocity of the link or of the rigid body* rather than about any particular point on it.

The fact that a general plane motion can be thought of as a superposition of translation and rotation is a special case of *Chasle's theorem*. The theorem, in general, states that any general motion of a rigid-body can be considered as an appropriate superposition of a translational motion and a rotational motion. In particular, if a rigid body is displaced from position 1 to position 2 in space, then it is possible to visualise the body to have undergone translation from position 1 to an intermediate position in regard to a certain point O on the body and then rotation about the point O. The extent of translation up to the intermediate position depends upon the choice of the reference point on the body whereas the extent of rotation is independent of the choice. Alternatively, a rigid body can be visualised to have rotated about a reference point first and then translated to the final position. In other words, *the order of superposition of the translational and rotational motions is immaterial.*

7.5 RIGID-BODY MOTION OF FLUIDS

It is possible that fluids, particularly liquids, may undergo rigid-body translation and rigid-body rotation.

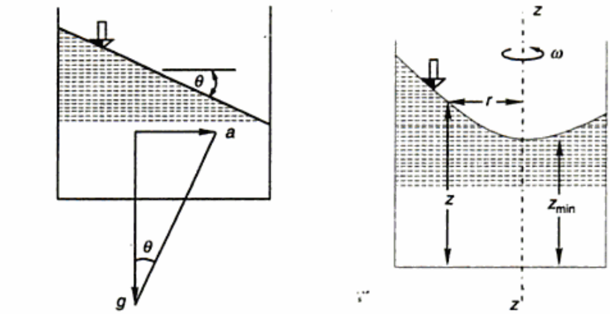
An example of translation is a liquid mass in a container subjected to a constant linear acceleration a along any direction. For the simplest case of constant horizontal acceleration a, the liquid orients itself with its free surface inclined at θ as shown in Fig. 7.9(a) such that

$$\boxed{\tan\theta = \frac{a}{g}} \tag{7.9}$$

An example of rotation is a forced-vortex flow of air in the core region of a tornado or cyclone. Another case of solid-body rotation is that of a liquid in a container subjected to a constant angular velocity ω about any axis, passing through the liquid or outside it. The surface of the liquid orients itself in the form of a paraboloid of revolution as shown in Fig. 7.9(b) such that

$$z = z_{min} + \frac{r^2 \omega^2}{2g} \tag{7.10}$$

where z is the depth of the liquid at a radial distance r from the axis of rotation and z_{min} is the minimum depth. Equations (7.9) and (7.10) have been stated without proofs.



(a) Plane or Rectilinear Translation (b) Fixed-axis Rotation

Fig. 7.9 ***Rigid Body Motion of Fluids***

7.6 INSTANTANEOUS CENTRE OF ROTATION

Recalling the statement that *a general plane motion of a rigid body may be considered as the sum of a plane translation and a rotation about an axis perpendicular to the plane of motion,* the velocity of a rigid body is completely specified by stating the translational velocity V_P of a point P and the rotational velocity ω about an axis through the point as shown in Fig. 7.10(a).

The fact that at an instant, the rotational velocity ω is the same for every point in the body and the extent of translation is different for different choices of the points of rotation suggests that a point *can* exist such that the body may be assumed to rotate about an axis though that point *at the instant*. Such a point is called the *instantaneous centre of rotation*. The velocity of the instantaneous centre of rotation is zero at that instant. Location of such an instantaneous point I requires that the perpendicular distance from it to the velocity at a point P in one of the parallel planes should be such that

$$V_P = IP \times \omega \quad \text{or} \quad V_P = r\omega \quad \text{and} \quad r = V_P/\omega \tag{7.11}$$

The velocity of the other points in the body can be determined by considering the body to rotate about an axis passing through I and normal to the parallel planes of motion. For example,

$$V_Q = (IQ) \times \omega \text{ perpendicular to } IQ$$

and $$V_R = (IR) \times \omega \text{ perpendicular to } IR$$

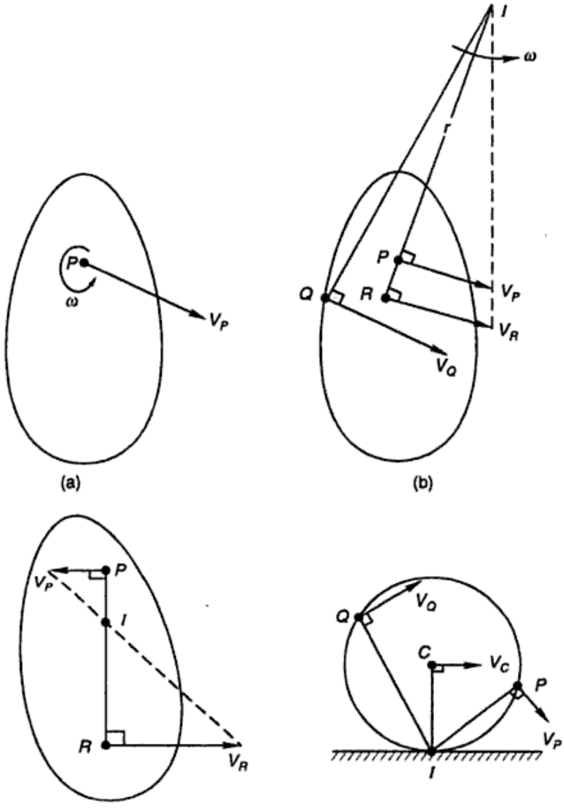


Fig. 7.10 ***Instantaneous Centre of Rotation***

This fact provides another method for locating the instantaneous centre of rotation I if the directions of the velocities at any two points on a rigid body are known. The point of intersection of the perpendiculars to the directions of the velocities must be I. In case the directions of the two points are the same, the magnitudes of the velocities are required to locate I by the join of the extremities of the velocity vectors as at P and R in Figs. 7.10(b) and (c). The instantaneous centre of rotation is a point identified with the body where the velocity is zero; conversely if a point identified with the rigid body is at rest at an instant *within or outside a rigid body*, it must be the instantaneous centre of rotation.

The instantaneous centre of rotation of a body undergoing pure translation must be at infinity since the directions of velocities of all the points in the body are the same in pure translation. The instantaneous centre of rotation of a cylindrical body

$$V_Q = 20\ IQ = 20 \times 2 \times 0.5 = 20 \text{ m/s}$$

perpendicular to IQ or parallel to V_c as shown in Fig. Ex. 7.1(b) (Solution).

$$V_R = 20\ \ IP = 20 \times 0.707 = 14.14 \text{ m/s}$$

perpendicular to IR as shown in Fig. Ex. 7.1(b) (Solution).

The motion of the wheel can alternatively be visualised as that of rotation about C and translation down the incline. In that case the velocity of a point on the periphery is composed of a component V_θ tangential to the periphery and a translational component of 10 m/s. The tangential component must also be 10 m/s for rolling without slip. Vector addition of the tangential velocity of 10 m/s to the down-slope component of 10 m/s at P, Q, R and I result in the velocities of 14.14 m/s, 20 m/s and 0 respectively.

***Example* 7.2** A straight rigid link AB 40 cm long has, at a given instant, end B moving along a line OX at 4 m/s and the other end A moving along YO, XOY being a right angle. Find the velocity of the end A and of the mid-point of the link when inclined at 30° with OX.

Solution The instantaneous centre of rotation of the link AB can be located by the knowledge of the directions of the velocities of the ends A and B. The point of intersection of the dotted lines drawn normal to the direction of velocities at A and B, as shown in Fig. Ex. 7.2 (Solution) should be the instantaneous centre I. The instantaneous angular speed with which the entire link AB rotates about the instantaneous centre I is given by

$$\omega = \frac{V_A}{IA} = \frac{V_B}{IB}$$

$$= \frac{V_C}{0.3464} = \frac{4}{0.20} = 20 \text{ rad/s}$$

The velocity of the point A must, therefore, be

$$V_A = 20 \times 0.3464 = 6.93 \text{ m/s}$$

Once the angular velocity of a link about an instantaneous centre is known, the velocity of any point on the link can be determined. The velocity of the midpoint C of the link is

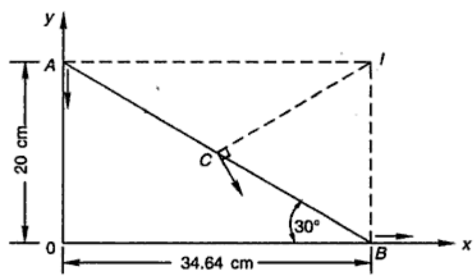


Fig. Ex. 7.2 ***(Solution)***

$$V_C = \omega \times (IC)$$
$$= 20 \times 0.20 = 4 \text{ m/s}$$

in magnitude and its direction is perpendicular to the join *IC*.

It may be noted that the lengths *IA*, *IB* and *IC* are determined either by measurement if the whole drawing is made to scale or by trigonometry.

7.7 RELATIVE VELOCITY AND ACCELERATION FOR POINTS ON A RIGID BODY

From the fact that a general plane motion is made up of a translation of a reference point *P* and a rotation about *P*, the absolute velocity of a point *Q* is given by

$$\mathbf{V}_Q = \mathbf{V}_P + \boldsymbol{\omega} \times \mathbf{r} \tag{7.12}$$

where **r** is the position vector of *Q* with respect to *P* as shown in Fig. 7.11.

Alternatively,

$$\mathbf{V}_Q = \mathbf{V}_P + \mathbf{V}_{QP}$$

Hence

$$\mathbf{V}_{QP} = \boldsymbol{\omega} \times \mathbf{r} \tag{7.13}$$

which implies that the velocity of a point *Q* with respect to a point *P* on a rigid link must be perpendicular to the plane containing $\boldsymbol{\omega}$ and **r**, i.e., must be in the plane of the motion and directed perpendicular to the line joining *P* and *Q*.

Similarly, the absolute acceleration of a point *Q* in terms of the acceleration of a reference point *P* is given by

$$\mathbf{a}_Q = \mathbf{a}_P + \mathbf{a}_{QP} \tag{7.14}$$

The acceleration of *Q* with respect to a point *P* may be made up of

Tangential component: $\boldsymbol{\alpha} \times \mathbf{r}$

Normal component: $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

In the plane motion of a rigid body, attention is focussed on one of the parallel planes; the velocity of a point in the plane must stay in that plane and the rotation of the body must be about an axis normal to that plane. It follows that the velocity of *P* and *Q* as well as the relative velocity of *Q* with respect to *P* must lie in that plane. Likewise, the accelerations of *P* and *Q* and the components of the relative acceleration of *Q* with respect to *P* must all lie in the same plane. From the fact that ω and α are collinear and *r* is perpendicular to either of them, the magnitudes of the vectors are identified easily:

Vector	*Magnitude*	*Direction*
$\mathbf{V}_{QP} = \boldsymbol{\omega} \times \mathbf{r}$	$r\omega$	Perpendicular to **r**
$\mathbf{a}_{QPt} = \boldsymbol{\omega} \times \mathbf{r}$	$r\alpha$	Perpendicular to **r**
$\mathbf{a}_{QPn} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$	$r\omega^2$	From *Q* to *P*

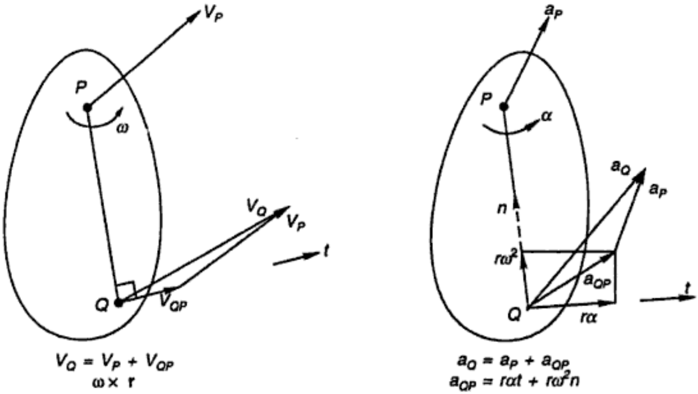


Fig. 7.11 ***Relative Velocity and Acceleration***

There may, in addition, be a Coriolis component of acceleration if there is a velocity of sliding $\mathbf{v}_B$, of an element S sliding over the link PQ and being coincident with Q at that instant. The Coriolis acceleration of S would be

$$\mathbf{a}_{cor} = 2\omega \times \mathbf{v}_B \tag{7.15}$$

Some facts regarding the graphical or vectorial representation of the velocities and accelerations for link-motions are summarised as follows:

1. The velocity of a point on a link with respect to another point on the same link must be perpendicular to the line joining the two points.
2. The acceleration of a point on a link with respect to another point on the same link may have a component perpendicular to the line joining the points and a component along the line joining them.
3. The velocity of a point on a member sliding relative to a link must be along the line and the velocity of a point on a link sliding relative to a surface must be along the tangent to that surface at that point.
4. If a member slides along a link rotating about a point of an axis in space, the member is subjected to the Coriolis component of acceleration in addition to the other components. If the member slides outward from the centre of rotation; the Coriolis component is along the direction of rotation and vice versa.
5. The velocity and acceleration diagrams are drawn for a known configuration of the linkage or mechanism and give results which are valid *for that instant only*. The velocity and acceleration diagrams are, therefore, instantaneous diagrams and keep changing from instant to instant.

***Example* 7.3** A straight bar AB is placed in a semi-cylindrical trough of radius 20 cm and released to slide in it such that the end A slides inside the trough as shown in Fig. Ex. 7.3, while the bar touches and slips at the corner O. At an instant when the bar makes 45° with the diametral axis Ox and the end A is known to slide at 5 m/s, determine the velocity of sliding of the bar at point P.

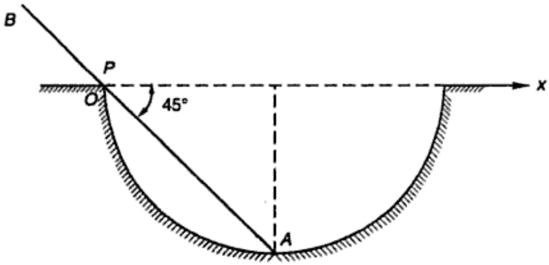


Fig. Ex. 7.3

Solution This example will be solved both graphically and vectorially to demonstrate the methods as well as to appreciate the simplicity offered by the graphical method for the plane motion of rigid bodies.

The bar *AB* slides at the corner point *O* of the trough. The point *P* on the bar is coincident with the point *O* on the trough such that *P* has a relative velocity with respect to *O* as shown in Fig. Ex. 7.3(a) (Solution). Since *O* is a stationary point, this is also the absolute velocity of *P*. It is related to the velocity of *A* as

$$\mathbf{V}_P = \mathbf{V}_A + \mathbf{V}_{PA}$$

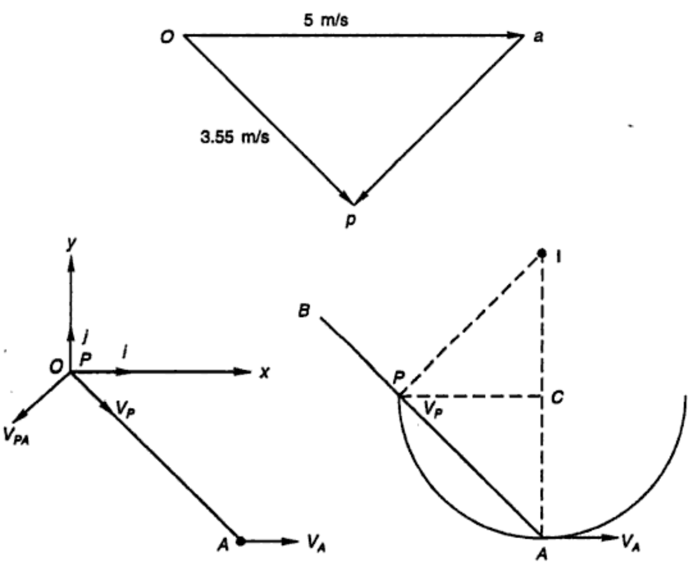


Fig. Ex. 7.3(a), (b), (c) ***(Solution)***

Finally, the velocity vector for P where the bar slides at the corner of the trough is

$$V_P = 3.54(\mathbf{i} - \mathbf{j})/\sqrt{2}$$
$$= 2.54\ (\mathbf{i} - \mathbf{j})$$

Comparison of the Two Methods

It is probably quite clear from this simple example that the graphical construction offers great simplicity. This is indeed true for the plane motion of rigid bodies in general. However, it may be remarked that the velocity and acceleration diagrams are only valid for the instant they are drawn. *For every instant of motion, we may require to draw separate diagrams.* Analytically, on the other hand, one can write the equations for the velocity and acceleration in terms of the variables and obtain the results by numerical substitution for different instants. For example, in this case, vector equations for V_P and V_{PA} can be written for any angle θ instead of 45° and at different instants, substitution for θ would give the results whereas the graphical construction will have to be repeated many times.

The same problem can be solved by employing the concept of the instantaneous centre of rotation. The centre of rotation can be located by knowing that the velocity of A is tangential to the trough at A and the velocity of P is along the bar at O. Lines IA and IP drawn normal to the directions of velocity at A and P respectively to locate the instantaneous centre of rotation I as shown in Fig. Ex. 7.3(c) (Solution).

By measurement from the figure drawn to scale or from trigonometry,

$$IA = 2\ CA = 40 \text{ cm}$$

$$IP = \sqrt{(20^2 + 20^2)} = 28.28 \text{ cm}$$

Considering the bar to be in pure rotation about I at this instant,

$$\frac{V_P}{IP} = \frac{V_A}{IA}$$

$$V_P = 5 \times 28.28/40 = 3.54 \text{ m/s}$$

It may also be remarked that the 'instantaneous centre of rotation' concept can be employed to advantage in some cases.

***Example* 7.4** A straight rigid link AB 40 cm long has, at a given instant, end B moving along a line OX at 0.8 m/s and accelerating at 4 m/s^2 and the other end moving along YO, XOY being a right angle as shown in Fig. Ex. 7.4. Find the velocity and the acceleration of the end A and of the mid-point C of the link when inclined at 30° with OX.

Solution

For the velocity diagram

$$\mathbf{V}_A = \mathbf{V}_B + \mathbf{V}_{AB}$$

$$\mathbf{V}_C = \mathbf{V}_B + \mathbf{V}_{CB}$$

The acceleration of mid-point C is, therefore, represented by oc. By measurement

$$\mathbf{a}_C = oc = 10 \text{ m/s}^2$$

directed at an angle of 79° with OX as shown in the acceleration diagram.

It may be appreciated that the point c can alternatively be located on the line ab which represents the relative acceleration of A with respect to B. Since C is the mid-point of AB, the point c must be the mid-point of ab. This is because both the centripetal and tangential components of the relative acceleration of A with respect to B are halved for the point C.

***Example* 7.5** A reciprocating engine mechanism shown in Fig. Ex. 7.5 has a crank OA of radius 150 mm rotating at 10 revolutions per second in the clockwise direction. The connecting rod AB is 700 mm long and its centre of gravity is 200 mm from A. Find the velocity and acceleration of the piston and of the centre of the connecting rod when the crank is 45° past the inner dead centre as shown. Find also the angular velocity and the angular acceleration of the connecting rod AB.

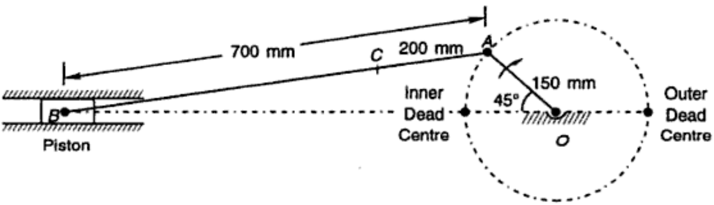


Fig. Ex. 7.5

Solution The solution is attempted both by the graphical method and by vectorial analysis.

By graphical method

The link diagram is first drawn to a suitable scale as shown. From the data,

$$\omega_{OA} = 2\pi \times 10 = 62.83 \text{ rad/s}$$

$$V_A = \omega_{OA} \times r = 62.83 \times 0.15 = 9.42 \text{ m/s}$$

The velocity diagram is now constructed to a suitable scale. Line oa is drawn perpendicular to OA to represent the absolute velocity of A. The velocity of B must be along BO and the velocity of B with respect to A on the link AB must be perpendicular to AB. Point b is thus located by drawing a line from o parallel to BO and a line from a perpendicular to AP as shown in Fig. Ex. 7.5(a) (Solution). The velocity of the piston at B is, therefore, given by

$$V_B = ob = 7.65 \text{ m/s}$$

directed towards O.

The angular velocity of AB is

$$\omega_{AB} = \frac{V_{AB}}{AB} = \frac{ab}{AB} = \frac{6.75}{0.7} = 9.64 \text{ rad/s}$$

$$a_B = ob = 418 \text{ m/s}^2$$

directed towards O.

The acceleration of the centre of gravity C is determined by locating the point c such that

$$\frac{ac}{ab} = \frac{AC}{AB}$$

The acceleration of the centre of gravity of the connecting rod is, therefore, $a_C = oc = 515 \text{ m/s}^2$ directed at $-35°$ with the line of dead centre.

By vectorial analysis

From the data and the geometry,

$$\boldsymbol{\omega}_{OA} = -2\pi \times 10\,\mathbf{k} = -62.83\,\mathbf{k} \text{ rad/s}$$

$$\mathbf{r}_{OA} = 0.15 \times (-\cos 45°\,\mathbf{i} + \sin 45°\,\mathbf{j})$$

$$= -0.106\,\mathbf{i} + 0.106\,\mathbf{j}$$

$$\mathbf{V}_A = \boldsymbol{\omega}_{OA} \times \mathbf{r}_{OA} = -62.83\,\mathbf{k} \times (-0.106\,\mathbf{i} + 0.106\,\mathbf{j})$$

$$= 6.66\,\mathbf{i} + 6.66\,\mathbf{j} \text{ m/s}$$

$$\mathbf{V}_B = V_B\mathbf{i}$$

At the instant of interest, the connecting rod is inclined at an angle ϕ with the x-axis such that

$$0.7 \sin \phi = 0.15 \sin 45°$$

$$\sin \phi = 0.1515$$

whence $\phi = 8.71°$ and $\cos \phi = 0.988$

The unit vector along AB is

$$\mathbf{e}_1 = \cos \phi\,\mathbf{i} + \sin \phi\,\mathbf{j} = 0.988\,\mathbf{i} + 0.1515\,\mathbf{j}$$

and the unit vector normal to AB is

$$\mathbf{e}_2 = -\sin \phi\,\mathbf{i} + \cos \phi\,\mathbf{j} = -0.1515\,\mathbf{i} + 0.988\,\mathbf{j}$$

The velocity of B must be given by

$$\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{BA}$$

Substituting the values of the terms to the extent these are known,

$$V_B\mathbf{i} = 6.66\,\mathbf{i} + 6.66\,\mathbf{j} + (-0.1515\,\mathbf{i} + 0.988\,\mathbf{j})\,V_{BA}$$

which may be written as two scalar equations,

$$V_B = 6.66 - 0.1515\,V_{BA}$$

$$0 = 6.66 + 0.988\,V_{BA}$$

whence $V_{BA} = -6.741 \text{ m/s}$

and $V_B = 7.68 \text{ m/s}$

It may be noted that the negative sign of V_{BA} implies that the velocity of B with respect to A is opposed to the unit vector $\mathbf{e}_2$ whereas the positive sign of V_B means that the velocity of B is indeed directed towards O.

The angular velocity of the connecting rod is given by

$$\omega_{BA} = \frac{V_{BA}}{BA} = -\frac{6.741}{0.7} = -9.63 \text{ rad/s}$$

The velocity of C must be given by

$$\begin{aligned}\mathbf{V}_C &= \mathbf{V}_A + \mathbf{V}_{CA} \\ &= 6.66\,\mathbf{i} + 6.66\,\mathbf{j} + \frac{CA}{BA}(-6.741) \times (-0.1515\,\mathbf{i} + 0.988\,\mathbf{j}) \\ &= 6.95\,\mathbf{i} + 4.76\,\mathbf{j}\end{aligned}$$

The magnitude of $\mathbf{V}_C$ is, therefore,

$$V_C = \sqrt{(6.95^2 + 4.76^2)} = 8.42 \text{ m/s}$$

and it is inclined to the x-axis at an angle

$$\theta = \tan^{-1}\left(\frac{4.76}{6.95}\right) = 34.4^\circ$$

The acceleration of B may be obtained from the relation

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA}$$

which, upon substitution of the known facts, becomes

$$a_B\,\mathbf{i} = \boldsymbol{\omega}_{OA} \times (\boldsymbol{\omega}_{OA} \times \mathbf{r}_{OA}) + \boldsymbol{\omega}_{BA} \times (\boldsymbol{\omega}_{BA} \times \mathbf{r}_{BA}) + \boldsymbol{\alpha} \times \mathbf{r}_{BA}$$

or $\mathbf{a}_B\,\mathbf{i} = -62.83\,\mathbf{k} \times (-62.83\,\mathbf{k} \times (-0.106\,\mathbf{i} + 0.106\,\mathbf{j})) + (-9.63\,\mathbf{k}) \times (-9.63\,\mathbf{k})$
$\times 0.7\,(-0.988\,\mathbf{i} - 0.1515\,\mathbf{j})) + \alpha\,\mathbf{k} \times 0.7(-0.988\,\mathbf{i} - 0.1515\,\mathbf{j})$

or $\mathbf{a}_B\,\mathbf{i} = 418.4\,\mathbf{i} - 418.4\,\mathbf{j} + 64.14\,\mathbf{i} + 9.84\,\mathbf{j} + 0.106\alpha\,\mathbf{i} - 0.69\alpha\,\mathbf{j}$

which may be written as two scalar equations,

$$a_B = 482.54 + 0.106$$

$$0 = -408.4 - 0.69\alpha$$

whence $\alpha = -592 \text{ rad/s}^2$

and $a_B = 420 \text{ m/s}^2$

The acceleration of C may be obtained by utilising the fact that

$$\mathbf{a}_c = \mathbf{a}_A + \mathbf{a}_{CA}$$

or

$$\begin{aligned}\mathbf{a}_c &= 418.4\,\mathbf{i} - 418.4\,\mathbf{j} + \frac{CA}{BA}(64.14\,\mathbf{i} + 9.84\,\mathbf{j} + 0.106\alpha\,\mathbf{i} - 0.69\alpha\,\mathbf{j}) \\ &= 418\,\mathbf{i} - 300\,\mathbf{j}\end{aligned}$$

which shows that the magnitude of the acceleration of C is

$$a_c = \sqrt{418^2 + 300^2} = 514.5\ \text{m/s}^2$$

and it is directed at an angle of

$$\beta = \tan^{-1}\left(\frac{-300}{418}\right) = -35.67^\circ$$

with the line of dead centres.

***Example* 7.6** A quick-return shaping mechanism consists of a crank CA rotating clockwise, as shown in Fig. Ex. 7.6, at 50 revolutions per minute. At an instant when the crank makes 30° with the x-axis, determine the velocity of the ram F moving in the horizontal direction. Determine also the stroke length of the ram and the velocity of the ram, which carries the cutting tool, during the cutting stroke of the ram when the oscillating link OE and the crank CA are vertical.

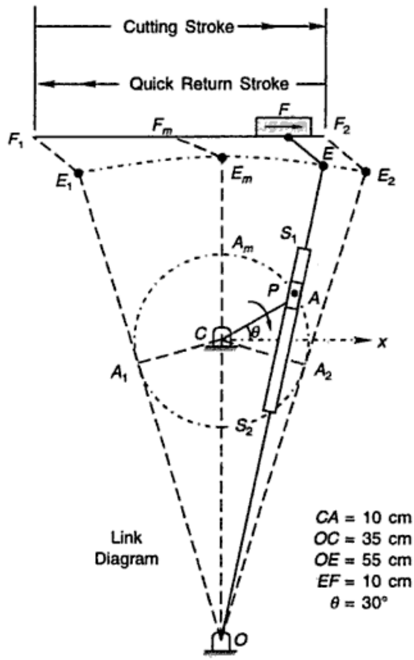


Fig. Ex. 7.6

Solution The link diagram is first drawn to a suitable scale as shown in Fig. Ex. 7.6 (Solution). The angular velocity of the crank CA is

$$\omega = \frac{2\pi \times 50}{60} = 5.24 \text{ rad/s}$$

and the linear velocity of point *A* is

$$V_A = 5.24 \times 0.1 = 0.524 \text{ m/s}$$

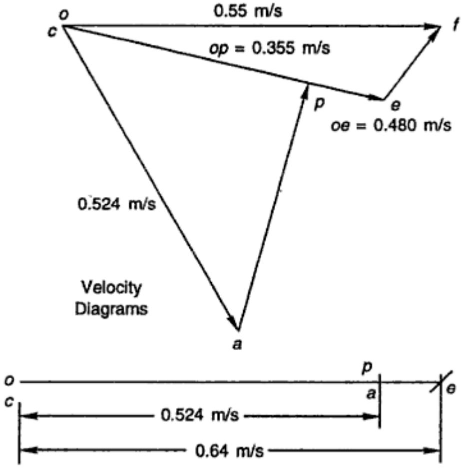


Fig. Ex. 7.6 ***(Solution)***

Consider *P* as a point on link *OE* coincident with the point *A* on the crank *CA*. The points *A* and *C* have, therefore, a relative motion of sliding in the slot s_1s_2 shown in the line diagram.

At the instant shown,

$$\theta = 30°$$

$$OA = OP = 41 \text{ cm}$$

For the velocity diagram, draw *ca* = 0.524 m/s perpendicular to *CA* to represent the velocity of *A*. Since *P* can slide with respect to *A* along the slot, draw *pa* parallel to the slot as shown in Fig. Ex. 7.6 (Solution). Knowing that *P* is on the rotating link *OPE*, the absolute velocity of *P* must be perpendicular to *OPE*; this is shown by drawing *op* perpendicular to *OPE* and thus locating *P*. From the link diagram,

$$\frac{V_E}{V_P} = \frac{OE}{OP}$$

oe is, therefore, mode 55/41 times *op*.

$$oe = 0.355 \times 55/41 = 0.48 \text{ m/s}$$

Since *F* can only move horizontally, *of* is drawn a horizontal line. The velocity of *F* with respect to *E* on the link *EF* must be perpendicular to *EF*; *ef* is drawn thus

The velocity of point S or of the string would be given by

$$\begin{aligned} \mathbf{V}_S &= \mathbf{V}_A + \omega \times \mathbf{r}_{SA} \\ &= 0 + (-20\ \mathbf{k}) \times 0.3\ \mathbf{j} \\ &= 6\ \mathbf{i}\ \text{m/s} \end{aligned}$$

The velocity of point B marked on wheel would be given by

$$\begin{aligned} \mathbf{V}_B &= \mathbf{V}_A + \omega \times \mathbf{r}_{BA} \\ &= 0 + (-20\ \mathbf{k}) \times 0.5\ (\cos 30\ \mathbf{i} + \sin 30\ \mathbf{j}) \\ &= 5\ \mathbf{i} - 8.66\ \mathbf{j}\ \text{m/s} \end{aligned}$$

***Example* 7.10** A link *OAR* rotates anticlockwise at an angular velocity of 1 rad/s. Another link *BCS* at right angles to it has a collar at *B* which slides over *OAR* at 1 m/s and decelerates at 2 m/s^2 with respect to *OAR* as shown in Fig. Ex. 7.10. A collar *D* slides over *BCS* with a velocity of 3 m/s and decelerates at 4 m/s^2 with

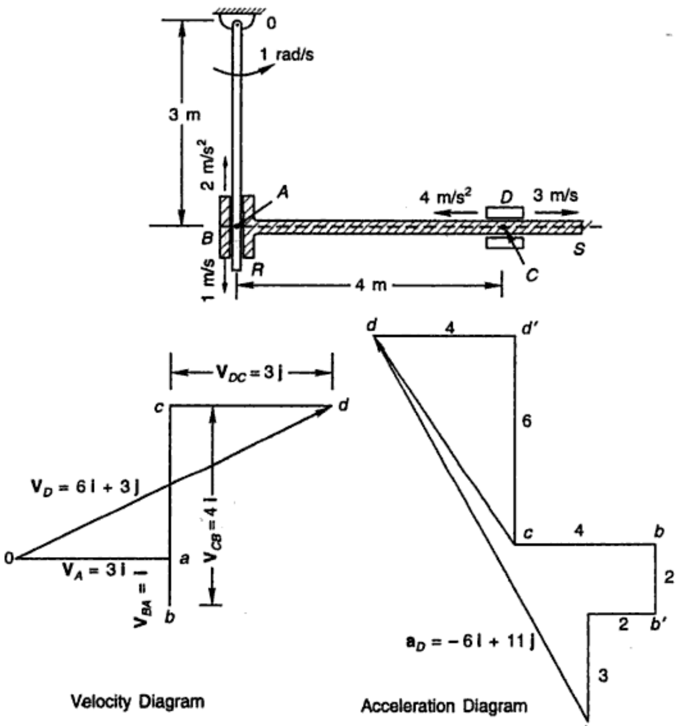


Fig. Ex. 7.10

respect to *BCS* as shown. Obtain the velocity and acceleration of the collar *D* with respect to ground reference at the instant of interest.

Solution There are many alternative methods of locating the moving frame of reference. Accordingly, the methods of solution differ.

Method I

The moving frame can be fixed at point *C* on the link *BCS* such that

$$\omega = 1\ \mathbf{k}\ \text{rad/s}$$

$$\alpha = 0$$

and $$\mathbf{r}_D = 0$$

since the points *C* and *D* are coincident.

$$\mathbf{V}_{Dm} = 3\ \mathbf{i}\ \text{m/s}$$

$$\begin{aligned} \mathbf{V}_{Df} &= \mathbf{V}_{Dm} + \mathbf{V}_C + \omega \times \mathbf{r}_D \\ &= 3\ \mathbf{i} + (1\ \mathbf{k} \times (4\ \mathbf{i} - 3\ \mathbf{j}) - 1\ \mathbf{j}) + 0 \\ &= 6\ \mathbf{i} + 3\ \mathbf{j}\ \text{m/s} \end{aligned}$$

Similarly, $$\mathbf{a}_{Dm} = -4\ \mathbf{i}\ \text{m/s}$$

and $$\begin{aligned} \mathbf{a}_{Df} &= \mathbf{a}_{Dm} + \mathbf{a}_C + \alpha \times \mathbf{r}_D + 2\omega \times \mathbf{V}_{Dm} + \omega \times (\omega \times \mathbf{r}_D) \\ &= -4\ \mathbf{i} + (2\ \mathbf{j} + 1\ \mathbf{k} \times (1\ \mathbf{k} \times (4\ \mathbf{i} - 3\ \mathbf{j})) + 2 \times 1\ \mathbf{k} \times (-1\ \mathbf{j})) \\ &\quad + 0 + 2 \times 1\ \mathbf{k} \times 3\ \mathbf{i} + 0 \\ &= -6\ \mathbf{i} + 11\ \mathbf{j}\ \text{m/s}^2 \end{aligned}$$

Method II

Let the moving frame be fixed at *O* on the link *OAR* such that

$$\omega = 1\ \mathbf{k}\ \text{rad/s}$$

$$\alpha = 0$$

and $$\mathbf{r}_D = 4\ \mathbf{i} - 3\ \mathbf{j}$$

The velocity of the collar *D* with respect to the link *OAR* is

$$\mathbf{V}_{Dm} = (3\ \mathbf{i} - \mathbf{j})\ \text{m/s}$$

and the acceleration of the collar with respect to the link *OAR* is

$$\mathbf{a}_{Dm} = (-4\ \mathbf{i} + 2\ \mathbf{j})\ \text{m/s}^2$$

Velocity of the collar *D* with respect to ground reference is

$$\begin{aligned} \mathbf{V}_{Df} &= \mathbf{V}_{Dm} + \mathbf{V}_0 + \omega \times \mathbf{r}_D \\ &= (3\ \mathbf{i} - \mathbf{j}) + 0 + 1\ \mathbf{k} \times (4\ \mathbf{i} - 3\ \mathbf{j}) \\ &= 3\ \mathbf{i} - \mathbf{j} + 4\ \mathbf{j} + 3\ \mathbf{i} = 6\ \mathbf{i} + 3\ \mathbf{j}\ \text{m/s} \end{aligned}$$

Acceleration of the collar *D* with respect to the ground reference is

$$\mathbf{a}_{Df} = \mathbf{a}_{Dm} + \mathbf{a}_0 + \alpha \times \mathbf{r}_D + 2\omega \times \mathbf{V}_{Dm} + \omega \times (\omega \times \mathbf{r}_D)$$

$$= (-4\,\mathbf{i} + 2\,\mathbf{j}) + 0 + 0 + 2 \times 1\,\mathbf{k} \times (3\,\mathbf{i} - \mathbf{j}) + 1\,\mathbf{k} \times (1\,\mathbf{k} \times (4\,\mathbf{i} - 3\,\mathbf{j}))$$

$$= -4\,\mathbf{i} + 2\,\mathbf{j} + 6\,\mathbf{j} + 2\,\mathbf{i} - 4\,\mathbf{i} + 3\,\mathbf{j}$$

$$= -6\,\mathbf{i} + 11\,\mathbf{j} \text{ m/s}^2$$

Method III

Let us now fix the moving frame at point *A* on the link *OAR* such that

$$\omega = 1\,\mathbf{k} \text{ rad/s}$$

$$\alpha = 0$$

and $$\mathbf{r}_D = 4\,\mathbf{i}$$

Velocity and acceleration of the collar *D* with respect to the link *OAR* are first obtained:

$$\mathbf{V}_{Dm} = (3\,\mathbf{i} - \mathbf{j}) \text{ m/s}$$

$$\mathbf{a}_{Dm} = (-4\,\mathbf{i} + 2\,\mathbf{j}) \text{ m/s}^2$$

Velocity of the collar *D* with respect to the ground reference is then expressed as

$$\mathbf{V}_{Df} = \mathbf{V}_{Dm} + \mathbf{V}_A + \omega \times \mathbf{r}_D$$

$$= (3\,\mathbf{i} - \mathbf{j}) + 1\,\mathbf{k} \times (-3\,\mathbf{j}) + 1\,\mathbf{k} \times (4\,\mathbf{i})$$

$$= 3\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{i} + 4\,\mathbf{j} = 3\,\mathbf{j} + 6\,\mathbf{i} \text{ m/s}$$

Acceleration of the collar *D* with respect to the ground reference is given by

$$\mathbf{a}_{Df} = \mathbf{a}_{Dm} + \mathbf{a}_A + \alpha \times \mathbf{r}_D + 2\,\omega \times \mathbf{V}_{Dm} + \omega \times (\omega \times \mathbf{r}_D)$$

$$= -4\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k} \times (1\,\mathbf{k} \times (-3\,\mathbf{j})) + 0$$

$$+ 2 \times 1\,\mathbf{k} \times (3\,\mathbf{i} - \mathbf{j}) + 1\,\mathbf{k} \times (1\,\mathbf{k} \times 4\,\mathbf{i})$$

$$= -4\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{j} + 6\,\mathbf{j} + 2\,\mathbf{i} - 4\,\mathbf{i}$$

$$= -6\,\mathbf{i} + 11\,\mathbf{j} \text{ m/s}^2$$

Method IV

The moving frame may alternatively be fixed on the intermediate collar *B* but at a location coincident with *A* such that

$$\omega = 1\,\mathbf{k} \text{ rad/s}$$

$$\alpha = 0$$

and $$\mathbf{r}_D = 4\,\mathbf{i}$$

The velocity and acceleration of the collar *D* with respect to the collar *B* on link *BCR* are

$$\mathbf{V}_{Dm} = 3\,\mathbf{i} \text{ m/s}$$

$$\mathbf{a}_{Dm} = -4\,\mathbf{i} \text{ m/s}$$

Method VI

There is no end to the choice and location of the moving frames. Moving frames may be fixed at intermediate locations on the links but no advantage can be gained by such choices. Finally, let us see how we can solve the problem by two extreme choices of the moving frames:

(a) Without fixing the moving frame anywhere.

(b) By fixing the moving frame at the collar D itself.

In case (a), the moving frame is coincident with the fixed frame.

$$\omega = 0$$

$$\alpha = 0$$

The velocity and acceleration in the moving and the fixed frame are such that

$$\mathbf{V}_{Dm} = \mathbf{V}_{Df}$$

$$\mathbf{a}_{Dm} = \mathbf{a}_{Df}$$

The entire thinking of the components and the constitution of $\mathbf{V}_{Df}$ and $\mathbf{a}_{Df}$ is, therefore, done mentally in one long step. This is clearly inconvenient and unmanageable.

In case (b), the moving frame is at the collar D itself such that

$$\mathbf{V}_{Dm} = 0$$

$$\mathbf{a}_{Dm} = 0$$

which leave the entire job of finding out $\mathbf{V}_{Df}$ and $\mathbf{a}_{Df}$ to the application of the formulae in one long step. The method is as inconvenient as that in case (a). Thus, we can state that if a moving frame is desired to be used to advantage then it may not be chosen to be at the object. The moving frame should be located at a convenient point intermediate between the object and the ground reference.

Concept Review Questions

1. State with justifications if the following statements are true or false:
   (a) A rigid body may move along a curved path but may not be in a state of rotation.
   (b) A general plane motion must be a combination of a translation and a rotational motion about an axis perpendicular to the plane.
   (c) There must always be an instantaneous centre of rotation and a centre of angular acceleration of a rigid body no matter what the mode of motion of the body may be.
   (d) The plane motion of a rigid body is characterised by the locus of the instantaneous centre of rotation to be a plane curve.
2. Classify the following motions as translation, rotation, plane motion or space motion:
   (a) A cone rolling on a flat surface.
   (b) A cone sliding on a flat surface.
   (c) A door being shut by turning about the hinges.
   (d) A spherical ball rolling down an incline.
   (e) An aeroplane banked and taking a steady turn.

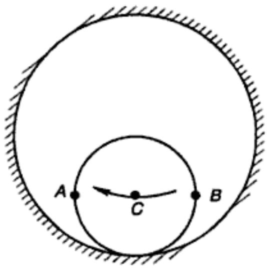


Fig. Prob. 7.2

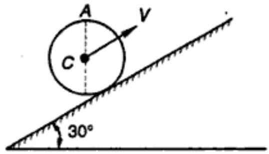


Fig. Prob. 7.3

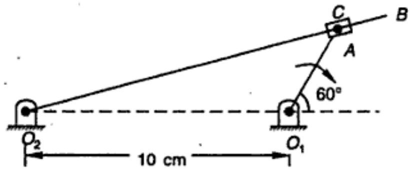
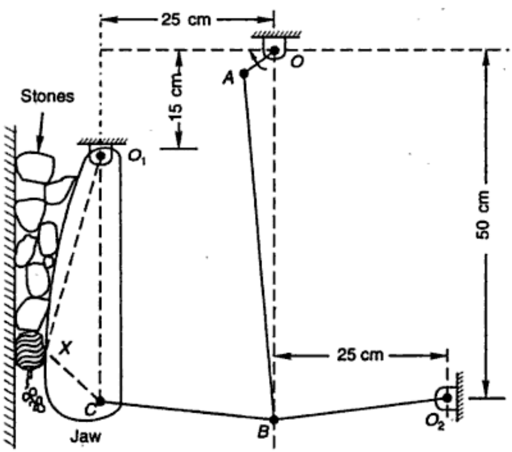


Fig. Prob.7.4

• 7.5 A stone-crusher mechanism is shown in Fig. Prob. 7.5. The crank OA rotates clockwise at 100 revolutions per minute. For the given configuration, determine the velocity and acceleration of the points marked on the crusher jaw. (**Ans.** 0.094 m/s)



OA	= 5 cm	O_2B	= 25.02 cm
AB	= 48 cm	BC	= 25.02 cm
O_1C	= 35 cm	O_1X	= 30 cm
CX	= 10 cm		

Fig. Prob. 7.5

7.6 A straight bar *OA* rotates and accelerates about a fixed axis through *O* as shown in Fig. Prob. 7.6. It carries a collar *C* which slides and accelerates with respect to the bar as shown. Find the total absolute acceleration of the collar.

(**Ans.** (– 260 **i** + 180 **j**) m/s^2)

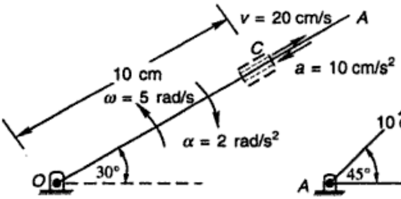


Fig. Prob. 7.6

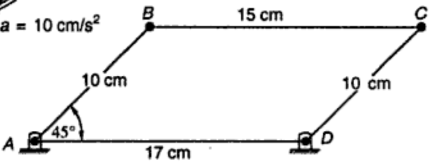


Fig. Prob. 7.7

7.7 In a four-bar mechanism *ABCD* shown in Fig. Prob. 7.7, link *AB* rotates anticlockwise at 5.25 rad/s and accelerates clockwise at 23 rad/s^2. Obtain the angular velocity and angular acceleration of the link *CD*.

(**Ans.** 4.05 rad/s anticlockwise and 17 rad/s^2 anticlockwise)

7.8 A slender bar *AB* slides down a circular surface and on a horizontal surface as shown in Fig. Prob. 7.8. At an instant when $\theta = 45°$ the velocity of the end *A* is 2 m/s. Determine the angular velocity of the bar and velocity of the point of contact on the circular surface. (**Ans.** 7.07 rad/s; 1.414 m/s)

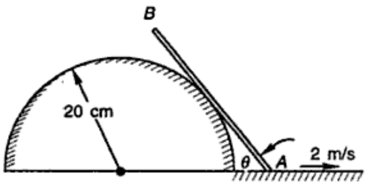


Fig. Prob. 7.8

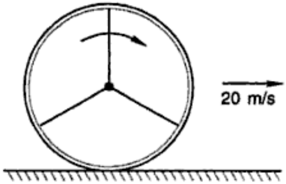


Fig. Prob. 7.9

7.9 A wheel of diameter 0.5 m with three equispaced spokes rolls without slip on a flat surface and proceeds to the right at 20 m/s as shown in Fig. Prob. 7.9. Determine the angular speed of the wheel and velocity of the point where the top spoke joins the rim when it is vertical. (**Ans.** 80 rad/s clockwise; 40 **i** m/s)

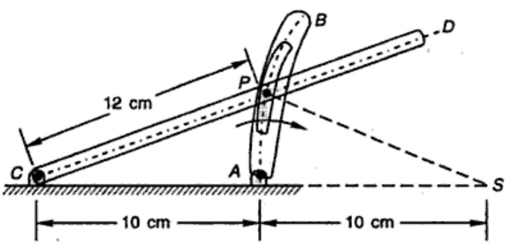


Fig. Prob. 7.10

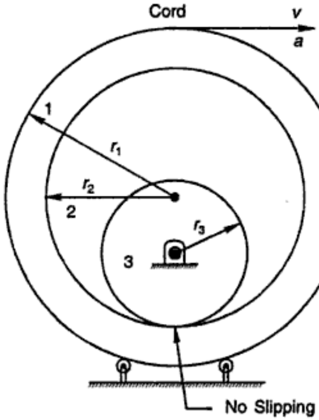


Fig. Prob. 7.13

7.15 The end A of a straight bar moves with a constant tangential velocity v along a semi-cylindrical trough as shown in Fig. Prob. 7.15. Find the velocity of the point B at the point of contact of the bar and the edge of the trough as a function of the angle ϕ between the bar and the horizontal. The bar moves in a plane normal to the axis of the trough. **(Ans.** $v/2 \sin \phi$**)**

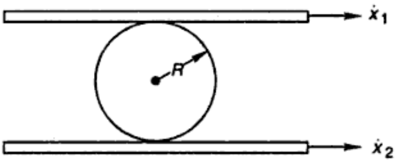


Fig. Prob. 7.14

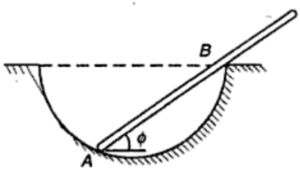


Fig. Prob. 7.15

7.16 In Fig. Prob. 7.16, C is a roller fixed to the link OB and sliding in a slot in QD. Determine the velocity of A when $\theta = 30°$ if QD rotates at constant speed 10 rad/s clockwise. **(Ans.** 4.28 m/s**)**

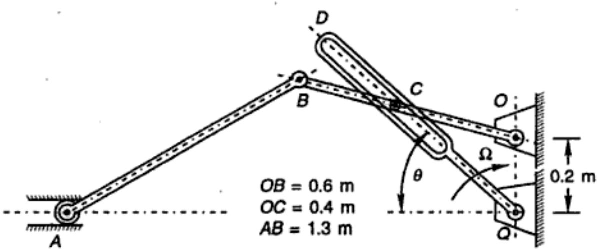


Fig. Prob. 7.16

7.17 Figure Prob. 7.17 shows diagrammatically the quick-return mechanism of a machine tool in which the driving crank *AB* rotates clockwise with a uniform speed of 150 revolutions per minute. What is the velocity and acceleration of the cutting tool at *E* when the crank *AB* makes an angle of 60° with the vertical line passing through the pivots *A* and *C*?

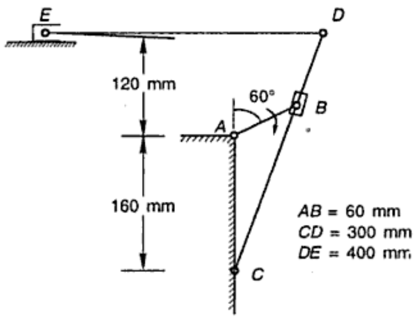


Fig. Prob. 7.17

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. Degrees of freedom of a rigid body imply the
   (a) angles that it may turn through
   (b) angular motions the body can have
   (c) constraints to its motion
   (d) total number of modes of displacement
2. A rigid body, in translation,
   (a) can only move in a straight line
   (b) may move along a straight or curved path
   (c) cannot move on a circular path
   (d) must undergo plane motion only
3. The instantaneous centre of rotation
   (a) should also be the instantaneous centre of acceleration
   (b) is a hypothetical concept to solve problems
   (c) can exist for any space motion
   (d) must exist for any plane motion
4. For the motion of a rigid link in any mechanism,
   (a) the velocity of one of the ends should be zero
   (b) the velocity of one end with respect to that of the other should be perpendicular to the link
   (c) the acceleration of one end with respect to that of the other should be perpendicular to the link
   (d) the two ends of the link may have different components of velocity along the link

Answers to Multiple-Choice Questions

1 (d), 2 (b), 3 (d), 4 (b)

8 Moment of Inertia: Area and Mass

8.1 INTRODUCTION

The area of a surface and mass of a body are important concepts but not less important are the concepts of relative distribution of area and mass over the domains. The shape and orientation of a surface with respect to some reference axes are as vital as the shape and orientation of a body relative to some reference frame in many circumstances. Quantitative estimates of the relative distribution of area and mass over the regions of interest are made by the concepts of 'moment of inertia' and 'radius of gyration'. The former is the second moment and the latter a length concept emanating from the second moment.

The concept of inertia is provided by Newton's first law of motion. The property of matter by virtue of which it resists any change in its state of rest or of uniform motion is called *inertia*. the translatory inertia is identified as *mass* whereas the rotational inertia is termed as *moment of inertia*. In other words, the moment of inertia is the rotational analogue of mass, i.e., it plays the role of resisting a change in rotational motion in quite the same sense as mass plays the role of resisting a change in translatory motion.

The concepts of the moment of inertia and radius of gyration are developed for an area and a mass in quite the same way. The area moment of inertia and mass moment of inertia will, therefore, be dealt with together. It is shown in the text that for thin bodies of uniform thickness and homogeneous density, the area moment of inertia and the mass moment of inertia are directly related.

It may appear, in the first instance, that the same notation for the area-inertia as for the mass-inertia is confusing particularly when the words 'area' and 'mass' are not used for specifying the moments of inertia. But there is no confusion because the moments of inertia refer to the area or mass in question and it is unnecessary to qualify the moment of inertia once it is known whether it refers to an area or a mass.

8.2 INERTIAL CONCEPTS: AREA

Consider a plane area A as shown in Fig. 8.1. Let the reference axes be xy in the plane of the area as indicated. The moments of inertia of the area about the x and y axes are defined as the second moments of the area about the x and y axes respectively.

$$\boxed{I_{xx} = \int_A y^2 \, dA} \tag{8.1}$$

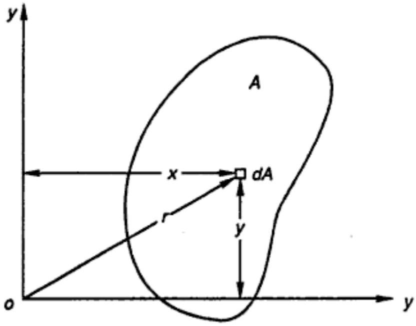


Fig. 8.1 ***Consideration for Inertial Concepts***

$$\boxed{I_{yy} = \int_A x^2 \, dA} \tag{8.2}$$

It follows from the definitions that the moments of inertia of an area cannot be negative whether the coordinates of its elements are positive or negative because it is the summation of the product of square quantities, i.e., x^2 or y^2 and dA. It can also be observed that the moment of inertia for an element farther from the axis is more and the moment of inertia of an element on the axis is zero.

The moments of inertia may also be written as

$$I_{xx} = k_x^2 A \tag{8.3}$$

$$I_{yy} = k_y^2 A \tag{8.4}$$

where k_x and k_y are called the corresponding radii of gyration. The coordinates k_x and k_y locate a point in the area which depend upon the shape of the area and its relationship with the reference axes. *The radius of gyration k is the effective distance where the entire area may be considered to be located with respect to the axis of rotation.*

Comparing the forms of writing the moments of inertia,

$$I_{xx} = k_x^2 \, A = \int_A y^2 \, dA$$

$$k_x = \sqrt{I_{xx} / A} = \sqrt{\int_A (y^2 \, dA)/A}$$

and

$$k_y = \sqrt{I_{yy} / A} = \sqrt{\left(\int_A x^2 \, dA\right)/A}$$

The product of inertia relates an area to a set of axes. For example,

$$I_{xy} = \int_A xy\, dA \tag{8.5}$$

which may be negative or positive depending upon the location of the area with respect to the reference axes. In particular, the product of inertia is zero if the area is symmetrical about any of the axes.

The polar moment of inertia of an area A about an axis normal to the area and passing through a pole O is defined as

$$I_0 = \int_A r^2\, dA$$

$$= \int_A (x^2 + y^2)\, dA$$

$$= \int_A y^2\, dA + \int_A x^2\, dA$$

or $$I_0 = J_0 = I_{xx} + I_{yy} \tag{8.6}$$

It is thus noted that, by definition, the polar moment of inertia about an axis through a pole must be the sum of the moments of inertia about the axes through the pole in the plane of the area. The polar moment of inertia is also denoted by J in literature.

Also,

$$J_0 = I_0 = k_0^2 A$$

$$= I_{xx} + I_{yy}$$

$$= k_x^2 A + k_y^2 A$$

and $$k_0^2 = k_x^2 + k_y^2 \tag{8.7}$$

***Example* 8.1** Determine the moment of inertia of a rectangle with sides b and h about an axis coincident with side b. Determine also the radius of gyration about this axis.

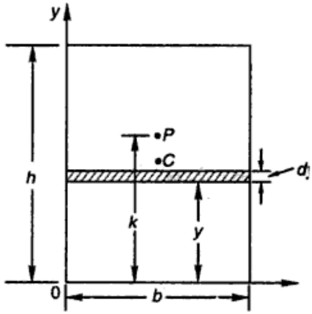


Fig. Ex. 8.1

Solution Let the x-axis be along the side b and y-axis along h. It is convenient to consider a strip-element of width dy and area.

$$dA = b\, dy$$

The moment of inertia of the strip-element about the x-axis is

$$dI_{xx} = y^2\, dA = y^2 b\, dy$$

and the moment of inertia of the entire rectangle is

$$I_{xx} = \int_0^h y^2 b\, dy = \frac{bh^3}{3}$$

Expressing it in terms of the radius of gyration,

$$I_{xx} = k^2 A = k^2 bh$$

and comparing with $bh^3/3$,

$$k = \frac{h}{\sqrt{3}}$$

The point P located by the radius of gyration is, in general, different from C, the centroid of the area because the latter is related to the first moment of area whereas the former depends upon the second moment of area. The point P is also a function of the axes whereas C is not. Moreover, the point P is not unique whereas C is uniquely located for a given area. An axis passing through the centroid and lying in the plane is called a *centroidal axis.*

***Example* 8.2** Determine the moments of inertia of a circular area about the centroidal axes. Determine also the radius of an equivalent cylindrical surface of the same area for the same polar moment of inertia.

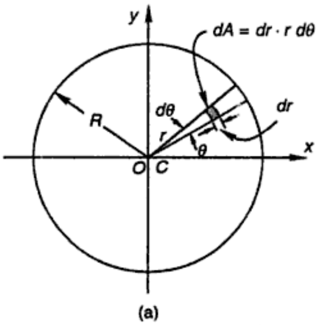


Fig. Ex. 8.2(a)

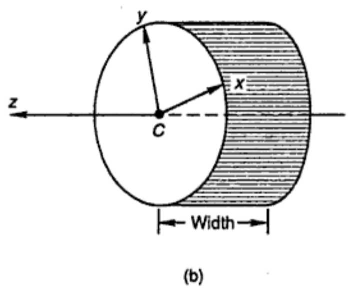


Fig. Ex. 8.2(b)

Solution Considering a small element of area

$$dA = dr \cdot r\, d\theta$$

as shown in the figure and noting its moments of inertia,

$$dI_{xx} = (r \sin\theta)^2\, dr \cdot r\, d\theta$$

Integrating over the circular area,

$$I_{xx} = \int_{r=0}^{R} \int_{\theta=0}^{2\pi} r^3 \sin^2\theta\, d\theta\, dr$$

$$= \pi \int_{r=0}^{R} r^3 \, dr = \frac{\pi R^4}{4}$$

Similarly,

$$I_{yy} = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r^3 \cos^2 \theta \, d\theta \, dr = \frac{\pi R^4}{4}$$

and $$I_{zz} = I_0 = I_{xx} + I_{yy}$$

$$= \frac{\pi R^4}{4} + \frac{\pi R^4}{4} = \frac{\pi R^4}{2}$$

The radii of gyration about the *x*, *y* and *r* axes are:

$$k_x = \sqrt{\left(\frac{\pi R^4}{4}\right) \Big/ \pi R^2} = \sqrt{\frac{R^2}{4}} = \frac{R}{2}$$

$$k_y = k_x = \frac{R}{2}$$

and $$k_z = \sqrt{\left(\frac{\pi R^4}{2}\right) \Big/ \pi R^2} = \sqrt{\frac{R^2}{2}} = \frac{R}{\sqrt{2}}$$

The radius of the equivalent cylindrical surface having the same polar moment of inertia as the circular area is, therefore, $R/\sqrt{2}$. The width of the cylinder is given by

$$2\pi \frac{R}{\sqrt{2}} \times \text{width} = \pi R^2$$

whence $$\text{width} = \frac{R}{\sqrt{2}}$$

which is the same as the radius of the cylinder.

***Example* 8.3** Calculate the polar area moment of inertia and the radius of gyration for the area of a ring of radii R_1 and R_2.

Solution The polar moment of inertia of a circular area can be determined conveniently by considering concentric ring-elements. For an elementary ring of width *dr* at a radius *r*, the elementary area is

$$dA = 2\pi r \, dr$$

and its polar moment of inertia is

$$dI_0 = r^2 \cdot 2\pi r \, dr = 2\pi r^3 \, dr$$

$$I_0 = \frac{\pi R_2^4}{2} - \frac{\pi R_1^4}{2}$$

$$= \frac{\pi (R_2^4 - R_1^4)}{2}$$

and the corresponding radius of gyration is again obtained as above.

8.3 PARALLEL-AXIS THEOREM: AREA

The parallel-axis theorem, also known as the transfer theorem, permits us to relate the moment of inertia I_{aa} of an area with respect to a given axis *aa* to the moment of inertia I_{cc} of the area with respect to a centroidal axis *cc* parallel to *aa*.

By definition,

$$I_{aa} = \int_A r^2 \, dA$$

Substituting $r = r_c + s$, as shown in Fig. 8.2,

$$I_{aa} = \int_A r_c^2 \, dA + \int_A 2r_c s \, dA + \int_A s^2 \, dA$$

The first term on the right-hand side represents the moment of inertia of the area about the centroidal axis *cc* and the second term vanishes because it is

$$\int_A 2sr_c \, dA = 2s \int_A r_c \, dA$$

and the first moment of area about the centroid is zero by the definition of centroid. The third term is

$$\int_A s^2 \, dA = s^2 \int_A dA = s^2 A$$

Finally,

$$\boxed{I_{aa} = I_{cc} + s^2 A} \quad (8.8)$$

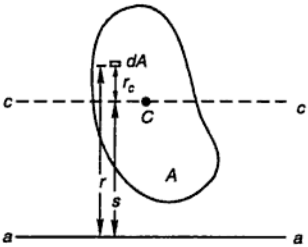


Fig. 8.2 ***Parallel Axes***

This is the statement of the parallel-axis theorem. In words, the moment of inertia of an area about an axis *aa* is in excess of the moment of inertia of the area about a parallel centroidal axis *cc* by a positive amount s^2A where s is the distance between the axes *aa* and *cc*. Obviously, the moment of inertia of an area is the least about an axis passing through the centroid.

The parallel-axis theorem for products of inertia appears in the form

$$\underset{\text{(for any set of axes)}}{I_{xy}} = \underset{\text{(for a parallel set of centroidal axes)}}{I_{x'y'}} + s_1 s_2 A \qquad (8.9)$$

where s_1 and s_2 are the perpendicular distances from the centroid to the x and y axes respectively.

The parallel-axis theorem for the polar moment of inertia of the area states that

$$I_0 = I_c + s^2 A \qquad (8.10)$$

where s is the distance between O and C.

8.4 MOMENT OF INERTIA OF COMPOSITE SECTIONS

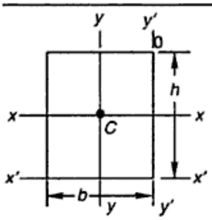
The inertial concepts can be applied to determine the moments and products of inertia of plane sections. The values for simpler geometrical shapes are obtained by integration and remembering these values, the moments and product of inertia for composite sections can be worked out. The moment of inertia of simple plane sections are tabulated in Table 8.1, whereas those for composite sections are computed by subdividing the area into its components. Let a composite area A have its components A_1, A_2, A_3, ... of which the moments of inertia about the axis in question are obtainable from Table 8.1 and by the application of the parallel-axis theorem. Let these moments of inertia be I_1, I_2, I_3, ... respectively. The moment of inertia of the entire area A about an axis is the algebraic sum of the moments of inertia of its component areas about the same axis:

$$I = I_1 + I_2 + I_3 + ... \qquad (8.11)$$

It may be mentioned that the composite area can be made up of additive or subtractive component areas. The moment of inertia of a component area may, therefore, be additive or subtractive in the algebraic summation to compute the moment of inertia of the composite area. It should be noted that *the radius of gyration k for the composite area about an axis is not equal to the sum of the radii of gyration of the component areas about the same axis*:

$$k \neq k_1 + k_2 + k_3 + ...$$

Table 8.1(a) Moment of Inertia of Plane Figures

Figures	*Description*	*Area*	*Moments of inertia*
	Rectangle (Upright)	bh	$I_{xx} = \frac{bh^3}{12}$; $I_{x'x'} = \frac{bh^3}{3}$ $I_{yy} = \frac{hb^3}{12}$; $I_{y'y'} = \frac{hb^3}{3}$ $J_c = \frac{bh}{12}(b^2 + h^2)$ $J_0 = \frac{bh}{3}(b^2 + h^2)$

$$I_{xx} = \frac{\pi(0.15^4)}{4} + (\pi \times 0.15^2) \times 0.85^2$$

(for circle removed)

$$= 0.0515 \text{ m}^4$$

$$I_{xx} = \frac{\pi \times 0.15^4}{4} + (\pi + 0.15^2) \times 1.15^2$$

(for circle added)

$$= 0.0939 \text{ m}^4$$

The moment of inertia for the composite area is, therefore,

$$I_{xx} = 0.1667 - 0.0515 + 0.0939 = 0.2091 \text{ m}^4$$

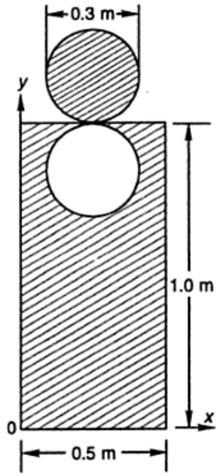


Fig. Ex. 8.4

Part(b)

The product of inertia about the base and left side can be calculated as follows:

$$I_{xy} = 0 + (1 \times 0.5) \times 0.5 \times 0.25$$

(for rectangle)

$$= 0.0625 \text{ m}^4$$

$$I_{xy} = 0 + (\pi \times 0.15^2) \times 0.85 \times 0.25$$

(for circle removed)

$$= 0.0150 \text{ m}^4$$

$$I_{xy} = 0 + (\pi \times 0.15^2) \times 1.15 \times 0.25$$

(for circle added)

$$= 0.0203 \text{ m}^4$$

For the composite area,

$$I_{xy} = 0.0625 - 0.0150 + 0.0203 = 0.0678 \text{ m}^4$$

Part (c)

The polar moment of inertia about the pole O can be calculated in a similar way.

$$I_0 = \frac{1}{12}(0.5 \times 1.0) \times (0.5^2 + 1^2) + (0.5 \times 1.0) \times (0.25^2 + 0.5^2)$$

(for rectangle)

$$= 0.2084 \text{ m}^4$$

$$I_0 = \frac{\pi \times 0.15^4}{2} + (\pi \times 0.15^2) \times (0.85^2 + 0.25^2)$$

(for circle removed)

$$= 0.0563 \text{ m}^4$$

$$I_0 = \frac{\pi \times 0.15^4}{2} + (\pi \times 0.15^2) \times (1.15^2 + 0.25^2)$$

(for circle added)

$$= 0.0983 \text{ m}^4$$

For the composite area,

$$I_0 = 0.2084 - 0.0563 + 0.0983 = 0.250 \text{ m}^4$$

***Example* 8.5** Determine the moments of inertia with respect to the centroidal axes of the wide-flange beam section shown in Fig. Ex. 8.5.

Solution From the symmetry of the section, the centroid can be located by inspection. The moment of inertia of the composite section can be determined by different choices of the subdivisions. Let us consider some choices for I_{xx}.

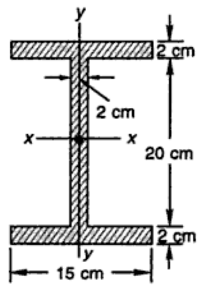


Fig. Ex. 8.5

Choice I

By subdividing it into three rectangles; 15 cm × 2 cm at top, 15 cm × 2 cm at the bottom and 20 cm × 2 cm in the middle.

$$I_{xx} = 2 \times \left[\frac{15 \times 2^3}{12} + (15 \times 2) \times 11^2\right]$$

(for the top and bottom rectangles)

$$= 7280 \text{ cm}^4$$

$$I_{xx} = \frac{2 \times 20^3}{12} = 1333 \text{ cm}^4$$

(for the middle rectangle)

I_{xx} of the composite section = 7280 + 1333 = 8613 cm^4

Choice II

By subdividing it into five rectangles, 24 cm × 2 cm in the middle and two each above and below of 6.5 cm × 2 cm

$$I_{xx} = 4 \times \left[\frac{6.5 \times 2^2}{12} + (6.5 \times 2) \times 11^2\right]$$

(for the four rectangles at top and bottom)

$$= 6309 \text{ cm}^4$$

$$I_{xx} = \frac{2 \times 24^3}{12} = 2304 \text{ cm}^4$$

(for the middle rectangle)

with respect to the ox'–oy' axes inclined at an angle θ with ox

$$I_{x'x'} = \int_A (y')^2 \, dA$$

$$= \int_A (-x \sin \theta + y \cos \theta)^2 \, dA$$

$$= I_{yy} \sin^2 \theta + I_{xx} \cos^2 \theta - 2 I_{xy} \sin \theta \cos \theta$$

or, by using the identities,

$$\cos^2 \theta = (1 + \cos 2\theta)/2$$

$$\sin^2 \theta = (1 - \cos 2\theta)/2$$

$$2 \sin \theta \cos \theta = \sin 2\theta$$

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\theta - I_{xy} \sin 2\theta \qquad (8.13)$$

Similarly,

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{yy} - I_{xx}}{2} \cos 2\theta + I_{xy} \sin 2\theta \qquad (8.14)$$

and

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} \sin 2\theta + I_{xy} \cos 2\theta \qquad (8.15)$$

These equations permit us to determine the moment of inertia and products of inertia of an area about any set of axes with an origin O from the knowledge of I_{xx}, I_{yy} and I_{xy} about a known set of axes with the same origin.

It can be appreciated that these equations are the parametric equations of a circle. Eliminating θ, by using the identity

$$\sin^2 2\theta + \cos^2 2\theta = 1$$

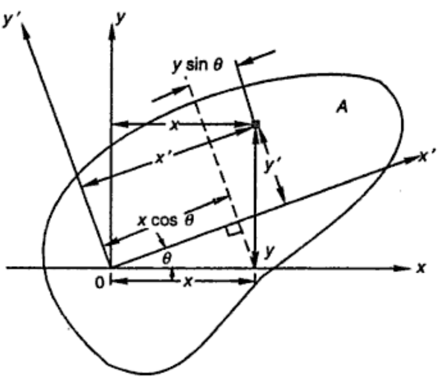


Fig. 8.4 Rotation of Axes

We obtain,

$$\left(I_{x'x'} - \frac{I_{xx} + I_{yy}}{2}\right)^2 + I_{x'y'}^2 = \left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2$$

Setting the average moment of inertia

$$I_{av} = \frac{I_{xx} + I_{yy}}{2} \tag{8.16}$$

and

$$R = \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2} \tag{8.17}$$

it can be rewritten as

$$(I_{x'x'} - I_{av})^2 + I_{x'y'}^2 = R^2 \tag{8.18}$$

which is the equation of a circle of radius R centered at a point located by $(I_{av}, 0)$ on the abscissa $I_{x'x'}$ and ordinate $I_{x'y'}$. This circle is often called *Mohr's circle*.

A typical point P on the circle denotes that the moment of inertia about an axis represented by radius vector CP is $I_{x'x'}$ and the product of inertia with reference to the axes is $I_{x'y'}$ as shown in Fig. 8.5. In particular, the product of inertia at A and B is zero. The moment of inertia $I_{x'x'}$ is also an extremum at these points; I_{max} at A and I_{min} at B. These points are located by setting

$$I_{x'y'} = 0$$

whence

$$\tan(2\theta_m) = -\frac{2I_{xy}}{I_{xx} - I_{yy}} \tag{8.19}$$

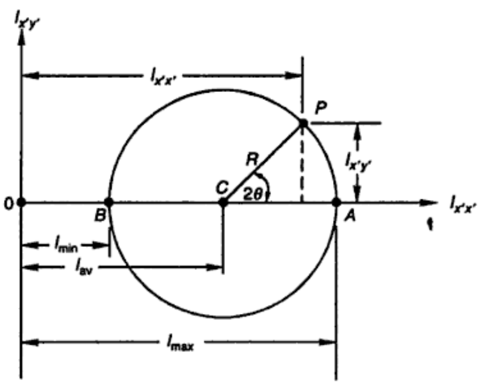


Fig. 8.5 ***Mohr's Circle Diagram***

If the sides of the rectangle were equal, i.e., if it was a square, i.e., $b = h$, then $\tan 2\theta_m$ would be infinite and $\theta_m = 45°$ and 135°, i.e., along the diagonal and perpendicular to it. The principal axes for a square at one of its corners must be along the diagonal through that corner and perpendicular to it. this is, however, not the case for a rectangle, as has been seen above. Let us calculate the moment of inertia about one of its diagonals, say *OP*. For this diagonal,

$$\tan \theta = \frac{h}{b} = 1.5$$

$$\theta = 56.31°$$

$$2\theta = 112.62°$$

$$\sin (2\theta) = 0.923 \qquad \cos (2\theta) = -0.385$$

Employing the expressions for the moments of inertia from Eqs. (8.13) and (8.14),

$$I_{x''x''} = 13 + 5 \times (-0.385) - 9 \times 0.923$$

$$= 2.77 \text{ m}^4$$

$$I_{y''y''} = 13 + 5 \times (0.385) + 9 \times 0.923$$

$$= 19.38 \text{ m}^4$$

It can be verified that

$$I_{x''y''} \neq 0$$

8.6 INERTIAL CONCEPTS: MASS

The mass *moment of inertia* is a measure of its inertial behaviour, i.e., resistance to the rotational acceleration of the mass of the body. Consider a body of mass m whose distribution with reference to a Cartesian frame of reference xyz is known.

Let an element of mass dm, also denoted by

$$dm = \rho \, dV$$

in terms of the mass density and its volume be located by a position vector $\mathbf{r}$,

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

as shown in Fig. 8.6.

The mass moment of inertia of the element about the x, y and z axes respectively are defined as follows:

$$dI_{xx} = (y^2 + z^2)\, dm \tag{8.23}$$

$$dI_{yy} = (x^2 + z^2)\, dm \tag{8.24}$$

$$dI_{zz} = (x^2 + y^2)\, dm \tag{8.25}$$

The moment of inertia of an element about an axis is given by the product of the mass element and the square of the perpendicular distance from the axis. The moment of inertia about the x-axis is shown in Fig. 8.6.

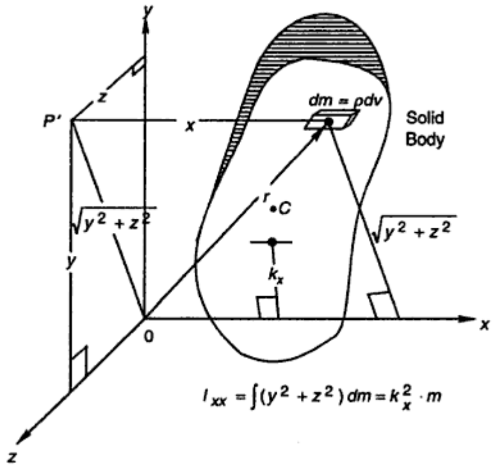


Fig. 8.6 ***Mass Moment of Inertia***

The mass moments of inertia of that entire body is, therefore,

$$I_{xx} = \int (y^2 + z^2)\,dm = \int_V (y^2 + z^2)\,\rho\,dv \tag{8.26}$$

$$I_{yy} = \int (x^2 + z^2)\,dm = \int_V (x^2 + z^2)\,\rho\,dv \tag{8.27}$$

$$I_{zz} = \int (x^2 + y^2)\,dm = \int_V (x^2 + y^2)\,\rho\,dv \tag{8.28}$$

The mass products of inertia of the body are similarly defined as

$$I_{xy} = \int xy\,\rho\,dv = I_{yx} \tag{8.29}$$

$$I_{xz} = \int xz\,\rho\,dv = I_{zx} \tag{8.30}$$

$$I_{yz} = \int yz\,\rho\,dv = I_{zy} \tag{8.31}$$

about the pairs of the axes specified in the indices.

For a given body, there are nine elements of moments and products of inertia defined above but only six of them are mutually independent; the set depends upon the mass distribution in the body and its relative orientation with respect to the reference axes These are

$$I_{xx}, I_{yy}, I_{zz} \quad \text{and} \quad I_{xy}, I_{yz}, I_{zx}$$

The sum of the moments of inertia at a point in space for a given body is invariant. This important property can be proved readily by addition and showing that the sum is

$$I_{xx} + I_{yy} + I_{zz} = \int_V 2\,|\mathbf{r}|^2\,\rho\,dv \tag{8.32}$$

which is constant for the chosen point *irrespective of the inclination of the reference axes at that point.*

It can be observed from the definitions that the moments of inertia must be positive quantities whereas the products of inertia can be positive or negative. If two axes of a body form a plane of symmetry for the mass distribution of the body, the products of inertia related to the normal to the plane of symmetry must be zero. The products of inertia may also vanish at a point for a pair of axes other than the axes of symmetry.

The moments of inertia may also be expressed in terms of the corresponding radii of gyration

$$I_{xx} = k_x^2\, m \tag{8.33}$$

$$I_{yy} = k_y^2\, m \tag{8.34}$$

$$I_{zz} = k_z^2\, m \tag{8.35}$$

The radius of gyration of a body is that distance from the axis of rotation where the entire mass of the body may be assumed to be concentrated for the same mass moment of inertia as the body offers.

The mass moments of inertia and the mass products of inertia of a body may be visualised as the nine components of a matrix called the inertia matrix

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

The diagonal components are the moments of inertia about the x, y and z axes whereas the other components are the products of inertia. As already shown,

$$I_{xy} = I_{yx} \qquad I_{xz} = I_{zx} \qquad \text{and} \qquad I_{yz} = I_{zy}$$

The inertia matrix is, therefore, expressed as

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

***Example* 8.7** A uniform slender rod of length l is supported and made to rotate about an axis through the bar at distance l_1 from one end. Obtain an expression for the moment of inertia about the axis and hence evaluate the moment of inertia if the rod is supported

(a) at its end
(b) at its mid-point.

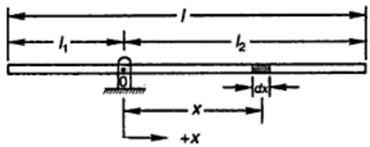


Fig. Ex. 8.7

Solution Consider an elementary length dx of the rod at a distance x from the axis of rotation. The mass of the element is

$$dm = \rho a \, dx$$

where ρ is the uniform density and a is the cross-section area

The mass of the rod can be seen to be

$$m = \rho a \, l$$

The moment of inertia of the element about the axis of rotation is

$$dI_0 = x^2 \, dm = x^2 \rho a \, dx \tag{i}$$

The moment of inertia of the entire rod is obtained by integrating it between the end limits of the rod.

$$I_0 = \int_{-l_1}^{+l_2} \rho a x^2 \, dx \tag{ii}$$

$$= \rho a \left| \frac{x^3}{3} \right|_{-l_1}^{+l_2}$$

$$= \rho a \frac{(l_1^3 + l_2^3)}{3} \tag{iii}$$

where $\quad l_1 + l_2 = l$ is the length of the rod.

In terms of the mass of the rod

$$m = \rho a \, l$$

the moment of inertia is expressed as

$$I_0 = \frac{1}{3} m \frac{(l_1^3 + l_2^3)}{l} \tag{iv}$$

(a) If the rod is supported at an end, either l_1 or l_2 is zero depending upon at which end it is supported. For support at its left end,

$$l_1 = 0 \quad \text{and} \quad l_2 = l$$

and $$I_0 = \frac{1}{3}\rho a l^3 = \frac{1}{3} m l^2 \qquad \text{(v)}$$

(b) If the rod is supported at its mid-point,

$$l_1 = l_2 = \frac{1}{2} l$$

and $$I_0 = \frac{1}{12}\rho a l^3 = \frac{1}{12} m l^2 \qquad \text{(vi)}$$

***Example* 8.8** A rectangular prism of cross-section ($a \times b$) and uniform density ρ has a length l. Determine its moments of inertia and the products of inertia about the longitudinal and transverse axes passing through the centre of mass.

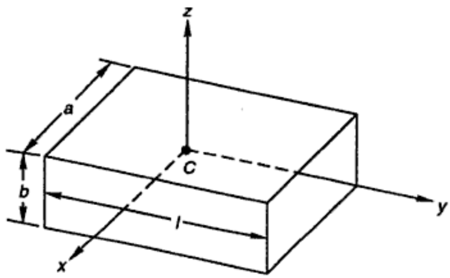


Fig. Ex. 8.8

Solution For the rectangular prism and coordinate axes through C as shown in Fig. Ex. 8.8.

$$I_{xx} = \int_V (y^2 + z^2)\rho \, dv$$

$$= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} \int_{-a/2}^{a/2} (y^2 + z^2)\rho \, dx \, dy \, dz$$

$$= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} (y^2 + z^2) a\rho \, dy \, dz$$

$$= \int_{-b/2}^{b/2} \left(\frac{l^3}{12} + z^2 l\right) a\rho \, dz$$

$$= \frac{\rho a b l^3}{12} + \frac{\rho a b^3 l}{12} = \rho a b l(l^2 + b^2)/12$$

$$= \rho v \frac{l^2 + b^2}{12} = m \frac{l^2 + b^2}{12}$$

Similarly,

$$I_{yy} = \int_V (x^2 + z^2)\rho \, dv$$

$$= m \frac{a^2 + b^2}{12}$$

$$I_{zz} = \int_V (x^2 + y^2)\rho \, dv$$

$$= m \frac{a^2 + l^2}{12}$$

The products of inertia are obtained as follows:

$$I_{xy} = \int_V xy\rho \, dv$$

$$= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} \int_{-a/2}^{a/2} xy\rho \, dx \, dy \, dz$$

$$= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} \frac{1}{2} y \left(\frac{a^2}{4} - \frac{a^2}{4} \right) dy \, dz$$

$$= 0$$

This result was expected from the symmetry; the above integration is only as an exercise. Similarly, by symmetry,

$$I_{yx} = I_{yz} = I_{xz} = 0$$

It may be verified that if three axes were drawn through a corner instead of the centre of mass C, the results would have been

$$I_{xx} = m \frac{l^2 + b^2}{3}$$

$$I_{yy} = m \frac{a^2 + b^2}{3}$$

$$I_{zz} = m \frac{a^2 + l^2}{3}$$

and the products of inertia would not vanish due to lack of symmetry

$$I_{xy} = m \frac{al}{4} = I_{yx}$$

$$I_{xz} = m\frac{ab}{4} = I_{zx}$$

$$I_{yz} = m\frac{bl}{4} = I_{zy}$$

8.7 MOMENTS OF INERTIA OF THIN PLATES

Let us now consider the moments of inertia of the uniform plates of homogeneous material where the reference axes x and y are contained in the plane and z is normal to the plane of the plate. Consider a plate of uniform thickness t and mass density ρ as shown in Fig. 8.7

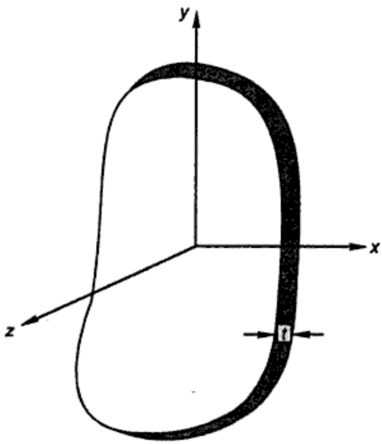


Fig. 8.7 ***A Thin Plate of Uniform Thickness***

$$\underset{\text{(mass)}}{I_{xx}} = \int_V (y^2 + z^2)\rho\, dv$$

$$= \rho \int_V (y^2 + z^2)\, dV = \rho \int_A (y^2 + z^2) t\, dA$$

$$= \rho t \int_A (y^2 + z^2)\, dA$$

Hence

$$\boxed{\underset{\text{(mass)}}{I_{xx}} = \rho t \underset{\text{(area)}}{I_{xx}}} \qquad (8.36)$$

and $$y = 4 \qquad \text{for } x = 0$$

it can be seen that

$$x = -0.75y + 3$$

The length of the strip is thus

$$2x = 6 - 1.5y$$

and the area of the strip is

$$2x\,dy = (6 - 1.5y)\,dy$$

whose moment of inertia is

$$dI_{xx} = (6 - 1.5y)\,y^2\,dy = (6y^2 - 1.5y^3)\,dy$$

Therefore, the moment of inertia of the entire area is

$$I_{xx} = \int_A dI_{xx} = \int_0^4 (6y^2 - 1.5\,y^3)\,dy$$

$$= \left|2y^3 - \frac{3}{8}y^4\right|_0^4 = 32 \text{ m}^4$$

$$I_{xx\,\text{mass}} = 5000 \times 0.1 \times 32 = 16{,}000 \text{ kg m}^2$$

$$\text{Mass of the sheet} = \frac{6 \times 4}{2} \times 5000 \times 0.1$$

$$= 6000 \text{ kg}$$

The radius of gyration is, therefore,

$$k = \sqrt{\frac{16{,}000}{6000}} = 1.63 \text{ m}$$

8.8 PARALLEL-AXIS THEOREM (MASS MOMENT OF INERTIA)

The parallel-axis theorem for the mass moment of inertia states that the mass moment of inertia with respect to any axis is equal to the moment of inertia with respect to a parallel axis through the centre of mass plus the product of the mass and the square of the perpendicular distance between the axes. Mathematically,

$$\boxed{I_{aa} = I_{cc} + s^2\,\text{m}} \tag{8.38}$$

where I_{aa} is the mass moment of inertia about an axis *aa* and I_{cc} is the mass moment of inertia about an axis parallel to *aa* and passing through the centre of mass *C*. The two axes are *s* apart and the mass of the body is *m*.

With reference to Fig. 8.8 where a thin strip of mass dm is taken parallel to either of the parallel axes and the distances $\mathbf{r}$, $\mathbf{r}_c$ and $\mathbf{s}$ are measured in a plane perpendicular to the parallel axes and passing through C

$$I_{aa} = \int r^2 \, dm$$

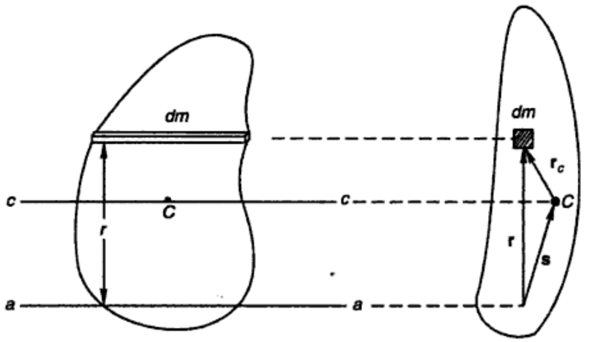


Fig. 8.8 ***Parallel Axes for Moments of Inertia***

Substituting $\mathbf{r} = \mathbf{r}_c + \mathbf{s}$ or $r^2 = \mathbf{r} \cdot \mathbf{r} = (\mathbf{r}_c + \mathbf{s}) \cdot (\mathbf{r}_c + \mathbf{s})$

$$= r_c^2 + 2\mathbf{r}_c \cdot \mathbf{s} + s^2$$

$$I_{aa} = \int r_c^2 \, dm + \int 2\mathbf{s} \cdot \mathbf{r}_c \, dm + \int s^2 \, dm$$

$$= I_{cc} + s^2 m$$

because the middle term vanishes by virtue of the definition of the centre of mass, i.e.,

$$\int 2\,\mathbf{s} \cdot \mathbf{r}_c \, dm = 2\mathbf{s} \cdot \int \mathbf{r}_c \, dm = 0$$

Similarly, the parallel axes theorem for the mass products of inertia states that

$$I_{xy} = I''_{xy} + s_1 s_2 m \tag{8.39}$$

(for any set of axes) (for a parallel set of axes at the centre of mass)

where s_1 and s_2 are the perpendicular distances from the centre of mass to the x and y axes respectively.

***Example* 8.10** Determine the mass moments of inertia for a hollow cylinder of radii R_1 and R_2 and axial length l about the longitudinal and transverse axes at the centre of mass.

and $$(R_2 - R_1) \ll R$$

such that $$R_1^2 + R_2^2 \approx 2R^2$$

$$I_{zz} = \frac{1}{2}m\, 2R^2 = mR^2$$

$$I_{xx} = I_{yy} = \frac{1}{12}\,(l^2 + 3 \times 2R^2) = \frac{1}{12}\, m(l^2 + 6R^2)$$

8.9 MOMENT OF INERTIA OF COMPOSITE BODIES

The moment of inertia of bodies composed of simpler homogeneous bodies can be determined from the knowledge of the corresponding values for the components about the same axes. The steps for computing the moment of inertia for a composite body are, therefore, as follows:

1. decompose the body into its simpler components, positive or negative depending upon the fact that the mass is additive or subtractive
2. look up or recall the moment of inertia of the component bodies about their centroidal axes
3. determine the moment of inertia of the component bodies about the desired axes by the application of the parallel-axis theorem
4. add algebraically the moment of inertia of the component bodies to obtain the moment of inertia of the composite body

$$I = I_1 + I_2 + I_3 + \ldots \tag{8.40}$$

The moments of inertia of some simple homogeneous bodies are tabulated in Appendices 2 and 3 for ready reference. It may be noticed that the moments of inertia of these bodies can be deduced from the values for a rectangular prism, hollow circular cylinder and hollow sphere which may be remembered.

It may be added that *the radius of gyration of a composite body cannot be obtained by adding the radii of gyration of the component bodies*

$$k \neq k_1 + k_2 + k_3 + \ldots$$

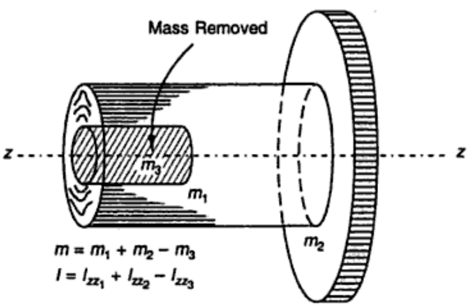


Fig. 8.9 ***A Composite Body***

but it may be determined from the moment of inertia of the composite body, i.e.,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{I_1 + I_2 + I_3 \ldots}{m_1 + m_2 + m_3 + \ldots}} \quad (8.41)$$

For example, the composite body shown in Fig. 8.9 consists of a mass m given by

$$m = m_1 + m_2 - m_3$$

***Example* 8.11** A clock pendulum consists of a slender rod and a circular disc with a hole in it as shown in Fig. Ex. 8.11. The rod has a density of 7000 kg/m^3 and cross sectional area of 50 mm^2 and the disc has a density of 8000 kg/m^3 and a thickness of 5 mm. Compute the moment of inertia of the pendulum about an axis of rotation perpendicular to the plane of oscillations.

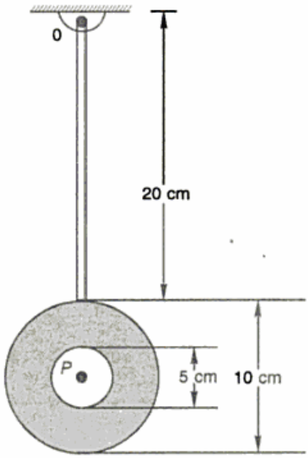


Fig. Ex. 8.11

Solution Let us consider the pendulum to consist of three composite parts:

(i) a slender rod 20 cm long

plus (ii) a solid disc 10 cm diameter and

minus (iii) a solid disc 5 cm diameter

for the sake of computing its moment of inertia.

For the slender rod,

$$I_0 = m\frac{l^2}{3} = (7000 \times 50 \times 10^{-6} \times 0.2^2 / 3$$

$$= 9.33 \times 10^{-4} \text{ kg m}^2$$

For the solid disc of 10 cm diameter,

$$m = 8000 \times 5 \times 10^{-3} \times \pi \times 0.1^2 = 1.257 \text{ kg.}$$

$$I_0 = \frac{1}{2} mR^2 + m\,(OP)^2$$

$$= \frac{1}{2} \times 1.257 \times 0.1^2 + 1.257 \times 0.25^2$$

$$= 0.085 \text{ kg m}^2$$

For the solid disc 5 cm diameter

$$m = 8000 \times 5 \times 10^{-3} \times \pi < (0.05)^2 = 0.314 \text{ kg}$$

$$I_0 = \frac{1}{2} mR^2 + m(OP)^2$$

$$= \frac{1}{2} \times 0.314 \times (0.05)^2 + 0.314 \times 0.25^2$$

$$= 0.02 \text{ kg m}^2$$

The moment of inertia of the pendulum about the axis of rotation through O is

$$I_0 = 9.33 \times 10^{-4} + 0.085 - 0.02 = 0.066 \text{ kg m}^2$$

Experiment E8

Moment of Inertia of a Body

OBJECTIVE

To determine the moment of inertia of a stepped pulley or a flywheel experimentally.

APPARATUS

A stepped pulley or flywheel mounted with its axle on bearings as shown in Fig. E8.1. Provision for a cord to be wound round it, to pass over a frictionless fixed pulley, to hold a known mass and to allow its fall through a known height.

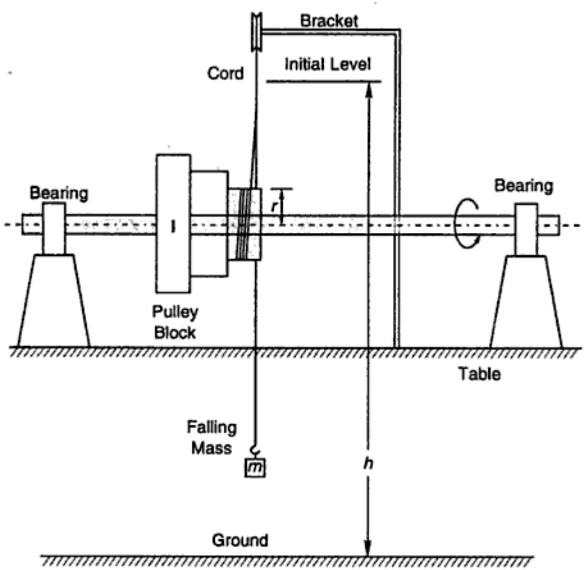


Fig. E8.1 ***Apparatus for Determining Moment of Inertia***

BACKGROUND INFORMATION

The equations expressing the moment of inertia in terms of the measurable parame-

With this substitution, Eq. (E8.3) becomes

$$I = mr^2\left(\frac{gt^2}{2h} - 1\right) = mr^2\left(\frac{g-a}{a}\right) \tag{E8.4}$$

Alternatively, if the time taken to complete N revolutions before the rotating body comes to stop after the instant the falling mass touches the ground is T, the average angular velocity would be

$$\omega_{\text{av}} = \frac{2\pi N}{T} \tag{ii}$$

Since the deceleration is assumed to be constant, the angular velocity at the instant of touchdown of the mass should be twice the average value

$$\omega = 2 \times \omega_{\text{av}} = \frac{4\pi N}{T}$$

If it is substituted in Eq. (E8.2), it reduces to

$$I = mr^2 \frac{\dfrac{ghT^2}{8\pi^2 N^2 r^2} - 1}{\left(1 + \dfrac{n}{N}\right)} \tag{E8.5}$$

and if n is negligible compared to N, i.e., if the frictional effect is neglected,

$$I = mr^2\left(\frac{ghT^2}{8\pi^2 N^2 r^2} - 1\right) \tag{E8.6}$$

It can be seen that Eqs. (E8.2) and (E8.5) are identical and that Eqs. (E8.3), (E8.4) and (E8.6) are also identical, the difference being the use of ω from Eq. (i) or Eq. (ii).

Observations and Calculations

The scheme of taking observations and calculations depends upon whether it is intended to account for friction or not and as to which of the two methods, (i) and (ii) for evaluating the rotational velocity ω at the instant of touchdown of the mass on the ground is preferred.

It is probably easiest to evaluate first and to substitute the same in either Eq. (E8.2) or Eq. (E8.3) accordingly depending on whether friction is to be accounted for or not. It may be noted that the fall h of the mass is related to the number of turns n the body makes before the touchdown of the falling mass as

$$h = 2\pi r\, n$$

A recommended tabulation of observation and calculations would be as follows:

S.No.	m	r	h	t	T	N	n	ω	I
Units	kg	m	m	s	s	—	—	rad/s	kgm^2
1									
2									
3									
4									

Results and Points for Discussion

1. Obtain the average moment of inertia I from the set of observations and calculations.
2. Measure the dimensions of the body and estimate its mass. From a knowledge of its moment of inertia and mass, calculate the radius of gyration k.
3. Comment on the accuracy in the measurements of time, length and mass in relation to the accuracy in the measurement of the moment of inertia. Would you recommend the use of a better stop watch, a more accurate scale or a better weighing machine in order to improve the accuracy of the value of the moment of inertia?
4. From a knowledge of the dimensions of the body, obtain its moment of inertia theoretically and compare it with the experimental result. Account for the difference.
5. What is the role of the moment of inertia of a body in its rotational motion? Explain why a flywheel should have a large moment of inertia.
6. Can you suggest some alternative methods for determining the moment of inertia of a body about a given axis if the body is cylindrical or irregular in shape. Examine the method of rolling a body down an inclined plane and the method of oscillating a body about a mean position.

***Example* 8.12** A spoked flywheel as shown in Fig. Ex. 8.12 consists of four spokes each 0.9 m long and of mass 50 kg which are cast with a rim of inner and outer radii 1 m and 1.5 m respectively and having a mass of 5000 kg. The shaft at the centre of the wheel has a diameter of 0.2 m and a mass of 1500 kg. Calculate the moment of inertia of the flywheel about the axis of rotation and also its radius of gyration.

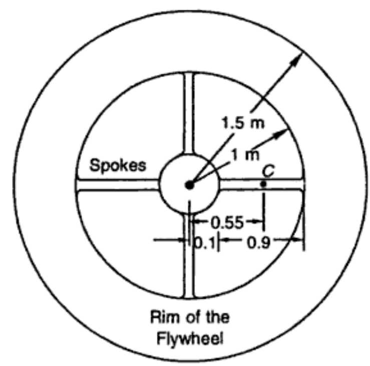


Fig. Ex. 8.12

Solution The moment of inertia of flywheel is the sum of the moments of inertia of the central shaft, spokes and rim. The greatest contribution is by the rim as is obvious from the figure and the values of the masses. Taking the components one by one,

$$\underset{\text{(shaft)}}{I_{oo}} = \frac{1}{2}mR^2 = \frac{1}{2} \times 1500 \times 0.1^2 = 7.50 \text{ kg m}^2$$

The moment of inertia of one spoke about an axis passing through *C* and parallel to the axis of rotation is

$$\underset{\text{(spoke)}}{I_{cc}} = \frac{ml^2}{12} = \frac{50 \times 0.9^2}{12} = 3.375 \text{ kg m}^2$$

and by the parallel-axis theorem.

$$\underset{\text{(spoke)}}{I_{oo}} = 3.375 + 50 \times 0.55^2 = 18.85 \text{ kg m}^2$$

For four spokes

$$\underset{\text{(four spokes)}}{I_{oo}} = 18.85 \times 4 = 75.40 \text{ kg m}^2$$

The moment of inertia of the rim is

$$\underset{\text{(rim)}}{I_{oo}} = \frac{1}{2}m(R_1^2 + R_2^2)$$

$$= \frac{1}{2} \times 5000 \times (1^2 + 1.5^2) = 8125 \text{ kg m}^2$$

The moment of inertia of the flywheel is, therefore,

$$7.50 + 75.40 + 8125 = 8207.9 \text{ kg m}^2$$

It can be seen that the rim alone contributes to the moment of inertia to the extent of

$$\frac{8125}{8207.9} \times 100 = 98.99\%$$

The radius of gyration of the flywheel can be determined from

$$I_{oo} = mk^2$$

whence $$k^2 = \frac{8207.9}{4 \times 50 + 5000 + 1500} = 1.225 \text{ m}^2$$

and $$k = 1.107 \text{ m}$$

***Example* 8.13** A three-bladed rotor of a helicopter consists of a 0.2 m diameter shaft, 1 m long and 1.5 m long radial blades. The shaft is made of steel, specific gravity 7, and the blades have a mass of 20 kg/m. Assuming the blades as uniform rods, estimate the polar moment of inertia of the rotor about the axis of rotation.

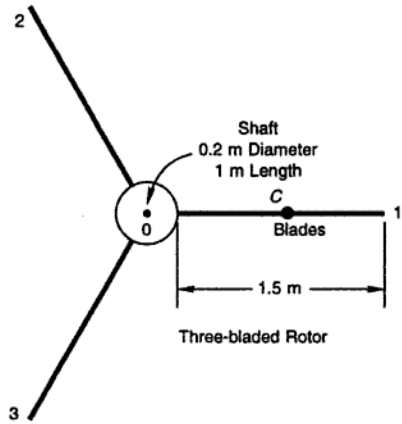


Fig. Ex. 8.13

Solution The polar moment of inertia of the shaft alone is

$$\underset{\text{(shaft)}}{I_{oo}} = \frac{mR^2}{2}$$

$$= \frac{\pi \times 0.2^2}{4} \times 1 \times 7 \times 1000 \times \frac{0.1^2}{2}$$

$$= 1.1 \text{ kg m}^2$$

The moment of inertia of one uniform-rod blade about an axis through C parallel to the axis of rotation is

$$\underset{\text{(blade)}}{I_{cc}} = \frac{ml^2}{12}$$

$$= (1.5 \times 20) \times \frac{1.5^2}{12} = 5.625 \text{ kg m}^2$$

By the parallel-axis theorem, for each blade

$$\underset{\text{(blade)}}{I_{oo}} = 5.625 + (1.5 \times 20) \times (0.75 + 0.1)^2$$

$$= 27.3 \text{ kg m}^2$$

For the three blades of the rotor,

$$\underset{\text{(three blades)}}{I_{oo}} = 27.3 \times 3 = 81.9 \text{ kg m}^2$$

For the complete rotor, then

$$I_{oo} = 1.1 + 81.9 = 83 \text{ kg m}^2$$

***Example* 8.14** Determine the moment of inertia of a hollow sphere of radii R_1 and R_2 and hence estimate the moments of inertia of

(a) a solid sphere of radius 0.5 m and mass 50 kg

(b) a thin spherical shell of mean radius 0.5 m and mass 20 kg.

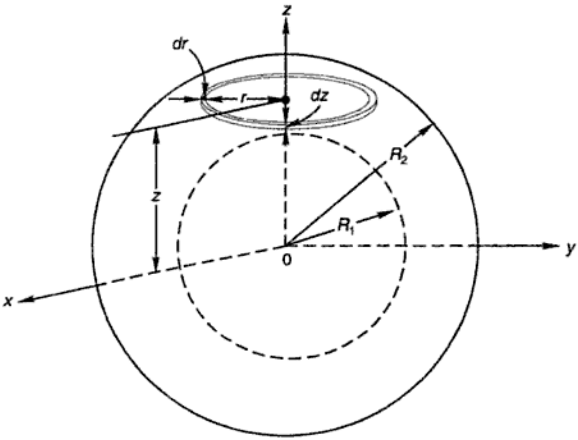


Fig. Ex. 8.14

Solution Consider a ring element of radius r, radial width dr and axial width dz at a distance z from the centre of the sphere. The mass of this element is

$$\rho \cdot 2\pi r \cdot dr\, dz$$

and its moment of inertia about the z-axis is

$$\underset{\text{(mass)}}{dI_{zz}} = \rho\, 2\pi r^3\, dr\, dz \qquad \text{(i)}$$

The moment of inertia of the hollow sphere is obtained by integration over the domain:

$$\left[\begin{array}{l} r \text{ varying from } 0 \text{ to } \sqrt{(R_2^2 - z^2)} \\ z \text{ varying from } R_1 \text{ to } R_2 \text{ and from } -R_1 \text{ to } -R_2 \end{array}\right.$$

and

$$\left[\begin{array}{l} r \text{ varying from } \sqrt{(R_1^2 - z^2)} \text{ to } \sqrt{(R_2^2 - z^2)} \\ z \text{ varying from } -R_1 \text{ to } R_1 \end{array}\right.$$

$$= \frac{8}{15} \rho\pi \cdot (R_2^5 - R_1^5) \qquad \text{(ii)}$$

Since the mass of the hollow sphere is

$$m = \frac{4}{3} \pi\rho (R_2^3 - R_1^3)$$

the moment of inertia can be expressed as

$$\boxed{\underset{\text{(hollow sphere)}}{I_{zz}} = \frac{2}{5} m \frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)}} \qquad \text{(iii)}$$

By symmetry about the axes and by interchangeability of the axes,

$$I_{xx} = I_{yy} = I_{zz}$$

Let us now observe two special cases as a matter of interest:

(*a*) *Solid Sphere*
Setting R_1 to zero and $R_2 = R$

$$\underset{\text{(solid sphere)}}{I_{zz}} = \frac{2}{5} mR^2 \qquad \text{(iv)}$$

In fact, the moment of inertia of a solid sphere could be determined *ab-initio* by considering elementary rings as for hollow sphere but with simple limits of integration, i.e.

r varying from 0 to $\sqrt{R^2 - z^2}$

and z varying from $-R$ to R

In that case,

$$\underset{\text{(sphere)}}{I_{zz}} = \int_{-R}^{R} \int_{0}^{\sqrt{R^2 - z^2}} \rho\, 2\pi r^3\, dr\, dz$$

$$= \int_{-R}^{R} \rho\, 2\pi \frac{(R_2^2 - z^2)^2}{4} dz$$

$$= \frac{\rho\pi}{2} \int_{-R}^{R} (R_2^4 + z^4 - 2R_2^2 z^2) dz$$

$$= \frac{8}{15} \rho\, \pi\, R^2$$

In terms of the mass of a sphere,

$$m = \frac{4}{3} \rho\, \pi\, R^3$$

5. What should be the length to radius ratio, i.e., l/R of a solid cylinder such that the moments of inertia about the longitudinal and transverse axes are all equal.

[**Ans.** $l/R = \sqrt{3}$]

6. From the fact that the moments of inertia for a hollow right circular cylinder are

$$I_{xx} = I_{yy} = \frac{1}{12} m(l^2 + 3(R_1^2 + R_2^2))$$

and

$$I_{xx} = m(R_1^2 + R_2^2)/2$$

Obtain the expression for the moment of inertia of
   (a) a solid cylinder of radius R
   (b) a slender rod of length l
   (c) a thin cylindrical shell of mean radius R
   (d) a thin disc of radius R
   (e) a thin ring of radii R_1 and R_2

7. You are given two spheres of the same mass, size and appearance but one of them is hollow at the centre and the other is solid throughout. How will you find out which is hollow and which is solid?
   (*Hint*: Consider their moments of inertia and try relating their accelerations when rolled down the same incline. The solid sphere accelerates faster and hence reaches down the incline first.)

8. What is the relative significance of the moment of inertia and the radius of gyration? With what intention are these defined? Under what circumstances is it desirable to have small and large values of the moment of inertia?

9. Would you imagine that the moment of inertia of the earth around its own axis is a negligible fraction of its moment of inertia about the axis of rotation around the sun?
   Take the mean radius of the earth as 6371 km and the mean radius of rotation around the sun as 149.7×10^6 km. [**Ans.** Yes, it is true]

Tutorial Problems

8.1 Determine the moments of inertia of an elliptical disc of mass m and the semi-major and semi-minor axes a and b respectively.

$$\left[\textbf{Ans.} \frac{1}{4} ma^2, \frac{1}{4} mb^2, \frac{1}{4} m(a^2 + b^2)\right]$$

8.2 An isosceles triangle of base b and height h is such that the moment of inertia about the base equals the moment of inertia about a perpendicular axis through the vertex. Determine the b to h ratio. (**Ans.** $b = 2h$)

8.3 Show that the moment of inertia about a centroidal axis parallel to a side for a cube of mass m is

$$I = \frac{1}{6} ma^2$$

where a is the length of a side.

8.4 Determine the moments of inertia about z axis of a right circular cone of mass m, base radius R and height h, as shown in Fig. Prob. 8.4

$$\left[\textbf{Ans.}\ I_{zz} = \frac{3}{10} mR^2\right]$$

8.9 A semicircle of radius r has been cut of a circle of radius R as shown in Fig. Prob. 8.9. Calculate the polar moment of inertia about O for the resulting section.

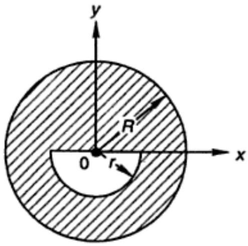


Fig. Prob. 8.9

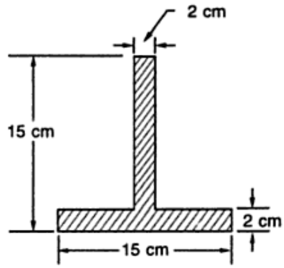


Fig. Prob. 8.10

8.10 A T-section is 15 cm × 15 cm × 2 cm as shown in Fig. Prob. 8.10. Calculate the moment of inertia of the section about an axis parallel to the base of the T and passing through its centroid. [**Ans.** 1160 cm^4]

8.11 Find the moment of inertia of a channel section shown in Fig. Prob. 8.11 about the centroidal axes. [**Ans.** 4558 and 760 cm^4]

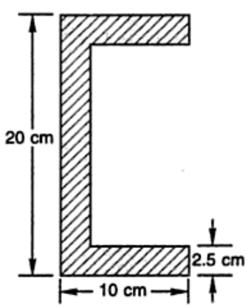


Fig. Prob. 8.11

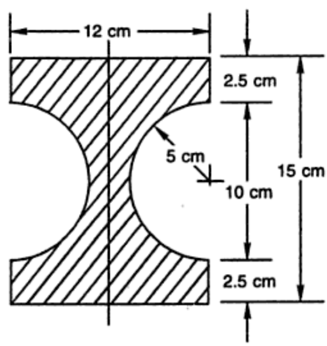


Fig. Prob. 8.12

8.12 The cross-section of a cast iron beam is shown in Fig. Prob. 8.12. Determine the moments of inertia about the centroidal axes. [**Ans.** 2885 and 340 cm^4]

8.13 Calculate the moment of inertia of a cast iron pulley with respect to its axis of rotation. Mass density of cast iron is 7200 kg/m^3.

8.14 A spherical bob of radius R and mass m_b is attached to a slender rod of length l and mass m_r. Calculate the moment of inertia of the assembly about the axis of rotation.

$$\left[\textbf{Ans.} \frac{1}{3}m_r l^2 + \frac{2}{5}m_b R^2 + (l + R/2)^2 m_b\right]$$

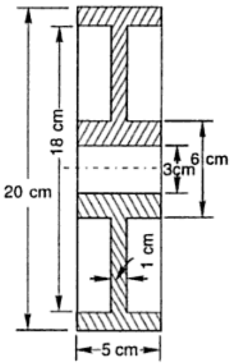


Fig. Prob. 8.13

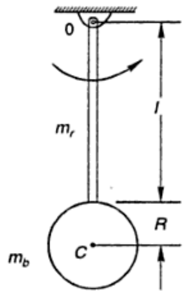


Fig. Prob. 8.14

8.15 A flywheel consist of a 1 m diameter plate 10 cm thick with four holes, each 20 cm in diameter cut at a pitch circle diameter of 40 cm symmetrically. Compute the mass moment of inertia of the flywheel about the axis of rotation. The material of the flywheel is cast iron with specific gravity 7.5. **[Ans.** 69.4 kg m^2**]**

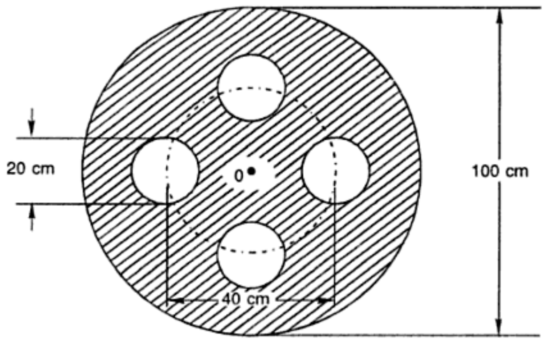


Fig. Prob. 8.15

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

1. The moment of inertia of a body is
   (a) the moment of its inertia
   (b) the rotational moment acting on the body
   (c) the rotational analogue of mass
   (d) the inertial moment acting on the body

9 DYNAMICS OF RIGID BODIES

9.1 INTRODUCTION

Further to the study of the general principle of dynamics and the kinematic behaviour of rigid bodies, it is our intention to study the dynamic behaviour of rigid bodies. A rigid body may be subjected to external forces and moments and kinematic parameters, such as velocity and acceleration may be predicted. Conversely, it may be required to evaluate the external action necessary for a desired set of kinematic conditions. This can be achieved by applying Newton's law of motion for the linear motion of the centre of mass of the rigid body and Euler's equation for the rotation of the rigid body. The purpose of this chapter is to present the methodology of applying the laws under different circumstances.

It is chosen, for the sake of simplicity of understanding, to study the dynamics of rigid bodies in steps, i.e., pure translation, fixed-axis rotation, plane motion and finally, space motion. The general form of the Euler's equation obtained for space motion is shown to contract for the special cases of translation, fixed-axis rotation, etc. There is no difficulty, therefore, if it is desired to reverse the approach, i.e., to proceed from general to particular cases.

A rigid-body model is not too hypothetical a model for many engineering situations. Most engineering materials maintain their shape and size and can be considered to be undeformed during their overall motions. It may also be appreciated that most engineering devices can be modelled as undergoing plane motion. More stress is, therefore, laid on the plane motion of rigid bodies than on their space motion.

9.2 TRANSLATION OF A RIGID BODY

Recalling the fact that the acceleration and velocity of each element on a rigid body must be the same in a pure translation, the translational motion must be governed by the Newton's law,

$$\mathbf{F} = m\,\mathbf{a} \tag{9.1}$$

where $\mathbf{F}$ is *the net external force* acting on the body and $\mathbf{a}$ is *the acceleration of any point on the body*. The net external force, sometimes written as $\Sigma\mathbf{F}$, must be the vector sum of all the external forces acting on the body

$$\mathbf{F} \quad \text{or} \quad \Sigma\,\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$$

as also shown Fig. 9.1.

There being no tendency of rotation, the angular velocity and the angular acceleration remain zero. In other words, the relative velocity of any point on the body referred to the centre of mass is zero and the angular momentum of the entire body with respect to the centre of mass is zero. It follows from the Euler's equation that the summation of moments about the centre of mass, C must also vanish:

$$\mathbf{M}_c = 0 \tag{9.2}$$

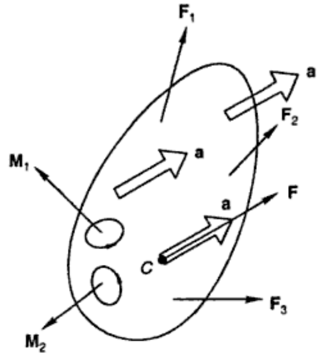


Fig. 9.1 ***A Rigid Body in Translation***

These relations govern the general translatory motion of a rigid body. If the body undergoes a plane motion, the parallel planes of motion being parallel to the x-y plane, the component equations governing the motion reduce to

$$F_x = ma_x$$

$$F_y = ma_y$$

where a_x and a_y are rectangular components of acceleration of the centre of mass, as also of any other point on the body and F_x and F_y are the rectangular components of the net force acting on the body. Also, the moment about the z-axis passing through the centre of mass must vanish:

$$M_z = 0$$

***Example* 9.1** A motor of mass 8000 kg resting on two supports A and B is pulled along a smooth horizontal surface by a string passing through a hook as shown in Fig. Ex. 9.1.

Calculate the acceleration of the motor and the reactions at the supports for a tension T applied in the string. Calculate the maximum tension in the string for the sliding motion.

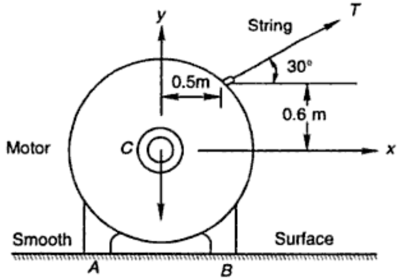


Fig. Ex. 9.1

Solution From the free-body diagram of the motor as shown in Fig. Ex. 9.1 (Solution),

$$F_x = T_x = T \cos 30° = 0.866\ T$$

$$F_y = T_y - mg + R_A + R_B = 0.5\ T - 78{,}500 + R_A + R_B$$

$$M_z = 0.5\ T \times 0.5 + R_B \times 0.5 - 0.866\ T \times 0.6 - R_A \times 0.5$$

$$= 0.5R_B - 0.5\ R_A - 0.27\ T$$

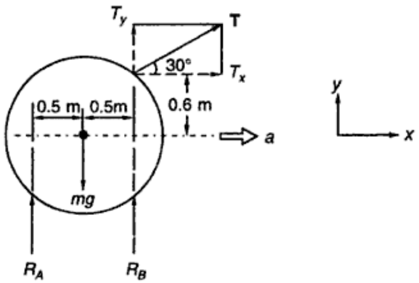


Fig. Ex. 9.1 ***(Solution)***

By the equations of motion for a rigid body in translation,

$$F_x = 0.866\ T = ma = 8000\ a$$

$$F_y = 0.5\ T - 78{,}500 + R_A + R_B = 0$$

$$M_z = 0.5R_B - 0.5R_A - 0.27\ T = 0$$

From these equations,

$$a = 0.000\ 108\ T \text{ m/s}^2$$

$$R_A = 39{,}250 - 0.52\ T$$

$$R_B = 39{,}250 + 0.02\ T$$

It can be observed that the reaction R_A decreases with the tension T increasing. In the limiting case of sliding,

$$R_A = 0 = 39{,}250 - 0.52\ T$$

and $$T = 39{,}250/0.52 = 75{,}480.8 \text{ N} = 75.48 \text{ kN}$$

beyond which the point A will not be restrained to move along the surface. The motor may then overturn forward.

Example **9.2** A horizontal uniform bar PQ of mass 100 kg and length 30 cm is supported by strings from A and B 30 cm apart and is released from rest when $\theta =$ 60° as shown in Fig. Ex. 9.2.

For a plane curvilinear motion of the bar, determine the tension in the string at the instant when

(a) it is released from rest
(b) it crosses the mean position
(c) $\theta = 30°$.

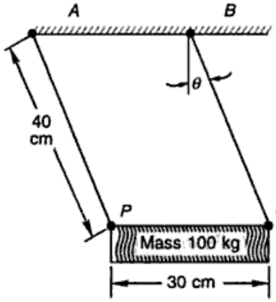


Fig. Ex. 9.2

Solution From the free-body diagram of the bar *PQ* at any instant as shown in Fig. Ex. 9.2 (Solution),

$$F_r = T_1 - T_2 + mg\cos\theta$$

$$F_\theta = mg\sin\theta$$

$$M_z = -T_1 p_1 + T_2 p_2$$

where p_1 and p_2 are the perpendicular distances from *C* to the lines of action of T_1 and T_2 respectively.

Since $$p_1 = p_2 = p$$

from the symmetry and

$$M_z = 0$$

for pure translation of the rigid body, it follows that

$$T_1 p - T_2 p = 0$$

or $$T_1 = T_2 = T$$

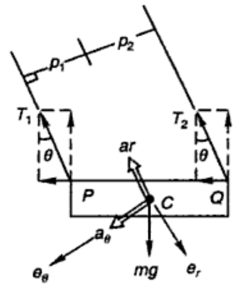


Fig. Ex. 9.2 ***(Solution)***

i.e., the tensions in the strings must be equal at all times. Then,

$$F_r = -2T + \text{mg}\cos\theta$$

$$F_\theta = mg\sin\theta$$

The acceleration of any point on the body is made up of two components; the radially inward component, i.e.,

$$-r\,\omega^2 = -0.4\,\omega^2$$

and the tangentially forward component, i.e.,

$$r\alpha = 0.4\alpha$$

By employing the equations of motion,

$$-2T + mg\cos\theta = 100 \times (-0.4\,\omega^2) = -40\,\omega^2 \quad \text{(i)}$$

or $$2T - mg\cos\theta = 40\,\omega^2$$

and $$mg\sin\theta = 100 \times 0.4\alpha$$

or $$mg\sin\theta = 40\alpha \quad \text{(ii)}$$

Case (a)

At the instant of release,

$$\theta = 60°,\ \omega = 0$$

from Eq. (i) $2T - mg/2 = 0$

$$T = mg/4$$

$$= 100 \times 9.81/4 = 245.25 \text{ N}$$

and the angular acceleration α is obtained from Eq. (ii) as

$$\alpha = \frac{100 \times 9.81 \times 0.866}{40} = 21.24 \text{ rad/s}^2$$

Cases (b) and (c)

At any instant when the strings are inclined at θ with the vertical, from Eq. (ii),

$$\alpha = \frac{mg \sin \theta}{40} = \frac{d\omega}{dt} = -\omega \frac{d\omega}{d\theta}$$

or $\omega / d\omega = -mg/40 \sin \theta \, d\theta$

and integrating each side,

$$\omega^2/2 = mg/40 \cos \theta + C$$

Employing the initial condition,

$$\omega = 0 \quad \text{at} \quad \theta = 60°$$

$$C = (-mg/40) \cos 60° = -mg/80$$

and $\omega^2 = mg/20 \cos \theta - mg/40$ (iii)

(b) When it crosses the mean position, $\theta = 0$

From Eq. (ii),

$$\alpha = -mg \sin \theta/40 = 0$$

and from Eq. (iii), $\omega^2 = mg/20 - mg/40 = mg/40$

$$\omega^2 = 100 \times 9.81/40 = 24.52$$

or $\omega = 4.95$ rad/s

From Eq. (i), $2T = mg \cos \theta = 40\omega^2$

$$2T = 40 \times 24.52 + 100 \times 9.81$$

$$= 1962 \text{ N}$$

and the tension $T = 981$ N

(c) When $\theta = 30°$

From (ii), $\alpha = -100 \times 9.81/80 = -12.26$ rad/s^2

and From (iii), $\omega^2 = 42.5 - 24.5 = 18.0$

$$\omega = 4.24 \text{ rad/s}$$

From (i), $2T - mg\cos\theta = 40 \times 18 = 720$

$$2T = 720 + 100 \times 9.81 \times 0.866$$

and the tension $T = 785$ N

It can be observed that the tension in each string is the least at the extremities of the oscillatory motion. It is the maximum at the mean position. At this position, the weight of the bar acting downward is

$$mg = 981 \text{ N}$$

but the total tension in the two strings upward is 1962 N. The vertically upward unbalanced force (1962–981) N is the one which is responsible for the radially inward (in this position, vertically upward) acceleration, i.e., $0.4\omega^2$. If, on the other hand, the mass was in equilibrium, supported by two equal vertical strings *AP* and *BQ*, the tension in each string would only be $mg/2$, i.e., 490.5 N each.

9.3 ROTATION OF A RIGID BODY ABOUT A FIXED PRINCIPAL AXIS

The case of a rigid body rotating about a fixed principal-axis is a special case of plane motion of a rigid body. Let the principal-axis of rotation be the *z*-axis passing through *O* such that *O* and *C* lie on a plane normal to the axis of rotation. With reference to the coordinate system shown in Fig. 9.2, the rotational velocity and acceleration are ω and α respectively.

An elementary strip of mass *dm* chosen parallel to the fixed-axis located at a position vector **r** as shown is then imparted a velocity and an acceleration given by

$$\mathbf{v} = r\omega\,\mathbf{e}_\theta$$

$$\mathbf{a} = -r\omega^2\,\mathbf{e}_r + r\alpha\,\mathbf{e}_\theta$$

The elementary mass must have been acted upon by an elementary force $d\mathbf{F}$ in accordance with Newton's law, i.e.,

$$d\mathbf{F} = dm\,\mathbf{a} = (-r\omega\,\mathbf{e}_r + r\alpha\,\mathbf{e}_\theta)\,dm$$

The moment of the elementary force about the axis of rotation passing through *O* is

$$d\mathbf{M}_z = \mathbf{r} \times d\,\mathbf{F}$$

$$= r\,\mathbf{e}_r \times (-r\omega^2\,\mathbf{e}_r + r\alpha\,\mathbf{e}_\theta)\,dm$$

$$= r^2\,\alpha\,dm\,\mathbf{k}$$

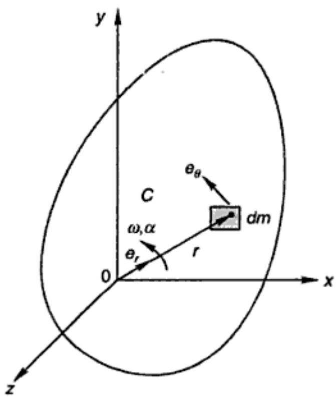


Fig. 9.2 ***Rotation about a Fixed Principal Axis***

This is the moment which must be exerted about *O* so as to accelerate the elementary mass at an angular acceleration α. The total moment required to rotate and accelerate the rigid body about an axis through O is, therefore,

$$M_z = \int d\,M_z = \alpha \int r^2\, dm$$

or
$$M_z = I_z \alpha \tag{9.3}$$

where I_z, the mass moment of inertia of the body about an axis through O is, by definition, given by

$$I_z = \int r^2\, dm$$

and the moment vector $\mathbf{M}_z$ is directed perpendicular to the parallel planes of rotation.

It should be recorded here that the moment M_z acting on the rigid body about an axis through O is due to the entire system of forces on the rigid body. For example, the weight of the body, reactions from the supports and other external actions as obtained by drawing a free-body diagram of the rigid body must be taken into account to obtain the net moment M_z.

The centre of mass denoted by C in the figure is imparted a velocity and a acceleration given by

$$\mathbf{v}_c = \boldsymbol{\omega} \times \mathbf{r}_c = r_c \omega\, \mathbf{e}_\theta$$

$$\mathbf{a}_c = -\, r_c \omega^2\, \mathbf{e}_r + r_c \alpha\, \mathbf{e}_\theta$$

The equation of motion for the centre of mass is given by

$$\mathbf{F} = m\, \mathbf{a}_c \tag{9.4}$$

The net external force $\mathbf{F}$ on the rigid body is due to the weight, reactions from the supports and other actions. This equation is useful if the desired goal is to find the reactions from the supports on a dynamic body.

In particular, if the axis of rotation passes through the centre of mass, i.e., O and C coincide,

$$\mathbf{F} = m\, \mathbf{a}_c = 0$$

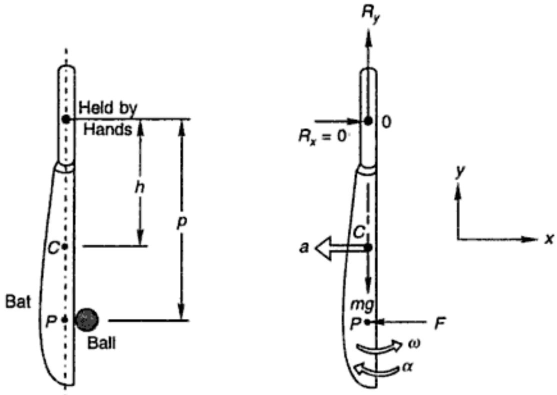
$$M_z = I_z \alpha$$

It is usual to define a point called *centre of percussion* as a location on a rigid body through which the resultant of the applied forces acts.

Consider, for example, a rigid-body model of a cricket bat free to rotate about an axis through O as shown in Fig. 9.3(a). Its centre of mass is at C but the centre of percussion is at a point P such that if the force due to a ball striking it passes through P, there is no horizontal reaction at O where the bat is held by the batsman and he can receive the ball comfortably, without jarring.

If the moment of inertia of the body about its axis of rotation is I_0 and the corresponding radius of gyration is k_0, the distance p at which the centre of percussion is located is given by

$$p = \frac{k_0^2}{h}$$



(a) Ball Hitting the Bat Held by a Player

(b) Free-body Diagram of the Bat at the Instant of Impact

Fig. 9.3 ***Concept of the Centre of Percussion***

This may be shown by considering the free-body diagram of the bat and by determining the condition of zero horizontal reaction at O. Referring to the free-body diagram shown in Fig. 9.3(b), the equation of motion for the centre of mass is

$$R_x - F = ma \quad \text{(i)}$$

and the Euler's equation for rotation about the axis of rotation through O is

$$-F \cdot p = I_0\alpha = (I_c - mh^2)\alpha$$

$$= mk_0^2\alpha = m(k_c^2 + h^2)\,\alpha \quad \text{(ii)}$$

Eliminating F between Eqs. (i) and (ii)

$$R_x = ma - \frac{mk_0^2\alpha}{p} = ma - \frac{m(k_c^2 + h^2)\alpha}{p}$$

Considering the fact that

$$a = h\alpha$$

the condition for R_x to be zero is

$$p = \frac{k_0^2}{h} = \frac{k_c^2 + h^2}{h} \quad (9.5)$$

***Example* 9.3** A rectangular plate 2 m × 3 m of mass 100 mg/m² is supported by hinges at A and B as shown in Fig. Ex. 9.3. If the support A is removed, determine the reaction at B, the angular acceleration of the plate and acceleration of the centre of mass.

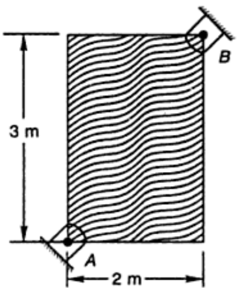


Fig. Ex. 9.3

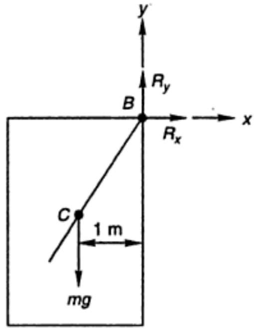


Fig. Ex. 9.3 *(Solution)*

Solution At the instant when the support A is removed, the plate tends to rotate about the axis passing through B and normal to the plane of the plate. From the free-body of the plate at that instant, as shown in Fig. Ex. 9.3 (Solution)

$$F_x = R_x$$

$$F_y = R_y - mg = R_y - 600 \times 9.81$$

$$= R_y - 5886$$

$$M_B = mg \times 1 = 5886 \text{ N m}$$

$$I_B = \frac{1}{12} \times 600 \times (3^2 + 2^2) + 600 \times (1.5^2 + 1^2)$$

$$= 2600 \text{ kg m}^2$$

By the equation of motion for a rigid-body rotation about a fixed axis,

$$M_B = I_B \alpha$$

or $$\alpha = 5886/2600 = 2.26 \text{ rad/s}^2$$

Also, $$M_c = I_c \alpha$$

$$1 \times R_y - 1.5 \times R_x = \frac{1}{12} \times 600 \times (3^2 + 2^2) \times 2.26$$

$$R_y - 1.5\, R_x = 1469$$

and $$F_x = R_x = ma_x = 600\, a_x$$

$$F_y = R_y - 5886 = 600\, a_y$$

At the initial instant, the body starts from rest, $\omega = 0$ although the acceleration α is finite as determined above. The acceleration of the centre of mass C is

$$\mathbf{a}_c = \boldsymbol{\alpha} \times \mathbf{r}$$

$$= 2.26\, \mathbf{k} \times (-1\, \mathbf{i} - 1.5\, \mathbf{j}) = 3.39\, \mathbf{i} - 2.26\, \mathbf{j}$$

where the negative sign shows that the bearing friction torque acts to resist the motion and to bring about deceleration of the disc.

This resisting torque is equivalent to a constant resisting force of

$$5.00/0.5 = 10.00 \text{ N}$$

acting over the rim of the disc.

(b) From $\alpha = -1.31 \text{ rad/s}^2$

$$\omega_1 = \frac{1500 \times 2 \times \pi}{60} = 157.1 \text{ rad/s}$$

and $\omega_2 = 0$

Since $\omega_2^2 - \omega_1^2 = 2\alpha\,\theta$

whence, $$\theta = \frac{0 - 157.1^2}{2 \times (-1.31)} = 9420 \text{ rad}$$

and $n = 9349/2\pi = 1499.2$ revolutions

9.4 PLANE MOTION OF A RIGID BODY

Plane translation, rectilinear or curvilinear, together with rotation about an axis perpendicular to the parallel planes in which translation takes place, constitutes a general plane motion. Conversely, a general plane motion can be regarded equivalent to a combination of translation of the centre of mass and rotation about an axis passing through the centre of mass. Correspondingly, the dynamic behaviour of a rigid body is governed by the set of equations

$$\boxed{\begin{aligned} F_x &= ma_x \\ F_y &= ma_y \end{aligned}} \tag{9.6}$$

for translation of the centre of mass

and $$\boxed{M_c = I_c \alpha} \tag{9.7}$$

for rotation about an axis passing through the centre of mass of the body.

***Example* 9.5** A cord passing over a light frictionless pulley carries a weight W_1 suspended vertically at one end and is wrapped around a cylinder of weight W_2 as shown in Fig. Ex. 9.5. Assuming that the cylinder can roll without slip on a horizontal plane, calculate the acceleration of the suspended weight.

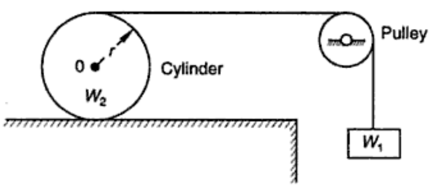


Fig. Ex. 9.5

Hence,

$$a_1 = \frac{g}{1 + \dfrac{3\,W_2}{8\,W_1}}$$

An observation of this equation shows that W_2/W_1 is the only factor which governs the acceleration of the suspended weight. The size of the cylinder is immaterial. When W_2/W_1 is zero, $a_1 = g$, the acceleration of the free-fall. As W_2/W_1 increases, the acceleration of the suspended weight decreases, tending to become zero as W_2/W_1 tends to infinity.

Example 9.6 A wheel and a differential axle assembly has a mass m and a radius of gyration k. The radii of the two parts of the axle are r_1 and r_2. Cords wrapped round these parts carry suspended weights W_1 and W_2 as shown in Fig. Ex. 9.6. Determine the acceleration of the suspended weights when the weight W_1 descends.

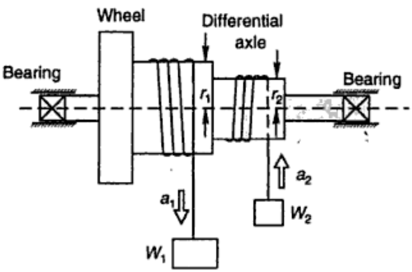


Fig. Ex. 9.6

Solution Let the acceleration of the suspended weights be a_1 and a_2 downwards and upwards respectively. The angular acceleration α of the wheel and differential axle assembly is

$$\alpha = \frac{a_1}{r_1} = \frac{a_2}{r_2}$$

With reference to the free-body diagrams of the suspended weights

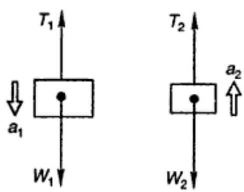


Fig. Ex. 9.6 ***(Solution)***

$$W_1 - T_1 = \frac{W_1}{g}a_1 \qquad \text{(i)}$$

$$T_2 - W_2 = \frac{W_2}{g}a_2 \qquad \text{(ii)}$$

The net external moment on the assembly due to the suspended weight is

$$M = T_1r_1 - T_2r_2$$

and solving for T and α simultaneously with Eq. (iii), remembering that

$$I_c = \frac{m(2L)^2}{12} = \frac{mL^2}{3}$$

the expressions for T and α appear as follows:

$$T = \frac{mg}{1 + 3\cos^2\theta}$$

and
$$\alpha = \frac{-3g\cos\theta}{L(1 + 3\cos^2\theta)}$$

It may be remarked that the positive sign of T shows that the string is indeed in tension and the negative sign of α indicates that the angular acceleration is clockwise and not anticlockwise as assumed in the free-body diagram.

***Example* 9.8** A uniform bar of length L and weight W rests on smooth surfaces as shown in Fig. Ex. 9.8. Obtain an expression for the angular velocity of the bar and determine the angle θ at which the bar no longer touches the vertical wall.

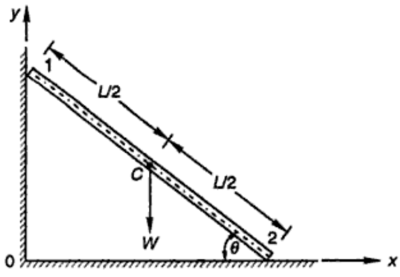


Fig. Ex. 9.8

Solution With reference to the free-body diagram of the bar where the reactions at the two surfaces are taken normal only due to the absence of friction, let us consider the motion of the bar with reference to its centre of mass C. Assuming the motion as a plane motion in the plane of the figure, the equations describing its motion are

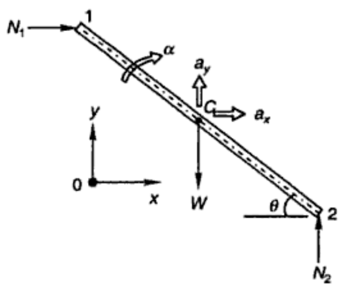


Fig. Ex. 9.8 ***(Solution)***

$$N_1 = ma_x \quad \text{(i)}$$

$$N_2 - W = ma_y \quad \text{(ii)}$$

$$N_1 \frac{L}{2}\sin\theta - N_2 \frac{L}{2}\cos\theta = I_c\alpha \quad \text{(iii)}$$

Taking $$I_c = \frac{mL^2}{12}$$

and $$W = mg$$

for the rod, and substituting N_1 and N_2 from Eqs. (i) and (ii) into Eq. (iii),

$$a_x \sin\theta - a_y \cos\theta = g\cos\theta + \frac{L}{6}\alpha \tag{iv}$$

The linear accelerations a_x and a_y of the centre of mass are related to its angular acceleration α by the geometry of the motion. From the coordinates of the centre of mass at any instant,

$$x = \frac{L}{2}\cos\theta \qquad \text{and} \qquad y = \frac{L}{2}\sin\theta$$

Differentiation with respect to t leads to

$$V_x = \frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} = -\frac{L}{2}\sin\theta\,\frac{d\theta}{dt}$$

$$a_x = \frac{dV_x}{dt} = -\frac{L}{2}\cos\theta\left(\frac{d\theta}{dt}\right)^2 - \frac{L}{2}\sin\theta\,\frac{d^2\theta}{dt^2}$$

Similarly,

$$a_y = \frac{dV_y}{dt} = -\frac{L}{2}\sin\left(\frac{d\theta}{dt}\right)^2 + \frac{L}{2}\cos\theta\,\frac{d^2\theta}{dt^2}$$

and $$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

Substituting these relations in Eq. (iv)

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2L}\cos\theta \tag{v}$$

the angular speed ω is obtained by integration,

$$\omega = \sqrt{\frac{3g}{L}(1 - \sin\theta)} \tag{vi}$$

assuming that the bar started from rest at $\theta = 90°$, i.e., when the bar was vertical.

The condition of it no longer touching the wall means

$$N_1 = 0$$

and from Eq. (i)

$$a_x = 0$$

or $$-\frac{L}{2}\cos\theta\left(\frac{d\theta}{dt}\right)^2 - \frac{L}{2}\sin\theta\,\frac{d^2\theta}{dt^2} = 0 \tag{vii}$$

Substituting for the first and second derivates of θ in Eq. (vii),

$$\sin\theta = \frac{2}{3} = 0.667$$

whence $$\theta = 41.8°$$

***Example* 9.9** Two homogeneous cylindrical discs, each of diameter 1 m and mass 10 kg connected by an axle of diameter 0.3 m and mass 15 kg lie on a rough surface as shown in Fig. Ex. 9.9.

A string wrapped round the axle as shown, exerts a force of 50 N at the mid-span of the axle. Analyse the motion to determine the acceleration of the system.

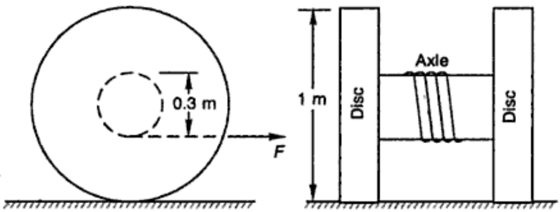


Fig. Ex. 9.9

Solution With reference to the free-body diagram of the system where the motion is assumed to be towards right and the frictional force f is shown to act in a direction to oppose the motion, as shown in Fig. Ex. 9.9 (Solution):

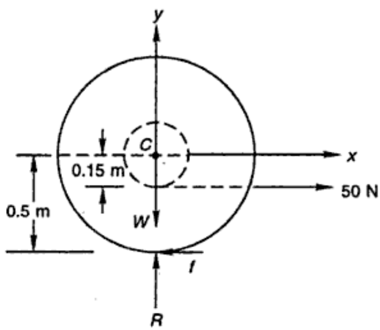


Fig. Ex. 9.9 ***(Solution)***

For the reference axes fixed at the centre of mass C, the equations of motion are:

$$F_x = 50 - f = ma_x$$

$$F_y = R - (10 + 10 + 15)\,g = ma_y$$

$$M_c = 50 \times \frac{0.3}{2} - f \times \frac{1}{2} = I_c\alpha$$

Using the fact that

$$a_y = 0$$

because the body is not being lifted along the y-axis and

$$I_c = 2 \times \left(\frac{1}{2} \times 10 \times 0.5\right) + \frac{1}{2} \times 15 \times 0.15^2 = 2.67 \text{ kg m}^2$$

and $$m = 2 \times 10 + 15 = 35 \text{ kg}$$

the equations of motion simplify to

$$50 - f = 35a_x \quad \text{(i)}$$

$$R - 35g = 0 \quad \text{(ii)}$$

$$7.5 - f/2 = 2.67\ \alpha \quad \text{(iii)}$$

Substituting

$$\alpha = \frac{a_x}{0.5}$$

in Eq. (iii) reduces it to

$$15 - f = 10.68a_x$$

which, together with Eq. (i), provides

$$a_x = 1.44 \text{ m/s}^2$$

Experiment E9

Looping the Loop

OBJECTIVE

To determine the minimum initial height of a ball in order that it may succeed in 'looping the loop'.

APPARATUS

A channel-track arranged in the form of a looping-the-loop apparatus as shown in Fig. E9.1, spherical ball and a metre scale.

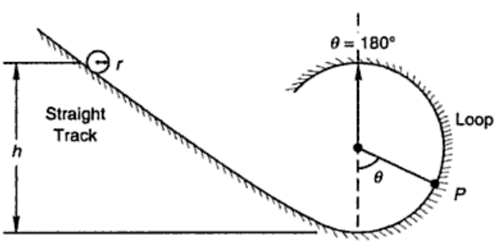


Fig. E9.1 *'Looping-the-loop' Apparatus*

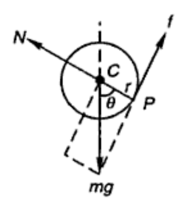


Fig. E9.2 *Free-body Diagram of the Ball*

BACKGROUND INFORMATION

When a spherical ball is released from rest from a position on the straight part of the track, it rolls down and gains speed until it reaches the bottom of the loop. Thereafter, it is subjected to a centripetal force and a varying reaction N normal to the track. Considering the free-body diagram of the ball at a point P, as shown in Fig. E9.2, the equations of motion for its centre of mass C are

Tangential: $$mg \sin \theta - f = ma \tag{E9.1}$$

Normal: $$N - mg \cos \theta = \frac{mv^2}{(R-r)} = \frac{mr\omega^2}{(R-r)} \tag{E9.2}$$

where v is the velocity of the centre of mass of the ball, $(R-r)$ is the radius of the path of the centre of mass as the ball of radius r moves within the circular loop of radius R and ω is the angular velocity of the centre of mass of the ball.

The Euler's equation for the rotation of the ball must be

$$fr = I_c\alpha = \frac{2}{5} mr^2 \frac{d\omega}{dt}$$

$$= \frac{2}{5} mr^2 \omega \frac{d\omega}{d\theta} \tag{E9.3}$$

Integrating Eq. (E9.3) and substituting for f from Eq. (E9.1) together with the fact that

$$a_t = r\alpha = r\frac{d\omega}{dt} = r\omega\frac{d\omega}{d\theta}$$

it is seen that

$$\omega = \frac{\log (R-r)}{7r^2} \cos \theta + \frac{\log (h - R + r)}{7r^2} \tag{E9.4}$$

where the initial height h is related to the velocity of the ball by the principle of energy conservation applied during the rolling of the ball down the incline without friction

$$mgh = \frac{1}{2} I_c \omega_0^2 + \frac{1}{2} mv_0^2 \tag{E9.5}$$

where v_0, the translational velocity of the centre of mass of the ball at $\theta = 0$ position equals r times the rotational velocity ω_0 of the ball at the same position.

Substituting the value of ω in Eq. (E9.2),

$$N = mg \cos \theta + \frac{mr^2}{R-r}\left[\frac{\log (R-r)}{7r^2} \cos \theta + \frac{\log (h - R + r)}{7r^2}\right] \tag{E9.6}$$

It is noticed that the reaction N from the track on the ball varies with θ, the angular position of the ball for a given initial height h, the other parameters, i.e., the radii of the ball and the track remaining the same. The position where the ball loses contact with the track must be such that the reaction N becomes zero. If the ball is to complete the loop, it must be able to go all the way up to the position $\theta = 180°$ because it would then continue on the track thereafter.

Substituting $\theta = 180°$, $\cos\theta = -1$ and $N = 0$ in Eq. (E9.6) and simplifying the result,

$$h_1 = 2.7(R - r) \tag{E.9.7}$$

This is the minimum initial height of the ball for it to succeed in looping the loop.

Alternatively, the analysis may be considerably simplified if the moment of inertia of the ball is ignored, i.e., if the ball is assumed to be represented by a mass sliding frictionlessly. In that case only the normal equation of motion, i.e., Eq. (E9.2) together with the energy conservation principle would provide the result

$$N - mg\cos\theta = \frac{mv^2}{R-r} \tag{i}$$

$$mgh = \frac{1}{2}mv^2 + mg(R - R\cos\theta) \tag{ii}$$

Substituting $\theta = 180°$; $\cos\theta = -1$ and $N = 0$ as well as the value of v from Eq. (ii) in Eq. (i),

$$mg = \frac{2mgh - 4mg(R-r)}{(R-r)}$$

whence $$h_2 = 2.5\,(R - r) \tag{E9.8}$$

It may be seen that this result is remarkably close to the one obtained above, Eq. (E9.7). *In practice, neither of them may be valid because the ball may roll as well as slide with friction simultaneously.*

OBSERVATIONS AND CALCULATIONS

The experiment is indeed very interesting. *The condition of success of the ball in looping the loop is independent of the mass of the ball, angle of inclination of the straight track and value of g at a particular place.* The minimum initial height h required for a ball to be able to reach the top position and to complete the loop is determined experimentally by hit and trial.

$R =$ $\qquad$ $r =$

S.N.	h_{exp}	$h_1 = 2.7(R-r)$ Eq. (E9.7)	$h_2 = 2.5(R-r)$ Eq. (E9.8)	$\frac{h_{exp} - h_1}{h} \times 100\%$	*Differences in* $\frac{h_{exp} - h_2}{h_2} \times 100\%$

RESULT

State the average of the minimum experimental heights obtained together with the percentage discrepancy in comparison with the theoretical results by the alternative analysis.

POINTS FOR DISCUSSION

1. Under what circumstances is a ball likely to roll without sliding and under what conditions is it likely to roll as well as slide?
2. Examine the percentage variation within the experimental results and compare the same with the percentage difference between the experimental and theoretical values. Which of the two theoretical results is closer to the actual result?
3. The energy conservation principle has been employed in the theoretical analyses. Discuss the validity of the principle for the case of a ball rolling without sliding as assumed in the first analysis. The validity of the principle for frictionless sliding as assumed in the second analysis is quite clear.
4. If a ball were allowed to complete the loop and made to go up another incline, would it attain a height equal to the initial height h?
5. Supposing it was desired to determine the dissipation of energy between the initial position of the ball and its top position in the loop in the limiting case, how would you proceed to estimate it?
6. If you were to improve the experimental equipment, would you recommend
   (i) the selection of a smoother ball, a lighter ball or a smaller ball?
   (ii) the improvement of the track by minimising friction, by decreasing the radius of the loop or by increasing the inclination of the straight track?
   (iii) the use of a small circular cylinder or a small rectangular block instead of a spherical ball?

***Example* 9.10** A string is wrapped around the periphery of a thin disc of radius 0.5 m and mass 10 kg as shown in Fig. Ex. 9.10. At an instant when the string is pulled up with a force of 200 N, determine the acceleration of the centre of the disc and the angular acceleration of the disc.

Solution Considering the freebody diagram of the disc, and taking C as the reference point,

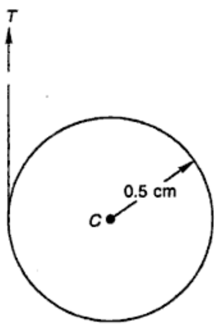


Fig. Ex. 9.10

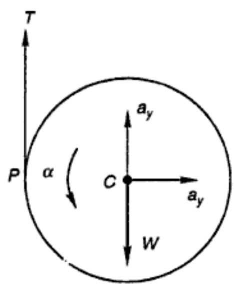


Fig. Ex. 9.10 *(Solution)*

$$T - W = ma_y \tag{i}$$

$$0 = ma_x \tag{ii}$$

$$-Tr = I_c\alpha = mr^2/2 \cdot \alpha \tag{iii}$$

From (i), $$a_y = \frac{200 - 10 \times 9.81}{10} = 10.2 \text{ m/s}^2$$

and from (ii) $$a_x = 0$$

From (iii) $$\alpha = \frac{-200 \times 0.5 \times 2}{10 \times 0.5^2} = -80 \text{ rad/s}^2$$

which means that the angular acceleration is 80 rad/s^2 in the clockwise direction.

One may also determine the acceleration of the string which is the tangential component of acceleration of the disc at point P.

$$a = 10.2 + 0.5 \times 80 = 50.2 \text{ m/s}^2 \text{ upwards.}$$

9.5 WORK-ENERGY FORMULATION FOR PLANE MOTION

Let us recapitulate the definitions of work and energy, obtain expressions for them and formulate the work-energy principle as an alternative principle for studying the dynamics of a rigid body.

The work done due to a force **F** acting at an arbitrary point P on a body equals the dot product of the force with the displacement of the point of application of the force:

$$dW = \mathbf{F} \cdot d\mathbf{r}_p$$

For a finite displacement of the point of application of force, i.e., from $\mathbf{r}_1$ to $\mathbf{r}_2$, the work done is given by

$$W = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}_p$$

If, instead, it is sought to evaluate the integral

$$W' = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}_c \tag{9.8}$$

as is the case in the work-energy formulation to follow, the integral does not represent the work done by the force on the body, it only represents the work which would have been done by the force on the body if the force acted at its centre of mass.

The action of a moment **M** due to a couple on a rigid body results in an angular displacement $\mathbf{d\theta}$ of the body. The work of the moment is expressed by

$$dW = \mathbf{M} \cdot \mathbf{d\theta}$$

In view of these facts, it may be stated that the kinetic energy of a rigid body in a general plane motion, as given by Eq. (9.11), may be thought of as the sum of

1. The kinetic energy $(1/2)mV_c^2$ associated with the motion of the centre of mass C of the body as if the total mass were concentrated at that point
2. The kinetic energy $(1/2)I_c\,\omega^2$ associated with the rotation of the body about an appropriate axis through C.

If a rigid body undergoes fixed axis rotation about an axis which does not pass through the centre of mass C, the expression for the kinetic energy of the body in plane may surely be employed but a simpler expression is more convenient. In order to arrive at it, consider a rigid body rotating about an axis through O as shown in Fig. 9.5. The expression for kinetic energy is

$$KE = \frac{1}{2} mV_c^2 + \frac{1}{2} I_c \omega^2$$

Substituting $V_c = r_c\omega$

$$KE = \frac{1}{2} mr_c^2\omega^2 + \frac{1}{2} I_c\omega^2$$

$$= \frac{1}{2}(I_c + mr_c^2)\omega^2$$

$$= \frac{1}{2} I_0 \omega^2 \qquad (9.12)$$

which implies that the kinetic energy of a rigid body during a non-centroidal fixed-axis rotation may also be written as $(1/2)\, I_0\, \omega^2$ provided the moment of inertia I_0 is taken about the axis of rotation. The same fact could have been established by considering a small element of mass dm at an arbitrary point P. The kinetic energy of the element is

$$KE = \int \frac{1}{2} V^2\, dm$$

$$= \int \frac{1}{2} (R\omega)^2\, dm$$

$$= \int \frac{1}{2} R^2\, dm \cdot \omega^2$$

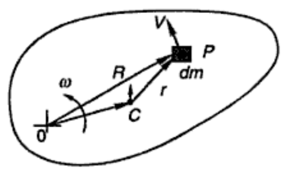


Fig. 9.5 ***Fixed-axis Rotation***

or

$$\boxed{KE = \frac{1}{2} I_0\, \omega^2}$$

where I_0, the moment of inertia of the entire body about the axis of rotation through O equals the integral $\int R^2\, dm$.

The work-energy equations for a rigid body may be obtained from the Newton's law and Euler's equation. From the Newton's law applied to the centre of mass of the body,

$$\mathbf{F} = m\,\mathbf{a}_c$$

$$\int_{r_1}^{r_2} \mathbf{F}\cdot d\mathbf{r}_c + \int_{\theta_1}^{\theta_2} M_c \, d\theta = \frac{1}{2} m(V_{c_2}^2 - V_{c_1}^2) + \frac{1}{2} I_c (\omega_2^2 - \omega_1^2) \tag{9.17}$$

which is termed as the total work-energy equation for the plane motion of a rigid body.

The total work-energy equation may be reduced to a conservation of energy equation in the absence of dissipative external forces and moments acting on a body or a system of bodies. The work done by a conservative force **F** during a displacement of the centre of mass from position 1 to 2 equals the change in potential energy and remembering that the action of a moment M_c about the centre of mass cannot bring about a change in its position, the total work done is

$$W_{total} = (PE_1 - PE_2)_{total}$$

Equating it to the total change in kinetic energy,

$$(PE_1 - PE_2)_{total} = (KE_2 - KE_1)_{total}$$

Rearranging the terms,

$$(KE_1 + PE_1)_{total} = (KE_2 + PE_2)_{total} \tag{9.18}$$

which shows that, under the action of a conservative force field, the sum of the total kinetic energy and potential energy of a rigid body is conserved. This is also referred to as the principle of conservation of mechanical energy for a rigid body.

***Example* 9.11** In a laboratory, a flywheel of diameter 0.5 m is made to rotate, starting from rest, by means of a suspended mass of 100 kg by an inextensible string wound around a concentric drum of 0.3 m diameter as shown in Fig. Ex. 9.11. If the frictional moment in the bearing is estimated to be 50 N m, determine the moment of inertia of the flywheel if the velocity of the suspended mass after a fall of 1 m is estimated to be 0.3 m/s.

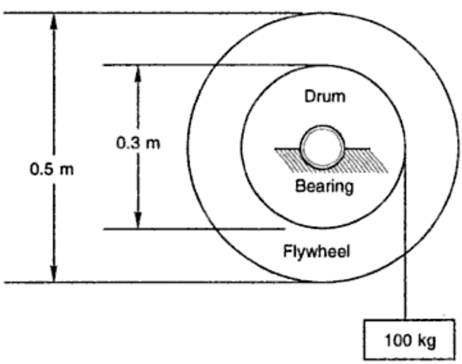


Fig. Ex. 9.11

Solution Let us consider the motion of the system of the flywheel together with the drum and suspended mass. The free-body diagram of the system is shown in Fig. Ex. 9.11 (Solution). Note that the tension in the string does not enter the picture.

The work done by the weight of the suspended mass over a fall of 1.0 m is given by

$$mgh = 100 \times 9.81 \times 1.0 = 981 \text{ N m}$$

and the negative work done by the frictional moment of 50 N m during an angular displacement of

$$\theta = \frac{1.0}{0.15} = 6.67 \text{ rad}$$

is given by

$$M\,\theta = -50 \times 6.67 = -333.5 \text{ N m}$$

Total work done by the net external force and the net external moment acting on the system is

$$W_{total} = 981 - 333.5 = 647.5 \text{ N m}$$

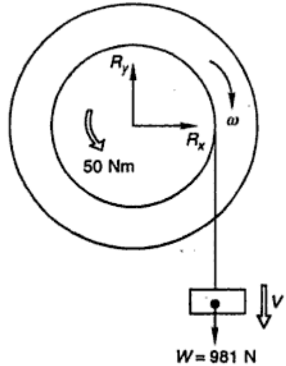


Fig. Ex. 9.11 *(Solution)*

The final total kinetic energy of the system must be the sum of the rotational kinetic energy of the flywheel together with the drum and the translational kinetic energy of the suspended mass.

$$KE_{2total} = \frac{1}{2} \times 100 \times (0.3)^2 + \frac{1}{2} I_c \left(\frac{0.3}{0.15}\right)^2$$

$$= 4.5 + 2\,I_c$$

Since the assembly started from rest, the initial kinetic energy must be zero. The total change in kinetic energy is also given by

$$(KE_2 - KE_1)_{total} = 4.5 + 2\,I_c$$

Using the work energy principle,

$$4.5 + 2\,I_c = 647.5$$

whence $\quad I_c = 321.5 \text{ kg m}^2$

***Example* 9.12** The following bodies are released from rest on an incline at the same elevation. In each case, the mass is m and the maximum radius is R. (a) A solid sphere, (b) a hollow sphere of inner radius $R/2$, (c) a solid cylinder, (d) a hollow cylinder of inner radius $R/2$ and (e) a hoop.

Determine the velocity of each body after it has rolled down the incline through the same distance s. What would be the velocity of each body if the incline was frictionless and the bodies slided instead of rolling?

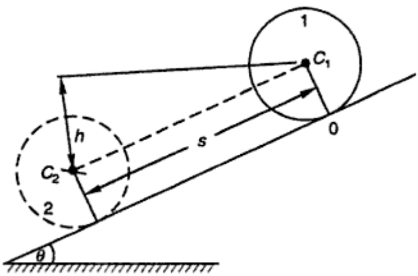


Fig. Ex. 9.12

Solution Consider the *rolling motion of a cylindrical body* in general. The motion takes place in the gravitational field of the earth and the frictional forces do no work in the rolling motion; the mechanical energy of the system is, therefore, conserved.

$$(KE + PE)_1 = (KE + PE)_2$$

$$KE_1 = 0$$

$$PE_1 = 0; \text{ initial position referred as datum}$$

$$KE_2 = \frac{1}{2} m V_{c_2}^2 + \frac{1}{2} I_c \omega_2^2$$

and $$PE_2 = -mgh$$

Substituting

$$\omega_2 = V_{c2}/R$$

$$h = s \sin \theta$$

and from the conservation of mechanical energy,

$$0 = \frac{1}{2} m V_{c_2}^2 + \frac{1}{2} I_c \frac{V_{c_2}^2}{R^2} - mgh$$

whence $$V_{c2} = \frac{\sqrt{2\,gh}}{\sqrt{1 + \dfrac{I_c}{mR^2}}} = \frac{\sqrt{2gs \sin \theta}}{\sqrt{1 + \dfrac{I_c}{mR^2}}}$$

Introducing the appropriate values of the moments of inertia for each case,

Case	*Body*	I_c	V_{c2}
(a)	Solid sphere	$\frac{2}{5} mR^2$	$0.845\sqrt{2gs \sin \theta}$
(b)	Hollow sphere	$\frac{2}{5} m \frac{R^5 - (R/2)^5}{R^3 - (R/2)^3}$	$0.832\sqrt{2gs \sin \theta}$

(Contd.)

(*Contd.*)

Case	Body	I_c	V_{c2}
(c)	Solid cylinder	$\frac{1}{2}mR^2$	$0.816\sqrt{2gs \sin\theta}$
(d)	Hollow cylinder	$\frac{1}{2}m(R^2 + (R/2)^2)$	$0.785\sqrt{2gs \sin\theta}$
(e)	Hoop	mR^2	$0.707\sqrt{2gs \sin\theta}$

If the incline was frictionless, the bodies would not be able to roll down. Frictionless sliding through an inclined distance s would be equivalent to a free fall through a vertical distance h

$$h = s \sin\theta$$

The velocity of each body would then be

$$\sqrt{2\,gh} = \sqrt{2\,gs \sin\theta}$$

A comparison of the final velocities of the bodies is indeed meaningful. Frictionless fall down the incline results in the maximum possible velocity. The final velocities in the other cases are less *not because of frictional loss of energy but because of the energy stored in rotation of the bodies.* The rotational energy depends on the moment of inertia of the bodies and for this reason their final velocities differ. Of the given bodies of the same mass m and same outer radius R, the solid sphere has the least moment of inertia and hence the greatest final velocity. On the other extreme, in the case of a hoop, the entire mass is concentrated at the radius R which makes its moment of inertia the maximum and its final velocity the least.

The results can be interpreted in another way, i.e., if the bodies in the list were allowed to start from rest at the same level simultaneously, the acceleration of the bodies would be such that the velocity at a later instant would be determined in the analysis. The body which accelerates most reaches first. The solid sphere will, therefore, reach the finishing point of the race first and then the hollow sphere, solid cylinder and hollow cylinder in that order. The hoop will be the last to reach. It is also interesting to note that the final velocities attained over a given distance along the slope or the accelerations acquired by the bodies are *independent of their mass and size*. In other words, whether a sphere is small or large, light or heavy, it will reach the finishing point before the other bodies do. It also follows that if a number of spheres of different sizes and mass densities are allowed to go down an incline starting from rest simultaneously, they will all acquire the same acceleration and reach the finishing point simultaneously.

Let us also demonstrate the application of the dynamical equations in terms of forces and moments for this problem. With reference to the free-body diagram shown in Fig. Ex. 9.12 (Solution) the dynamical equations may be written as

$$F_x = mg \sin\theta - f = ma_{cx} = ma_c \quad \text{(i)}$$

$$F_y = mg \cos\theta - R = ma_{cy} = 0 \quad \text{(ii)}$$

support gently and allowed to move for t seconds further, estimate its angular speed and the velocity of the centre of mass at that instant.

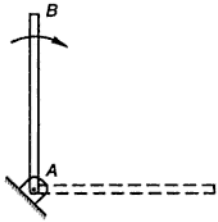


Fig. Ex. 9.13

Solution Applying the energy conservation principle between the vertical position (1) and the horizontal position (2),

$$(KE + PE)_1 = (KE + PE)_2$$

$$(0 + mgL/2) = \left(\frac{1}{2} I_c \omega^2 + \frac{1}{2} m V_c^2 + 0\right)$$

Using the facts that $I_c = \frac{mL^2}{12}$ and $V_c = \omega L/2$,

$$\omega^2 = 3g/L \quad \text{or} \quad \omega = \sqrt{3g/L}$$

in the clockwise direction.

Considering the free-body diagram at the instant when it just acquires the horizontal position (Fig. Ex. 9.13 (Solution)).

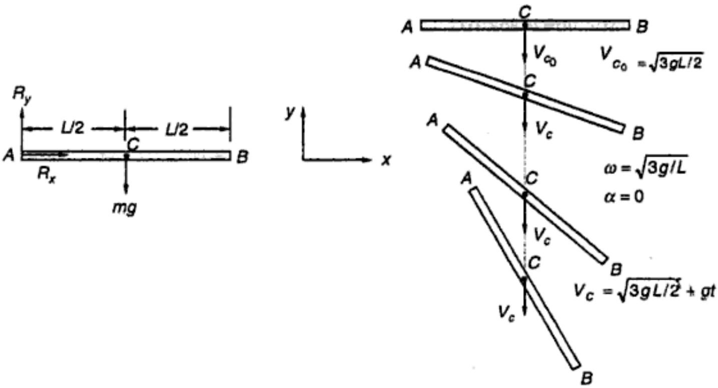


Fig. Ex. 9.13 ***(Solution)***

For the centre of mass of the bar,

$$R_x = ma_{cx} \qquad \text{(i)}$$

$$R_y - mg = ma_{cy} \qquad \text{(ii)}$$

and by the Euler's equation applied at A,

$$-mg\ L/2 = I_A\ \alpha = \frac{mL^2}{3}\alpha \qquad \text{(iii)}$$

From Eq. (iii), the angular acceleration

$$\alpha = -3g/2L$$

Potential energy at the point of interest

$$= M/4\ g\ R/2 = MgR/8$$

because the rolling mass must be quarter fraction of the initial mass.

Release of potential energy = 7/8 $Mg\ R$

The kinetic energy at this instant must be given by

$$KE = \frac{1}{2}\frac{M}{4}\cdot V_c^2 + \frac{1}{2} I\,\omega^2$$

$$= \frac{MV_c^2}{8} + \frac{1}{2}\frac{M}{4}\cdot\frac{(R/2)^2}{2}\cdot\left(\frac{V_c}{R/2}\right)^2 = 3/16\ MV_c^2$$

Hence, $3/16\ MV_c^2 = 7/8\ Mg\ R$

and $V_c = \sqrt{14/3\ gR}$

***Example* 9.15** A small sphere rolls down without slipping from the top of a track in a vertical plane. The track has an elevated section and a horizontal part as shown in Fig. Ex. 9.15. The horizontal part is 1.0 metre above the ground level and the top of the track is 2.4 metres above the ground. Find the distance on the ground with respect to the point B (which is vertically below the end of the track as shown) where the sphere lands. During its flight as a projectile, does the sphere continue to rotate about its centre of mass? Explain.

Solution At the initial point P,

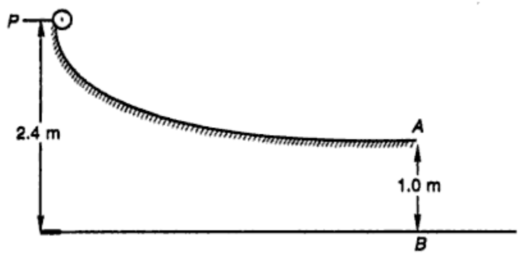


Fig. Ex. 9.15

$KE_1 = 0,\ PE_1 = 2.4\ mg$

At the end point A,

$$KE_2 = \frac{1}{2}\ mv^2 + \frac{1}{2}\ I\ \omega^2$$

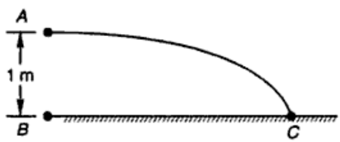


Fig. Ex. 9.15 ***(Solution)***

$$= \frac{1}{2} m V^2 + \frac{1}{2}\left(\frac{2}{5} m R^2\right)\frac{V^2}{R^2}$$

$$= \frac{1}{2} mV^2 + \frac{1}{5} mV^2 = \frac{7}{10} mV^2$$

$$PE_2 = 1.0\ mg$$

By conservation of energy in the absence of dissipation,

$$0 + 2.4\ mg = \frac{7}{10} mV^2 + 1.0\ mg;$$

whence, $V = \sqrt{2g}$

After leaving A, the sphere would continue to rotate at the same rotational speed in the absence of my external moment acting on it.

Then, $z = -\dfrac{g x^2}{2V_0^2}$

Therefore, $x = \sqrt{2 \times 2g \times 1/g} = 2.0$ m.

***Example* 9.16** State whether the following statement is TRUE or FALSE. Give very brief reasons in support of your answer.

A ring of mass 0.3 kg and radius 0.1 m and a solid cylinder of mass 0.4 kg and of the same radius are given the same kinetic energy and released simultaneously on a flat horizontal surface such that they begin to roll as soon as released towards a wall which is at the same distance from the ring and the cylinder as shown in Fig. Ex. 9.16. The rolling friction in both cases is negligible. The cylinder will reach the wall first.

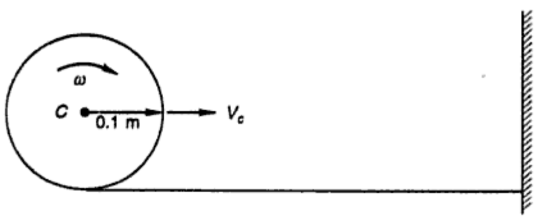


Fig. Ex. 9.16

Solution

For the ring,

$m = 0.3$ kg

$r = 0.1$ m

$I_c = 0.3 \times 0.1^2 = 0.003$ kg m^2

For the cylinder,

$m = 0.4$ kg

$r = 0.1$ m

$I_c = 0.4 \times \dfrac{0.1^2}{2} = 0.002$ kg m^2

For the ring,

$$KE = \frac{1}{2} \times 0.3 \times V_a^2 + \frac{1}{2} \times 0.003 \times \left(\frac{V_c}{0.1}\right)^2.$$

$$= 0.5\, V_c^2 + 0.15\, V_c^2 = 0.3\, V_c^2$$

For the cylinder,

$$KE = \frac{1}{2} \times 0.4 \times V_c^2 + \frac{1}{2} \times 0.002 \times \left(\frac{V_c}{0.1}\right)^2$$

$$= 0.2\, V_c^2 + 0.1\, V_c^2 = 0.3\, V_c^2$$

Since they have the same kinetic energy at the start, the expressions for *KE* require that they have the same velocity as well. There being no change in potential energy and negligible friction, they will reach the wall simultaneously.

The given statement is, therefore, FALSE.

***Example* 9.17** A sphere of radius 0.5 m and mass 10 kg is released gently from rest on a 30° incline as shown in Fig. Ex. 9.17. If it rolls without slipping, determine the minimum coefficient of friction compatible with the rolling motion. What would be the velocity of its centre of mass after it rolled down 5 m.

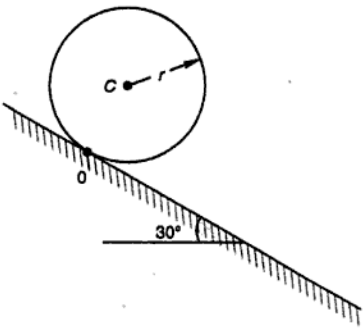


Fig. Ex. 9.17

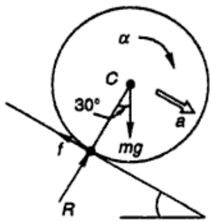


Fig. Ex. 9.17 ***(Solution)***

Solution Considering the free-body diagram of the sphere as shown in Fig. Ex. 9.17 (Solution) rolling without slipping,

$$a = r\alpha$$

$$R = mg \cos \theta \qquad \text{(i)}$$

$$mg \sin \theta - f = ma \qquad \text{(ii)}$$

$$mg \sin \theta \cdot r = ma \cdot r + I\alpha$$

or

$$mg \sin \theta\, r = m\, \alpha\, r^2 + 2/5\, mr^2 \cdot \alpha$$

whence,

$$\alpha = \frac{5g \sin \theta}{7r} = \frac{5 \times 9.81 \times \sin 30^\circ}{7 \times 0.5} = 7 \text{ rad/s}^2$$

and $$a = 0.5 \times 7 = 3.5 \text{ m/s}^2$$

From (ii), $$f = 10 \times 9.81 \times \sin 30 - 10 \times 3.5 = 14.05 \text{ N}$$

For $$f = \mu R = \mu \cdot mg \cos \theta$$

$$\mu = 14.05/(10 \times 9.81 \times \cos 30) = 0.165$$

Since the acceleration of the centre of mass is constant, 3.5 m/s², the velocity 5 m down the plane is such that

$$v^2 - 0 = 2 \times 3.5 \times 5$$

whence $$v = 5.91 \text{ m/s}$$

Let us determine the velocity of the sphere if it was sliding all the way on a frictionless plane. Then $f = 0$,

$$mg \sin \theta = ma, \qquad \alpha = 0$$

$$a = g \sin \theta$$

$$v^2 - 0 = 2 \times 9.81 \times \sin 30° \times 5$$

$$v = 7 \text{ m/s}$$

This is more than the velocity of the rolling sphere but there is no rotational velocity in this case. Let us compare the kinetic energies in the two cases, i.e., with rotation and frictionless sliding. In the former,

$$KE = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} mv^2 (1 + 2/5) = \frac{7}{10} \times 10 \times 5.91^2 = 245 \text{ J}$$

In the latter,

$$KE = \frac{1}{2} \times 10 \times 7^2 \approx 245 \text{ J}$$

The kinetic energy (as also the potential energy) is the same in both cases. It is indeed so because pure rolling does not entail loss of energy.

Let us do a little reflection. If it was a cylinder instead of a sphere, the moment equation would be

$$mg \sin \theta \cdot r = m\,\alpha\, r^2 + m\, r^2/2 \cdot \alpha = 3/2\, mr^2\, \alpha$$

whence $$\alpha = \frac{2g \sin \theta}{3r} = \frac{2 \times 9.81 \times \sin 30}{3 \times 0.5} = 6.54 \text{ rad/s}$$

$$a = 0.5 \times 6.54 = 3.27 \text{ m/s}^2$$

Then, $$f = 10 \times 9.81 \times \sin 30 - 10 \times 3.27 = 16.35 \text{ N}$$

and $$\mu = 16.35/(10 \times 9.81 \times \cos 30) = 0.192$$

which means that the surface must be more rough for the rolling of a cylinder than for a sphere. In other words, if an incline is rough enough to cause a cylinder to roll without slip, a sphere of the same mass and radius would surely roll without slip! Extending the same analysis to a hoop of the same mass and radius, $\mu = 0.288$ implies that the hoop is even more likely to slip than a cylinder on the same slope.

9.6 IMPULSE-MOMENTUM FORMULATION FOR PLANE MOTION

The equations of motion due to Newton and Euler may be integrated with respect to time to constitute an alternative set of equations for studying the dynamics of a rigid body.

The statement of the Newton's law

$$\mathbf{F} = m\,\mathbf{a} = m\frac{d\mathbf{V}}{dt}$$

integrated with respect to time t over the limits t_1 to t_2 results in

$$\boxed{\int_{t_1}^{t_2} \mathbf{F}\, dt = m\,|\mathbf{V}|_1^2 = m(\mathbf{V}_2 - \mathbf{V}_1)} \tag{9.19}$$

The left-hand side of Eq. (9.19) is the linear impulse due to the net external force $\mathbf{F}$ over the period of action whereas the right-hand side is the change in linear momentum of the centre of mass over the same interval of time. The equation is, therefore, called the *linear impulse momentum equation*, sometimes referred simply as the impulse-momentum equation.

The Euler's equation,

$$M_c = I_c\,\alpha = I_c\,\frac{d\omega}{dt}$$

on integration with respect to time t over the limits t_1 and t_2 results in

$$\boxed{\int_{t_1}^{t_2} M_c\, dt = I_c\,|\omega|_1^2 = I_c(\omega_2 - \omega_1)} \tag{9.20}$$

The left-hand side of Eq. (9.20) is the angular impulse due to the net moment M_c acting about the centre of mass C over the period of action, whereas the right-hand side is the change in angular momentum about the centre of mass over the same interval of time. The equation is, therefore, known as the *angular impulse-momentum principle*. It may be noted that for the plane motion of a rigid body, the angular impulse as well the angular momentum about the centre of mass must only be about the axis of rotation through C.

There is no doubt that *the principles of linear impulse-momentum and angular impulse-momentum applied to a rigid body are mutually independent principles*; one not derivable from the other and both of them are required jointly to tackle a problem on rigid-body dynamics in just the same way as Newton's law and Euler's equation are used. Preference for the impulse-momentum principles over the said laws is natural when concerned with the gross effect of the change in linear and angular velocity over a period of time. In particular, when the net external force or net external moment is zero, the corresponding impulse-momentum principle reduces to momentum conservation and may be used more conveniently. A system of two bodies in an encounter, say an impact, have no external force or moment acting upon them; the momentum conservation principle may be applied to advantage to the system of the two bodies. The momentum conservation principles for two bodies may be stated as follows:

Linear-momentum conservation principle:

$$m_1 V_{c_1} + m_2 V_{c_2} = m_1 V'_{c_1} + m_2 V'_{c_2} \tag{9.21}$$

Angular-momentum conservation principle:

$$I_{c_1} \omega_1 + I_{c_2} \omega_2 = I_{c_1} \omega'_1 + I_{c_2} \omega'_2 \tag{9.22}$$

where suffices 1 and 2 refer to the two bodies; unprimed variables for conditions just before and primed variables for conditions just after the phenomenon.

***Example* 9.18** A uniform circular cylinder of mass m and radius r is given an initial angular velocity ω_0 and no initial translational velocity as shown in Fig. Ex. 9.18. It is placed in contact with a plane inclined at α to the horizontal and it moves up. Assuming the coefficient of friction μ for sliding between the cylinder and the plane, find the distance the cylinder moves before sliding stops. Assume that μ is greater than tan α.

Solution Considering the free-body diagram of the cylinder as shown in Fig. Ex. 9.18. (Solution). Let us choose the centre of mass O as the reference point and use the angular impulse momentum equation to determine the time t required to decrease the velocity to a value ω.

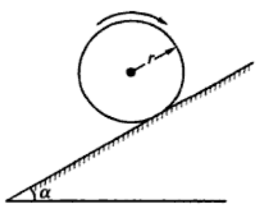


Fig. Ex. 9.18

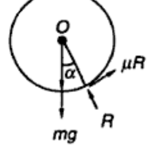


Fig. Ex. 9.18 *(Solution)*

$$-\mu R r t = I(\omega - \omega_0) = \frac{mr^2}{2}(\omega - \omega_0)$$

Since $\quad R = mg \cos \alpha$

$$t = \frac{r(\omega_0 - \omega)}{2\mu g \cos \alpha} \tag{i}$$

The velocity of the centre O parallel to the plane is obtained by applying the linear impulse momentum equation.

$$mg\,(\mu \cos \alpha - \sin \alpha)t = m(v - 0) \qquad \text{(ii)}$$

with positive direction up the plane.

At the instant sliding stops and pure rolling begins,

$$v = r\,\omega\,.$$

From (i) and (ii),

$$\omega = \frac{\mu \cos \alpha - \sin \alpha}{3\,\mu \cos \alpha - \sin \alpha} \cdot \omega_0$$

and
$$t = \frac{r\,\omega_0}{g(3\mu \cos \alpha - \sin \alpha)}$$

The forces acting on the cylinder remain constant during sliding and hence acceleration is constant. The distance travelled while sliding must be

$$d = 1/2\ vt$$

$$= \frac{r^2\ \omega_0^2\ (\mu \cos \alpha - \sin \alpha)}{2g\,(3\,\mu \cos \alpha - \sin \alpha)^2}.$$

***Example* 9.19** Determine the angular momentum of a uniform thin bar of length L and mass m when it is rotating at a constant angular velocity ω

(a) about its centre of mass

(b) about one end

provided that it stays in the same plane.

Solution The mass per unit length of the bar is m/Lh

Fig. Ex. 9.19 ***(Solution)***

For case (a), the element of mass $m/L \cdot dr$ shown in Fig. Ex. 9.19 (Solution) rotates at a velocity $r\omega$; its angular momentum is $r\ m/L\ r\omega\ dr$.

Integrating from $-L/2$ to $+L/2$, for the whole bar,

$$H_c = \int_{-L/2}^{+L/2} \frac{m\omega}{L} r^2 dr = \frac{m\omega}{L}\left(\frac{r^2}{3}\right)_{-L/2}^{+L/2} = 1/12\ mL^2\omega$$

$$= 1/12\ m\,L^2\ \omega = I_c\ \omega$$

where I_c is the moment of inertia of the bar about its centroid.

Similarly, for case (b), considering the origin at O and locating the element dr at a distance r for O,

$$H_c = \int_0^L \frac{m\omega}{L} r^2 \, dr = 1/3 \, m L^2 \, \omega = I_0 \, \omega.$$

***Example* 9.20** A uniform bar AD of length 2 m and mass 5 kg hanging freely from a frictionless pivot at A is struck by a 25 g bullet approaching at a velocity of 500 m/s as shown in Fig. Ex. 9.20. The bullet pierces through the bar and emerges with a velocity 40% of its initial value.

Determine the angular velocity of the bar just after the bullet emerges and the maximum angle through which the bar would swing. Comment on the total loss of energy in the process.

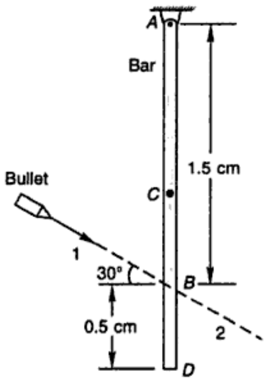


Fig. Ex. 9.20

Solution The free-body diagrams are drawn or simply visualised in order to decide whether to apply the linear impulse-momentum principle or the angular impulse-momentum principle (Fig. Ex. 9.20 (Solution)).

The fact that there is going to be an unknown impulsive reaction at the pivot A rules out the conservation of linear momentum along any axis. From the fact that the moment of impulsive reaction at A taken about A is zero, the angular impulse momentum principle about A can be employed with advantage. It should be understood that there is no external force acting at B as far as the system of bullet and bar is

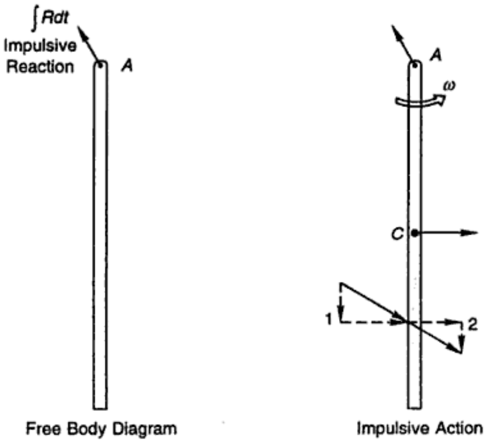


Fig. Ex. 9.20 *(Solution)*

concerned. The bullet exerts a force on the bar and the bar reacts it undiminished on the bullet.

The net angular impulse must equal the change in the angular momentum. Initially, the angular impulse or the moment of linear momentum taken about A is

$$0.025 \times 500 \cos 30° \times 1.5 = 16.24 \text{ kg m}^2/\text{s}$$

Finally, the angular impulse or the summation of the moments of the linear moment of the system about A is

$$0.025 \times (500 \times 0.4) \cos 30° \times 1.5 + 5 \times 1^2 \times \omega$$

$$= 6.5 + 5\omega \quad \text{kg m}^2/\text{s}$$

Net angular impulse about A is

$$16.25 - (6.5 + 5\omega) = 9.75 - 5\omega \quad \text{kg m}^2/\text{s} \qquad \text{(i)}$$

Starting from rest, the bar acquires an angular velocity so that the change in the angular momentum is

$$I_c\omega = \frac{1}{12} \times 5 \times 2^2 \omega = 1.67\omega \qquad \text{kg m}^2/\text{s} \qquad \text{(ii)}$$

Equating Eqs. (i) and (ii)

$$9.75 - 5\omega = 1.67\omega$$

whence $\omega = 1.46$ rad/s

Alternatively, the angular impulse-momentum principle applied to the bar can be interpreted as,

Net angular impulse imparted by the bullet on the bar = Change in angular momentum of the bar about its centre of mass

The left-hand side term is

$$0.025 \times (500 - 0.4 \times 500) \cos 30° \times 1.5 = 9.75 \qquad \text{kg m}^2/\text{s}$$

and the right-hand side term is

$$\frac{1}{3} \times 5 \times 2^2(\omega - 0) = 6.67\omega \qquad \text{kg m}^2/\text{s}$$

which when equated, provide $\omega = 1.46$ rad/s, the same result as obtained earlier.

The maximum swing of the bar can be calculated by applying the energy conservation principle because there are no dissipative actions from the instant the bullet emerges out of the bar to the instant of maximum swing. Considering the initial position as the reference position,

$$PE_1 = 0$$

$$KE_1 = \frac{1}{2} \times \left(\frac{1}{3} \times 5 \times 2^2\right) \times 1.46^2 = 7.1 \text{ J}$$

Finally, the bar swings through an angle θ and the centre of gravity is lifted by

$$h = \frac{2}{2} \times (1 - \cos\,\theta) = (1 - \cos\,\theta)$$

$$PE_2 = 5 \times 9.81 \times (1 - \cos\,\theta)$$

$$= 49.05(1 - \cos\,\theta)$$

and $$KE_2 = 0$$

because the bar comes to momentary rest at the position of maximum swing.

By energy conservation,

$$PE_1 + KE_1 = PE_2 + KE_2$$

$$7.1 = 49.05(1 - \cos\,\theta)$$

whence $$\theta = 31.2°$$

In order to estimate the total loss of energy in the process, consider the energy in the system just before and after impact.

Just before the impact, only the bullet has kinetic energy given by

$$\frac{1}{2} \times 0.025 \times 500^2 = 3125 \text{ J}$$

Just after the impact, the bullet has a part of its kinetic energy and the bar has kinetic energy given by

$$\frac{1}{2} \times 0.025 \times 200^2 + \frac{1}{2} \times \left(\frac{1}{3} \times 5 \times 2^2\right) \times 1.46^2$$

$$= 507.1 \text{ J}$$

Loss in energy of given by

$$3125 - 507.1 = 2617.9 \text{ J}$$

which is

$$\frac{2617.9}{3125} \times 100 = 83.8\%$$

of the initial energy of the bullet.

The percentage energy imparted to the pendulum in moving the pendulum is

$$\frac{7.1}{3125} \times 100 = 0.23\%$$

which is indeed a very small fraction of the initial energy of the bullet.

It is the mechanical energy which is lost; in fact, it is converted into heat and sound and thus dissipated.

If it is desired to determine the impulsive reaction at A, it is now possible to do so by applying the linear impulse momentum principle.

$$\int R_x\, dt = 0.025(500 - 200) \cos 30° - 5 \times 1 \times 1.46$$

It shows that the safe velocity limit is higher for

- large b, i.e., wide-base vehicles
- low h, i.e., low centre of mass
- high r, i.e., curve of large radius

and for a location with a higher value of g!

The vehicle would slip side ways when the lateral force equals the frictional resistance.

$$m\frac{v^2}{r} = \mu (R_1 + R_2)$$

$$= \mu mg$$

whence $$v = \sqrt{\mu g r} \tag{ii}$$

Comparing (i) and (ii), it is observed that on a smooth road, i.e., μ less than b/h, the vehicle will skid rather than overturn. For a rough road, μ greater than b/h, the vehicle tends to overturn than slip sideways.

Let us now consider the free body diagram of the vehicle turning on a banked road (as shown in Fig. Ex. 9.22 (Solution)).

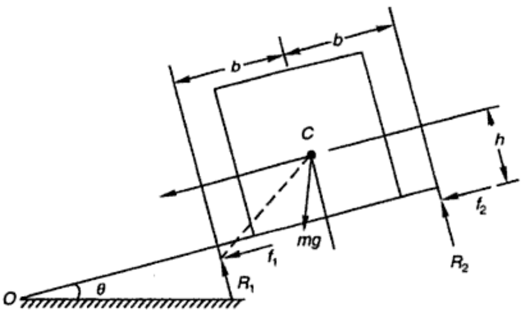


Fig. Ex. 9.22 *(Solution)*

$$f_1 + f_2 + mg \sin\theta = m\frac{v^2}{r}\cos\theta$$

$$R_1 + R_2 - mg\cos\theta = m\frac{v^2}{r}\sin\theta$$

$$R_2 \cdot 2b - mg\,(b\cos\theta - h\sin\theta) = m\frac{v^2}{r}\,(b\sin\theta + h\cos\theta)$$

For the condition of overturning, $R_1 = 0$,

$$v = \sqrt{\frac{1 + h/b\tan\theta}{1 - b/h\tan\theta}\cdot b/h\, gr}$$

9.7 GENERAL MOTION OF A RIGID BODY

Let us now establish the dynamical equations of the general motion of a rigid body. By general motion, we imply a combination of translation in space and rotation about the coordinate axes.

Consider a rigid body in general motion referred to the fixed reference frame *XYZ* as shown in Fig. 9.6. Let a set of body axes *xyz* be fixed at the point *A*. Let the

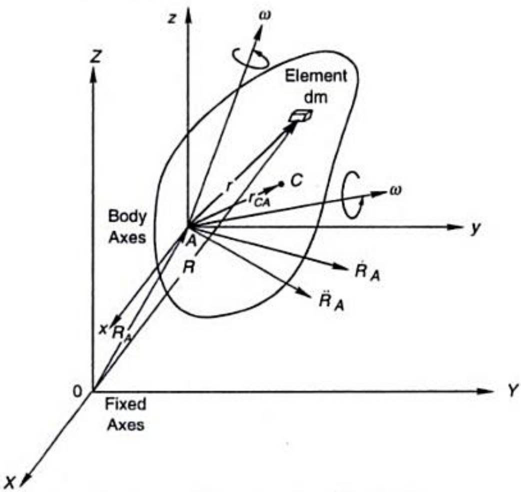


Fig. 9.6 ***General Motion of a Rigid Body***

$$H_x = \omega_x \int (y^2 + z^2)\, dm - \omega_y \int xy\, dm - \omega_z \int xz\, dm$$

$$H_y = \omega_y \int (z^2 + x^2)\, dm - \omega_z \int yz\, dm - \omega_x \int yx\, dm$$

$$H_z = \omega_z \int (x^2 + y^2)\, dm - \omega_x \int zx\, dm - \omega_y \int zy\, dm$$

Recalling the definitions of the moments of inertia, e.g.,

$$I_{xx} = \int (y^2 + z^2)\, dm$$

$$I_{xy} = \int xy\, dm$$

....

the angular momenta can be expressed as

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= I_{yy}\omega_y - I_{yz}\omega_z - I_{yx}\omega_x \\ H_z &= I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y \end{aligned} \tag{9.29}$$

In matrix form,

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

or

$$\boxed{\mathbf{H}_A = \mathbf{I}_A\, \omega} \tag{9.30}$$

The components of the inertia matrix are defined with respect to axes attached to and rotating with the body; these are invariant with respect to the body axes. If the body axes are selected to be the axes of symmetry or if these are the principal axes at the reference point, then the products of inertia vanish i.e.,

$$I_{xy} = I_{yz} = I_{zx} = I_{yx} = I_{xz} = I_{zy} = 0$$

and the principle moments of inertia are

$$I_x = I_{xx},\ I_y = I_{yy} \quad \text{and} \quad I_z = I_{zz}$$

Then the angular momentum is expressed as

$$\mathbf{H} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

where

$$\begin{aligned} H_x &= I_x\omega_x \\ H_y &= I_y\omega_y \\ H_z &= I_z\omega_z \end{aligned} \tag{9.31}$$

Case VI: Translation of a Rigid Body

$$\omega = 0 \qquad \text{and} \qquad \dot{\omega} = 0$$

whence, $\omega_x = \omega_y = \omega_z = 0$

and $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$

The Euler's equations are then reduced to naught and the dynamical equations for the body are:

$$F_x = ma_x$$

$$F_y = ma_y$$

$$F_z = ma_z \tag{9.44}$$

where a_x, a_y and a_z are the acceleration components for any point of the body, as also for the centre of mass.

For a plane translatory motion, therefore,

$$F_x = ma_x$$

$$F_y = ma_y$$

if the parallel planes of motion are parallel to the *xy* plane.

For a rectilinear motion along the *x*-axis, the sole equation of motion must be

$$F_x = ma_x$$

9.8 GYROSCOPIC ACTION

If an axisymmetric rigid body such as a plate or wheel spins about its axis of symmetry and this axis is precessing with a uniform angular velocity about an axis perpendicular to that of spin, then a couple called a gyroscopic couple acts on the body which is directed normal to the axes of spin and precession.

If a body spins at a constant angular velocity ω about its *x*-axis as shown in Fig. 9.7(a) and it has a moment of inertia I about this axis then the angular momentum possessed by it is given by

$$\boxed{H = I\omega}$$

which is shown by a vector *oa*. In a short interval of time Δt, the axis of spin precesses about the *z* axis by an angle of $\Delta\phi$. The angular momentum of the body remains the same in magnitude, i.e., $I\omega$ but changes in direction through an angle $\Delta\phi$ as represented by a vector *Ob*. The change from the initial to the final position over the time Δt is given by $I\omega\,\Delta\phi$ and the rate of change of angular momentum,

$$\dot{H} = \underset{\Delta t \to 0}{\text{Lt}} \frac{I\omega\,\Delta\phi}{\Delta t} = I\omega \frac{d\phi}{dt} = I\omega\omega_p$$

because the precessional angular velocity is given by

$$\omega_p = \frac{d\phi}{dt}$$

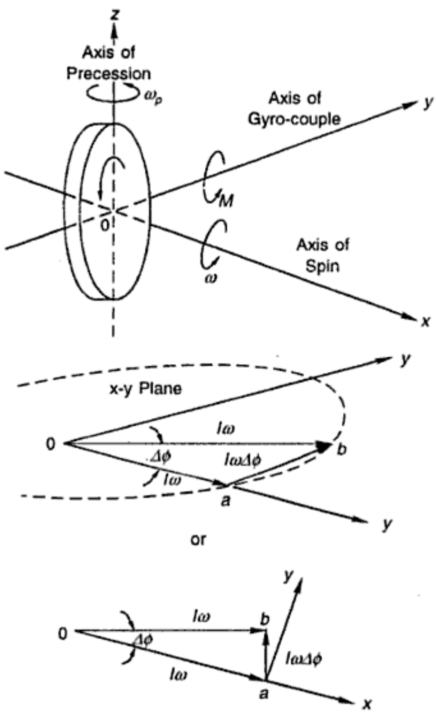
The rate of change of angular momentum, by Euler's equation, must equal the external moment acting on the body

$$\boxed{M = I\,\omega\,\omega_p} \tag{9.45}$$

This moment is also termed the gyro-couple.

It may be noted that the gyro-couple acts in accordance with the right-handed triad. The rule to determine the direction of the gyro-couple may be stated as follows:

The spin ω, gyro-couple M and precession ω_p are consistent if all of them are along the positive axes or if two of them are along the negative axes.



Vector Diagram in the x-y Plane for Gyroscopic Phenomenon

Fig. 9.7 ***(a) Gyroscope and Gyroscopic Action***

By the principle of the gyroscope,

$$M = I\omega\,\omega_p$$

$$\omega = \frac{M}{I\omega_p} = \frac{2.94}{0.005 \times 0.524}$$

$$= 1122 \text{ rad/s}$$

$$= 1122 \times \frac{60}{2\pi} = 10{,}720 \text{ revolutions per minute}$$

As is typical of a gyroscopic action, the spinning speed of the gyro-rotor is extremely high. The high speed of rotation of 10,720 revolutions per minute is, hence, not unexpected.

Another fact that emerges from this example is that the gyroscopic couple can be large to be able to precess a rotating body. If the polar moment of inertia of the gyro-rotor is large and the spinning velocity is made extremely high, then the gyro-couple required to precess the rotor by a small angle can be very large. This fact about a gyroscope is exploited by using it to provide stability to vehicles both on the earth and in space. *But for gyroscopic stability, it would have been impossible to drive a two-wheeler vehicle such as a bicycle or motorcycle.* Some other examples of the gyroscope are gyrocompass, rate-of-turn gyro, artificial horizon for aircraft and stabilization of aero-engines, rockets and ships.

Concept Review Questions

1. State, giving reasons, why
   (a) the reference point on a rigid body is usually chosen as the centre of mass or a point fixed in space.
   (b) the reference axes are attached with a rigid body oriented to coincide with the principal axes.
   (c) Newton's law and Euler's equation together govern the motion of rigid bodies.
   (d) the kinetic energy of a rigid body in general motion is the sum of the translational kinetic energy associated with the centre of mass and the rotational kinetic energy of the mass rotating about the centre of mass.
2. Under what conditions and for what states of motion is the simple form
   $$M = I\alpha$$
   adequate to describe the motion of a rigid body?
3. Write down the set of equations governing the dynamic behaviour of
   (a) a rigid body in rectilinear translation, plane translation, and space translation.
   (b) a rigid body in general plane motion.
   (c) a rigid body rotating about a fixed axis which is also its principal axis.
4. Why is it that
   (a) Euler's equations do not contain the products-of-inertia terms?
   (b) the components of the inertia matrix do not change with time during the motion of a rigid body?
   (c) the reference point has to be chosen with restrictions for the application of Euler's equations?
5. State the assumptions made in the development of Euler's equations and trace the origin of these assumptions.
6. State the form of the following principles as applied to the dynamics of a rigid body.
   (a) Impulse-momentum principle

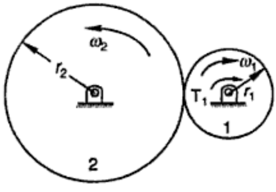


Fig. Prob. 9.6

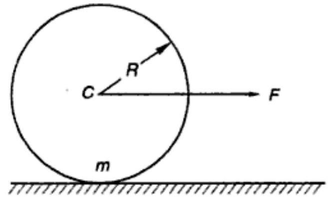


Fig. Prob. 9.7

9.7 A horizontal force F applied to the centre of mass of a sphere of radius R and mass m causes it to roll without slip (Fig. Prob. 9.7).

(a) Find the acceleration of the sphere.

(b) If there was a cylinder of the same radius and mass instead of a sphere and the force was applied centrally on it, would it have accelerated more or less and why?

Assume the friction coefficient as μ in each case.

(**Ans.** $(5F/7)$ m and $(2F/3)$ m; less)

9.8 At what distance p should the horizontal force F be applied to the homogeneous bar, homogeneous cylinder and homogeneous sphere so that the horizontal component of the reaction at the point of suspension is zero (ref. Fig. Prob. 9.8).

$\left(\textbf{Ans.}\ \frac{2}{3}d, \frac{3}{4}d \text{ and } \frac{7}{10}d\right)$

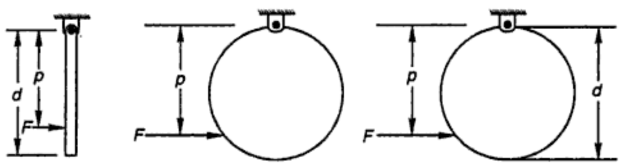


Fig. Prob. 9.8

9.9 An aeroplane is flying at 500 km/hr. Find the angle with the horizontal at which it must bank in order to turn without slip sideways, in a circular path of radius 5 km. Assume that the resultant air pressure on it acts through its centre of gravity at right angles to the angle of banking. (**Ans.** 21.5°)

9.10 A uniform rigid cylinder of mass m and radius r rolls without slip on a horizontal surface. What is its kinetic energy when its centre has a speed V_c?
Would a sphere of the same radius, same mass and same velocity of the centre of mass have less or more kinetic energy?

$\left(\textbf{Ans.}\ \frac{3}{4}mV_c^2\text{; less, }\frac{7}{10}mV_c^2\right)$

9.11 A bar AB of mass M and length L is pinned to a disc of mass m and radius r and the two are suspended at point A to hang freely. If a horizontal force F is applied at the centre of the bar, determine the angular accelerations of the bar and disc at the instant as shown in Fig. Prob. 9.11.

$\left(\textbf{Ans.}\ \frac{24f}{13mL}, \frac{12f}{13mr}\right)$

D
APPLICATIONS IN DYNAMICS

This section consists of some salient applications in dynamics under the following two chapters:

- ❑ D1 IMPACT OF TWO BODIES
- ❑ D2 CENTRAL FORCE MOTION

D1.2 COEFFICIENT OF RESTITUTION

The energy dissipation effects are assumed to be described by a single scalar parameter e, the *coefficient of restitution* defined as the ratio of the impulses of recovery and deformation for either body. Considering the equation of motion for each body, the impulse of contact force equals the change in momentum for the body along the line of impact in the vicinity of the contact point.

With reference to an oblique central impact, as shown in Fig. D1.3(b), together the force-time plot,

$$e = \frac{\text{Impulse of recovery}}{\text{Impulse of deformation}} = \frac{I_r}{I_d}$$

$$= \int_{\Delta t_1}^{\Delta t_2} F\,dt \Bigg/ \int_{0}^{\Delta t_1} F\,dt$$

For body 1,

$$e = \frac{m_1 (V_1' - V_c)\cdot \mathbf{e}_n}{m_1 (V_c - V_1)\cdot \mathbf{e}_n} \tag{i}$$

For body 2,

$$e = \frac{m_2 (V_2' - V_c)\cdot \mathbf{e}_n}{m_2 (V_c - V_2)\cdot \mathbf{e}_n} \tag{ii}$$

Combining Eqs. (i) and (ii)

$$e = \frac{(V_2' - V_1')}{(V_1 - V_2)} = -\frac{V_{sn}}{V_{an}} \tag{D1.3}$$

$$e = -\frac{\text{Velocity of separation along the line of impact}}{\text{Velocity of approach along the line of impact}}$$

The coefficient of restitution e may now be related to the dissipation of energy during a direct central impact. Assuming that the potential energy of each body remains the same during the infinitesimal time of impact, the kinetic energy just before the impact is

$$KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \tag{D1.4}$$

and the kinetic energy just after the impact is

$$KE' = \frac{1}{2} m_1 V_1'^2 + \frac{1}{2} m_2 V_2'^2 \tag{D1.5}$$

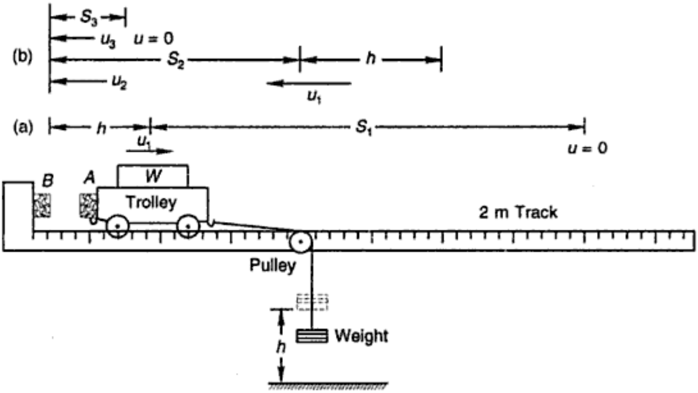


Fig. E12.1 ***Apparatus for Coefficient of Restitution***

BACKGROUND INFORMATION

The trolley is placed on the bumper end of the track. The cord and weight are tied to it at the other end and released. As soon as the weight drops to the ground the trolley is let go freely on the track. It decelerates and comes to rest after a distance S_1 given by

$$u_1^2 = 2aS_1 \tag{E12.1}$$

where u_1 is the speed of the trolley at the instant the weight just touches the ground.

If the trolley is now placed on the other side of the pulley and the weight, attached to the same length of the cord, is dropped by the same height h, the speed acquired by the trolley would be u_1 and it would go on for a distance S_1. However, after the trolley travels a distance S_2, an impact takes place, a fraction of energy is dissipated and the trolley retracts by a distance S_3 instead of the distance $(S_1 - S_2)$ as expected in the absence of an impact.

The speed of the trolley just before the impact is given by

$$u_1^2 - u_2^2 = 2aS_2$$

whence $$u_2^2 = u_1^2 - 2aS_2 = 2aS_1 - 2aS_2 = 2a\,(S_1 - S_2)$$

and $$u_2 = \sqrt{2a(S_1 - S_2)} \tag{E12.2}$$

The speed of the trolley just after the impact would be such that

$$u_3^2 = 2aS_3; \qquad u_3 = \sqrt{2aS_3} \tag{E12.3}$$

By the definition of coefficient of restitution,

$$e = \frac{u_3}{u_2} = \frac{\sqrt{2aS_3}}{\sqrt{2a(S_1 - S_2)}} = \sqrt{\frac{S_3}{S_1 - S_2}} \tag{E12.4}$$

The value of e is unity if the collision is elastic, i.e., if energy is not dissipated during the collision. Minimum value of e is zero for plastic collision, i.e., if the energy is entirely dissipated. Typical values of e are given in Table D1.1.

Table D1.1 Typical Values of Coefficients

Value of e	*Type of Materials*
1	Perfectly elastic materials
0.5 to 1	Steel, cast iron, brass
0 to 0.5	Plasticine, rubber, wood
0	Perfectly plastic materials

D1.3 PLANE CENTRAL COLLISION

Let us now analyse a plane central collision of two bodies with reference to Fig. D1.4.

	Body 1	*Body 2*
Precollision velocity	$\mathbf{V}_1$	$\mathbf{V}_2$
Post-collision velocity	$\mathbf{V}'_1$	$\mathbf{V}'_2$

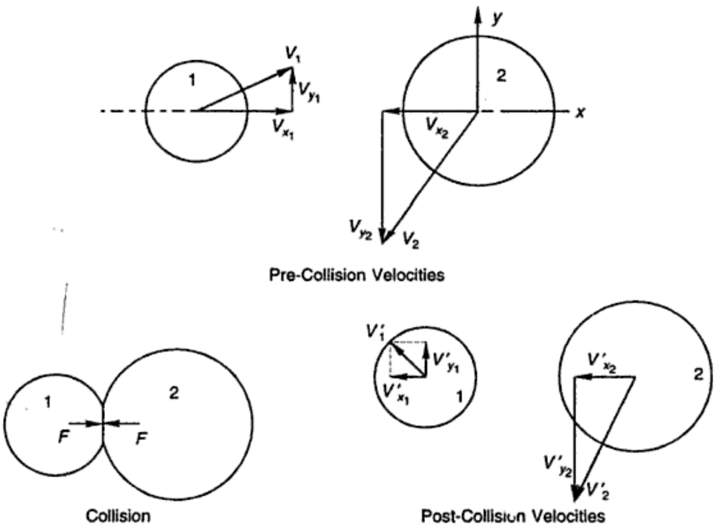


Fig. D1.4 ***Central Collision of Two Bodies***

With the knowledge of the initial conditions, computation of post-collision velocities implies the evaluation of four velocity components. Four scalar equations must, therefore, be set up:

impact when the precollision velocities of the bodies are in the same direction and sense as shown in Fig. D1.5.

By the conservation of momentum principle,

$$m_1 V_1 + m_2 V_2 = m_1 V_1' + m_2 V_2' \tag{D1.13}$$

and by definition of e,

$$e = -\frac{V_{sn}}{V_{an}} = -\frac{V_1' - V_2'}{V_1 - V_2} \tag{D1.14}$$

This is a set of equations for the two unknowns, say the final velocities after the impact.

It is interesting to see the change of kinetic energy during the impact:

Before the impact

$$KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$$

After the impact

$$KE' = \frac{1}{2} m_1 V_1'^2 + \frac{1}{2} m_2 V_2'^2$$

By employing Eqs. (D1.13) and (D1.14), the expression for KE' can be reduced in terms of V_1 and V_2. The amount of kinetic energy dissipated is given by

$$KE - KE' = \frac{1 - e^2}{2} \frac{m_1 m_2}{m_1 + m_2} (V_1 - V_2)^2 \tag{D1.15}$$

It may be noted that the dissipation of kinetic energy depends upon the coefficient of restitution e, masses m_1 and m_2 and the initial difference of velocities of the bodies. The dissipation is zero if $e = 1$, i.e., when the impact is elastic.

Perfectly Elastic Direct Central Impact

For a perfectly elastic direct central impact, as shown in Fig. D1.5(a), the coefficient of restitution is unity

$$e = 1 = -\frac{V_{sn}}{V_{an}} = -\frac{V_1' - V_2'}{V_1 - V_2} \tag{D1.16}$$

It follows that

$$V'_1 - V'_2 = V_2 - V_1 = -(V_1 - V_2) \tag{D1.17}$$

which means that the velocity of separation equals the velocity of approach in magnitude but is opposed to it in direction.

From Eq. (D1.15), for $e = 1$, the kinetic energy dissipated is

$$KE - KE' = 0 \tag{D1.18}$$

Since the potential energy remains unaltered during a collision, Eq. (D1.18) may be interpreted to imply that the mechanical energy of the system of bodies is unaltered during a direct central elastic collision. If the surfaces are frictionless, the mechanical energy will also be conserved in an indirect central elastic collision because the tangential velocity component of each body will then remain unaltered during the collision.

$$KE_1 + KE_2 = KE'_1 + KE'_2$$

or
$$\frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2 = \frac{1}{2}m_1 V_1'^2 + \frac{1}{2}m_2 V_2'^2 \tag{D1.19}$$

Perfectly Plastic Direct Central Impact

For a perfectly plastic direct central impact, as shown in Fig. D1.5(b).

$$e = 0 = -\frac{V_{sn}}{V_{an}} = -\frac{V_1' - V_2'}{V_1 - V_2} \tag{D1.20}$$

It follows that

$$V_1' - V_2' = 0;\ V_1' = V_2' = V' \tag{D1.21}$$

which means that the two bodies move together at a common post-collision velocity V' following a direct central plastic impact.

From Eq. (D1.15), for $e = 0$, the kinetic energy dissipated is

$$KE - KE' = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(V_1 - V_2)^2 \tag{D1.22}$$

This is incidentally not the entire kinetic energy possessed by the bodies before the collision. The kinetic energy KE before the collision was

$$KE = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2$$

and that after the collision is

$$KE' = \frac{1}{2}(m_1 + m_2)V'^2$$

The percentage dissipation of energy is given by

$$\frac{KE - KE'}{KE} \times 100\,\%$$

D1.4 COLLISION OF A SMALL BODY WITH A MASSIVE BODY

If a small body collides with a massive rigid body of a flat or curved surface, as shown in Fig. D1.6, the velocity of the massive body remains unaltered because of its large mass and the large momentum required to produce a small change in its velocity. If a normal is drawn at the point of impact, the velocity component of the small body in the direction of the normal must obey the restitution hypothesis

$$e = -\frac{V_n'}{V_n}$$

or $$V_n' = e\, V_n \qquad \text{(D1.23)}$$

which implies that V_n' is opposed to V_n and is reduced in magnitude by a factor e. The velocity component tangential to the surface remains unaltered in the absence of frictional forces,

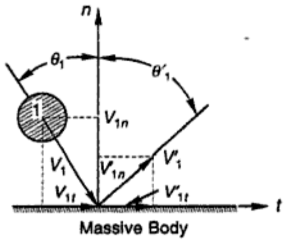


Fig. D1.6 ***Collision of a Small Body with a Massive Rigid Body***

$$V_t' = V_1 \qquad \text{(D1.24)}$$

The angle of incidence θ_1 is given by

$$\tan \theta_1 = \frac{V_t}{V_n} \qquad \text{(D1.25)}$$

and the angle of deflection θ' is given by

$$\tan \theta_1' = \frac{V_t'}{V_n'} = -\frac{V_t}{V_n e} = -\tan \theta_1 / e \qquad \text{(D1.26)}$$

This implies that θ_1' must be reverse in sign to that of θ_1, i.e., the angles of incidence and deflection must be subtended on either side of the normal direction. Since e may lie within O and l for different materials, numerically

$$\tan \theta_1' \geq \tan \theta_1$$

or $$\theta_1' \geq \theta_1$$

In particular, for an elastic impact, $e = 1$

$$\tan \theta'_1 = \tan \theta_1; \qquad \theta'_1 = \theta_1$$

the angles of incidence and deflection must be equal.

For a plastic impact, $e = 0$

$$\tan \theta_1' \to \infty; \quad \theta_1' = 90^\circ$$

The small body must be deflected tangential to the massive body at the point of contact.

***Example* D1.1** (a) A small steel ball is dropped on to a plane surface from a height h and it rebounds to a height h_1 after impact. If the ball is allowed to drop and rebound repeate dly, determine the height of rebound after n impacts.

$$e = \sqrt{\frac{h-d}{h}}$$

or
$$e^2 h = h - d$$

whence
$$h = \frac{d}{1-e^2}$$

It has been assumed that the ball drops approximately vertically on each step and that the step width is small.

***Example* D1.2** A bullet of mass 20 g moving with a velocity of 100 m/s hits a 2 kg bob of a simple pendulum horizontally as shown in Fig. Ex. D1.2. Determine the maximum angle through which the pendulum string 0.5 m long may swing if

(a) the bullet get embedded in the bob
(b) the bullet escapes from the other end at 20 m/s
(c) the bullet is rebounded from the surface of the bob at 20 m/s

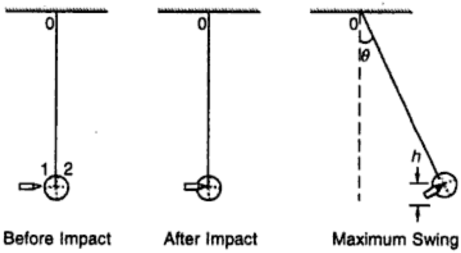


Fig. Ex. D1.2

Solution

Just before the impact, for the bullet,

$$V_1 = 100 \text{ m/s} \quad \text{and} \quad m_1 = 0.02 \text{ kg}, \quad m_1V_1 = 2 \text{ kg m/s}$$

and for the bob

$$V_2 = 0 \quad \text{and} \quad m_2 = 2 \text{ kg}; \quad m_2V_2 = 0$$

Now,
$$m_1V_1 + m_2V_2 = 2 \text{ kg m/s}$$

Case (a)

Just after the impact, the bullet and the bob become united to travel at a velocity of V' m/s and their momentum is

$$(2 + 0.02)V' = 2.02V' \text{ kg m/s}$$

Equating the initial and final momenta,

$$2.02\ V' = 2, \qquad V' = 0.99 \text{ m/s}$$

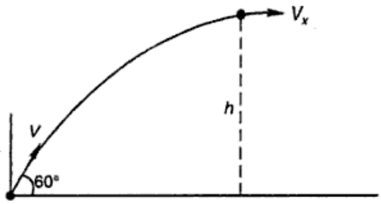


Fig. Ex. D1.7 *(Solution)*

During explosion, the momentum of the system must be conserved,

$$m_1 \mathbf{V}_1' + m_2 \mathbf{V}_2' = m\mathbf{V}$$

$$4V_1' + 1\,V_2' = 5 \times 10 = 50 \qquad \text{(i)}$$

Before the explosion, the kinetic energy is

$$\frac{1}{2}\, m\, V^2 = \frac{1}{2} \times 5 \times 10^2 = 250 \text{ J}$$

and the potential energy is

$$mgh = 5 \times 9.81 \times 15.29 = 750 \text{ J}.$$

After the explosion, the kinetic energy becomes

$$\frac{1}{2}\cdot 4 \cdot V_1'^2 + \frac{1}{2}\cdot 1 \cdot V_2'^2 = 2 \times 250 = 500$$

$$4V_1'^2 + V_2'^2 = 1000 \qquad \text{(ii)}$$

From (i) and (ii)

$$V'_1 = 5 \text{ or } 15 \text{ m/s} \quad \text{and} \quad V'_2 = -10 \text{ and } 15 \text{ m/s}$$

Feasible sets of answers are

$$V'_1 = 5 \quad \text{and} \quad V'_2 = 30 \text{ m/s} \qquad \text{Set I}$$

and

$$V'_1 = 15 \quad \text{and} \quad V'_2 = -10 \text{ m/s} \qquad \text{Set II}$$

The horizontal separation between the two on the ground equals the sum of their ranges.

Time taken to reach the ground is given by

$$0 + \frac{1}{2}\, g\, t^2 = 15.29;\; t = 1.766 \text{ seconds}$$

$$D = 15 \times 1.766 + 10 \times 1.766 = 44.15 \text{ m} \qquad \text{for set I}$$

Alternatively,

$$D = 30 \times 1.766 - 5 \times 1.766 = 44.15 \text{ m} \qquad \text{for set II}$$

The two answers happen to be indentical!

***Example* D1.9** A simple pendulum is suspended from a peg on a vertical wall. The pendulum is pulled away from the wall to a horizontal position and released (see Fig. Ex. D1.9). The ball hits the wall, the coefficient of restitution being $\frac{2}{\sqrt{5}}$. What is the minimum number of collisions after which the amplitude of oscillation becomes less than 60 degrees?

Solution In the initial state, with P as datum,

$$PE = mg\,L,\; KE = 0$$

Just before hitting the wall,

$$PE = 0,\; KE = \frac{1}{2}\,mV^2$$

By energy conservation,

$$\frac{1}{2}\,mV^2 = mgL\;;\; V = \sqrt{2gL}$$

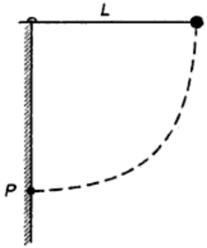


Fig. Ex. D1.9

Now, $e = \dfrac{\text{Velocity of separation}}{\text{Velocity of approach}} = \dfrac{V'}{V} = \dfrac{2}{\sqrt{5}}$

It also implies that the kinetic energy becomes e^2, i.e., 4/5 times i.e., 80% with every impact.

At the position of 60° oscillation,

$$PE = mg\,L(1 - \sin 30) = mg\,L/2$$

which corresponds to a kinetic energy 50% of the original value. The kinetic energy would be less than 50% after 3 collisions since

$$80\% \times 80\% \times 80\% \text{ is just less than } 50\%$$

The minimum number of collisions is, therefore, three.

***Example* D1.10** A block A of mass 2 m is placed on another block B of mass 4 m which in turn is placed on a fixed table. The two blocks have the same length $4d$ and they are placed as shown in Fig. Ex. D1.10. The coefficient of friction (both static and kinetic) between the block B and the table is μ. There is no friction between the two blocks. A small object of mass m moving horizontally along a line passing through the centre of mass (c, see figure) of the block B and perpendicular to its face with a speed V collides elastically with the block B at a height d above the table.

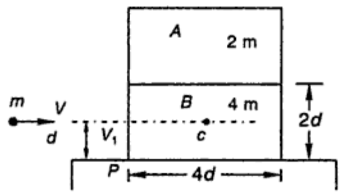


Fig. Ex. D1.10

(a) What is the minimum value of V (call it V_0) required to make the block A topple?

(b) If $V = 2V_0$, find the distance (from the point P in the figure) at which the mass m falls on the table after collision. Ignore the role of friction during the collision.

Solution The block A would topple if the block B below it slides past it half way, i.e., a distance of $2d$ to the right. The frictional force f between the table and the block B at the state of impending motion and thereafter is given by

$$f = 6\, m\, \mu g$$

opposing the direction of motion of the block. The energy required for the block to move a distance of $2d$ is, therefore,

$$E = 6\, m\, \mu g \times 2d = 12\, m\, \mu g\, d$$

(a) The minimum value of velocity of the block B just after impact should be such that

$$\frac{1}{2}\,(4\, m)\cdot V_B^2 = 12\, m\, \mu\, g\, d$$

whence $$V_B = \sqrt{6\mu g d}$$

During the impact, the momentum is conserved,

$$4m\, V_{Bf} + V_{Af} = mV_0$$

$$4V_{Bf} + V_{Af} = V_0$$

and, by the definition of restitution,

$$e = -\left(\frac{V_{Bf} - V_{Af}}{0 - V_0}\right) = 1$$

From these two equations

$$V_{Bf} = \frac{2}{5}\,V_a = \sqrt{6\,\mu g d};\ V_{1f} = -3/5\, V_0$$

or $$V_0 = \frac{5}{2}\sqrt{6\mu g d}$$

(b) In this case,

$$V = 2V_0 = 5\sqrt{6\mu g d}$$

For momentum conservation during the impact,

$$4\, m\cdot V_{Bf}\, m\cdot V_{Af} = mV$$

and by virtue of elastic impact

$$e = -\left(\frac{V_{Bf} - V_{Af}}{0 - V}\right) = 1;\ V_{Bf} - V_{Af} = V$$

whence, $$V_{Af} = -\frac{3}{5}V$$

$$= -\frac{3}{5}(2V_0) = -\frac{3}{5}\times 2\times\frac{5}{2}\sqrt{6\mu g d}$$

$$= -3\sqrt{6\mu g d}$$

***Example* D1.11** A drop hammer 1 with a mass of 6 Mg falls from rest 0.8 m onto a forged anvil 2 mounted on springs which have a composite stiffness 2 MN/m as shown in Fig. Ex. D1.11. Find the maximum compression of the springs after the impact if the anvil has a mass of 4 Mg and the coefficient of restitution between the hammer and the anvil is $e = 0.5$. Neglect the friction along the vertical guide posts.

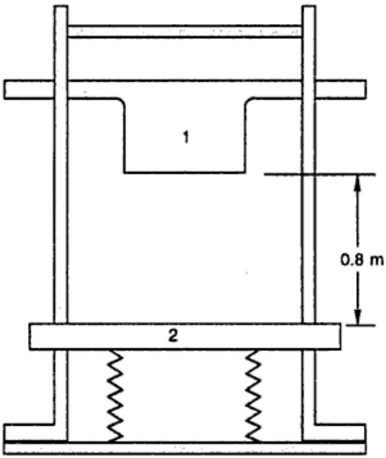


Fig. Ex. D1.11

Solution The velocity of the hammer just before the impact is given by

$$\frac{1}{2}m_1 V_1^2 = m_1 gh$$

$$V_1 = \sqrt{2\times 9.81\times 0.8} = 3.96 \text{ m/s}$$

During the impact, momentum is conserved,

$$m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2'$$

whence $$6\times 10^3\times 3.96 + 0 = 6\times 10^3\ V_1' + 4\times 10^3\ V_2'$$

or $$3V_1' + 2V_2' = 11.88 \quad \text{(i)}$$

(*Note*: 1 Mg = 1 × 10^6g = 1000 kg)

Also, from the definition of coefficient of restitution,

$$e = -\frac{V_s}{V_a} = \frac{V_2' - V_1'}{V_1 - V_2}$$

$$0.5 = \frac{V_2' - V_1'}{3.96 - 0}$$

or $$V_2' - V_1' = 1.98 \quad \text{(ii)}$$

From (i) and (ii)

$$V_2' = 3.564 \text{ m/s}; \quad V_1' = 1.584 \text{ m/s}.$$

The kinetic energy possessed by the anvil is used up in creating the potential energy of the springs:

$$\frac{1}{2} \times 4000 \times 3.564^2 = \frac{1}{2} \times 2 \times 10^6 x^2$$

The spring compression is given by

$$x = 0.16 \text{ m}$$

provided that the hammer does not hit the anvil again.

***Example* D1.13** A ball of radius r hits the ground with an initial velocity V_1 and top-spin, i.e., angular velocity ω_1. Determine an expression for its linear velocity V_2 just after the impact (see Fig. Ex. D1.13).

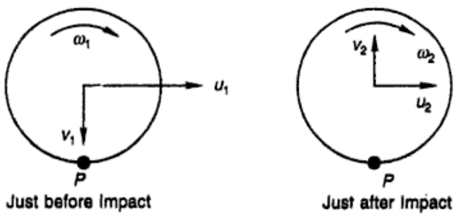


Fig. Ex. D1.13

Solution Just before the impact, the ball may have a horizontal component and a vertical component of velocity, say u_1 and v_1 respectively.

The vertical rebound and change in the vertical component of velocity from V_1 to V_2 takes place depending upon the coefficient of restitution between the ball and the ground. It is not effected by the spin of the ball.

The horizontal component of velocity changes from u_1 to u_2 and the spin changes from ω_1 to ω_2. It is reasonable to assume that the slipping at the point of contact will cease, the earth being fairly rough and sticky. Consequently, the angular velocity of the ball just after the impact will only be u_2/r;

$$\omega_2 = u_2/r$$

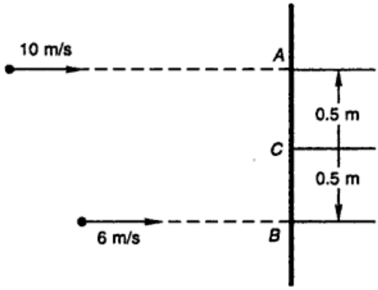


Fig. Ex. D1.14

Momentum of the upper particle = 0.08 × 10 = 0.8 kg m/s
and for the lower particle, 0.08 × 6 = 0.48 kg m/s.

For the system, in the absence of external forces, the linear momentum is conserved.

$$0.8 + 0.48 = (0.16 + 0.08 + 0.08)\, V_c$$

whence $\quad V_c = 4$ m/s

Also, in the absence of external moments, the moment of momentum of the system must be conserved.

$$0.8 \times 0.5 - 0.48 \times 0.5 = 0.16 \times \left(\sqrt{3}\right)^2 \big/ 12 \times \omega + 2 \times 0.08 \times (0.5)^2$$

Therefore, $\quad \omega = 2$ rad/s

The final kinetic energy is given by

$$\frac{1}{2}\,(0.16 + 0.08 + 0.08) \times 4^2 + \frac{1}{2} \times 0.08 \times (2)^2$$

$$= 2.56 + 0.16 = 2.72 \text{ J}$$

Loss of kinetic energy in the process is, therefore, 5.44 – 2.72, i.e., 2.72 J.

***Example* D1.15** A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses m and $2m$ moving in the same horizontal plane with speeds $2V$ and V, respectively, strike the bar (as shown in Fig. Ex. D1.15) and stick to the bar after collision. Denoting angular velocity (about the centre of mass), total energy and centre of mass velocity by ω, E and V_c respectively, which of the following is (are) true after collision?

(A) $V_c = 0$

(B) $\omega = \dfrac{3V}{5a}$

(C) $\omega = \dfrac{V}{5a}$

(D) $E = \dfrac{3mV^2}{5}$

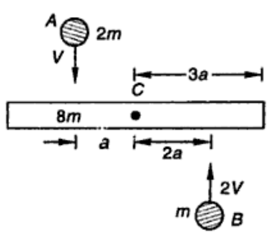


Fig. Ex. D1.15

which, approximated by the binomial theorem, becomes

$$g_r/g = 1 - 2\ h/R \tag{D2.6}$$

or
$$g_r = (1 - 2\ h/R)\ g = g - 2\ h/R\ g$$

or
$$g_r - g = -2\ h/R\ g \tag{D2.7}$$

It implies that the reduction in acceleration due to gravity at a point h above the surface of the earth in comparison with that on the surface is $2h/R\ g$, for small height h.

Let us look at the variation of the acceleration due to gravity below, i.e., inside the surface of the earth at a point P as shown in Fig. D2.3. In this case, the mass of the earth inside it is

$$M' = (b/R)^3\ M$$

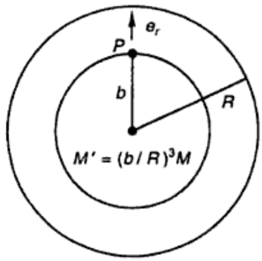


Fig. D2.3

where M is the mass of the earth.

This is because the mass is proportional to volume for constant density and the volume is proportional to the cube of the radius of a sphere.

We are neglecting the gravitational effect of hollow sphere of outer radius R and inner radius b or assuming that the net effect of that is zero.

Then, the gravitational force is given by

$$\mathbf{F} = -\frac{Gm\,M'}{b^2}\,\mathbf{e}_r$$

and
$$g_b = -\frac{GM'}{b^2}\,\mathbf{e}_r = -\frac{GM}{R^2}\,b/R\,\mathbf{e}_r$$

whence
$$g_b = g\ b/R \quad \text{or} \quad g_b/g = b/R \tag{D2.8}$$

which means that, below the surface of the earth, the acceleration due to gravity varies directly as the radial distance from the centre of the earth as also shown graphically in Fig. D2.2.

From the above simple analysis, we also conclude that the acceleration due to gravity is the maximum at the surface of the earth itself ignoring the variation in density of the earth.

It is also interesting to consider the variation of the acceleration due to gravity with latitude θ. The variation is due to the differential contribution of acceleration by virtue of the spinning motion of the earth.

The point P located by R and θ as shown in Fig. D2.4 is in a circular motion of radius r (= $R\cos\theta$) with a rotational speed ω. A body placed at P experiences a real acceleration $g = GM/R^2$ towards the centre of the earth and a pseudo acceleration $g_s = \omega^2 r$ as shown. The resultant acceleration g' is given by

$$g' = \left[g^2 + g_s^2 - 2g\,g_s\,\cos\theta\right]^{\frac{1}{2}}$$

$$= g\left[1 + (R\omega^2/g)^2\,\cos^2\theta - 2\,R\omega^2/g\cos^2\theta\right]^{\frac{1}{2}}$$

if the polar axis is chosen so that $\theta_0 = 0$. This is the equation of a conic section in the polar coordinates with the origin O located at the centre of the earth. Comparing it with the standard form of a conic section

$$\frac{1}{r} = u = \frac{1}{ep} + \frac{1}{p}\cos\theta \tag{D2.13}$$

it is seen that

$$D = \frac{1}{p}$$

$$ep = \frac{C^2}{GM}$$

or

$$e = \frac{DC^2}{GM} \tag{D2.14}$$

which is the eccentricity of the conic section.

Four cases are given in Table D2.1.

Table D2.1 Trajectories Under Central Force

Case	*Value of e*	*Features*	*Type of Conic Section*
1	$e = 0$	$D = 0,\ r = \frac{C^2}{GM}$	Circle
2	$e < 1$	Finite r for all θ	Ellipse
3	$e = 1$	$r \to \infty$ at $\theta = \pi$	Parabola
4	$e > 1$	$r \to \infty$ for two values of θ	Hyperbola

The eccentricity of the orbit

$$e = \frac{DC^2}{GM}$$

depends upon the constant quantities G and M as well as the constants of motion, C and of the conic section, D. The state of a satellite can be predicted with the help of Eq. (D2.13) if the conditions at any one state, say the moment of burnout, are known.

Assuming that the burnout occurs at the end of a powered flight at a position P in space such that the velocity of the rocket is parallel to the surface of the earth, as shown in Fig. D2.5, the satellite or space vehicle is said to begin its free-flight at the vertex P. If the satellite, at this instant, is located at a distance r_0 from the centre of the earth and has a velocity V_0 parallel to the surface of the earth

$$C = r_0 V_0$$

and

$$\frac{1}{r_0} = \frac{GM}{C^2} + D \tag{D2.15}$$

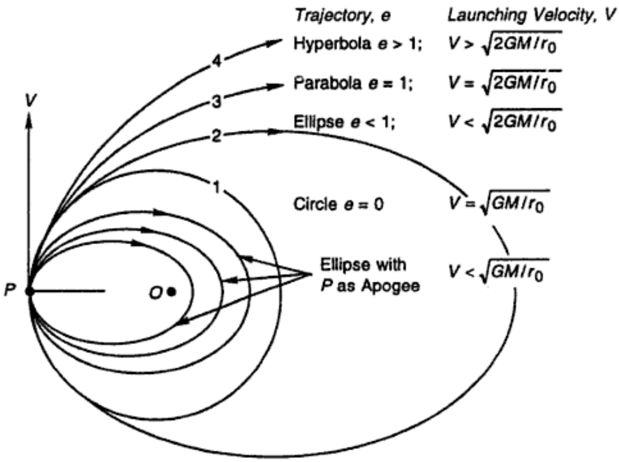


Fig. D2.5 ***Possible Satellite Trajectories***

whence
$$D = \frac{1}{r_0} - \frac{GM}{C^2} = \frac{1}{r_0} - \frac{GM}{r_0^2 V_0^2}$$

$$= \frac{1}{r_0} - \frac{gR^2}{r_0^2 V_0^2}$$

since g the acceleration at the surface of the earth is given as

$$g = \frac{GM}{R^2} \tag{D.2.16}$$

where R is the radius of the earth.

Let us again discuss the four possible cases in terms of the launching parameters. Reference is made in Figs. D2.2 and D2.3.

Circular Orbit

$$e = 0 = \frac{DC^2}{GM}; \qquad D = 0$$

The path is given by

$$\frac{1}{r} = \frac{GM}{C^2} = \frac{GM}{(r_0 V_{\theta_0})^2} = \frac{1}{r_0} = \text{Constant}$$

whence
$$V_{\theta_0} = \sqrt{\frac{GM}{r_0}} = \sqrt{g_0 r_0} \tag{D2.17}$$

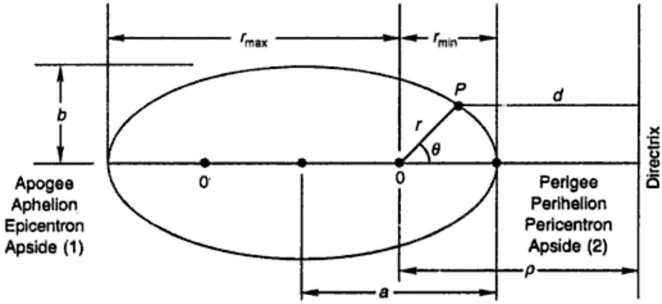


Fig. D2.6 ***Features of an Ellipse***

This is the velocity required to launch the satellite for a circular orbit. It can be seen that the velocity of a satellite in a circular orbit is more if it is closer to the earth and less if away from it.

Modern-day communication satellites are geo-stationary or synchronous i.e., remain fixed in location relative to the earth spinning about their own axes; the period of revolution of a communication satellite should also be 24 hours. The orbit of a communication satellite must be circular and it must be in the equational plane of the earth. The rotational speed of the earth is

$$\omega = \frac{2\pi}{24 \times 60 \times 60} = 0.000073 \text{ rad/s}$$

The rotational speed of a geo-stationary satellite must also be the same.

$$\omega = V_\theta / r = 0.000\,073 \text{ rad/s}$$

or $$V_\theta = 0.000\,073\, r \text{ m/s} \quad \text{(i)}$$

But, for a circular orbit

$$V_\theta = \sqrt{GM/r} = \sqrt{3.9860 \times 10^{14} / r} \quad \text{(ii)}$$

From Eqs. (i) and (ii)

$$r = 42{,}180{,}000 \text{ m} = 42{,}180 \text{ km}$$

and $$V_\theta = 3080 \text{ m/s} = 11{,}100 \text{ km/h}$$

Geo-stationary satellites must, therefore, be located at a radial distance of 42,180 km or at an altitude h given by

$$h = 42{,}180 - 6370 = 35{,}810 \text{ km}$$

above the mean surface of the earth and must move with a velocity of 11,100 km/h in circular orbits in an equatorial plane of the earth and in the same sense as shown in Fig. D2.7. The period of rotation of a communication satellite must be 24 hours.

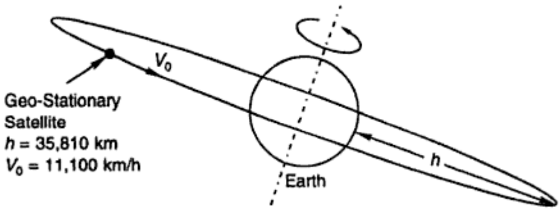


Fig. D2.7 *A Communication Satellite*

Parabolic Trajectory

$$e = 1 = \frac{DC^2}{GM}, \quad D = \frac{GM}{C^2}$$

The path is given by

$$\frac{1}{r} = \frac{GM}{C^2} + D\cos\theta$$

For launching at $\theta = 0$ and $r = r_0$

$$D = \frac{1}{r_0} - \frac{GM}{C^2} = \frac{GM}{C^2}$$

Substituting for $C = r_0 V_{\theta_e}$ in this equation

$$V_{\theta_e} = \sqrt{\frac{2GM}{r_0}} = \sqrt{2g_0 r_0} = \sqrt{2}\, V_{\theta c} \quad \text{(D2.18)}$$

$= \sqrt{2}$ times the launching velocity for a circular orbit

If this velocity is imparted to a particle, it will follow a parabolic path, i.e., escape from the gravitational field of the earth. For such a path

$$\theta \to \pi, \frac{1}{r} \to 0, \; r \to \infty$$

Since GM is a constant for the earth, the escape velocity V_{θ_0} is maximum at the surface of the earth and it decreases if the satellite is launched from higher altitudes. This fact explains why satellites are launched after reaching very high altitudes by powered flights!

Elliptical Orbit

$$0 < e < 1, \quad \frac{DC^2}{GM} < 1$$

The launching velocity is obviously between the values

$$V_{\theta_c} = \sqrt{\frac{GM}{r_0}} \text{ for a circular orbit}$$

and

$$V_{\theta_e} = \sqrt{\frac{2GM}{r_0}} \text{ for a parabolic escape trajectory}$$

If, however, the launching velocity is less than that for a circular orbit, i.e.,

$$0 < V_\theta < \sqrt{\frac{GM}{r_0}}$$

the satellite still goes on an elliptical orbit with the centre of the earth as the second focus instead of the first. It may also be appreciated that the launching velocity can be adjusted to touch the surface of the earth at the diametrical opposite point.

If the launching velocity is less than this value, the object will hit the surface of the earth while tending to complete the elliptical orbit. A bullet fired from a gun or stone thrown by hand are examples of this case. In the first instance, it may appear contrary to our belief that the trajectory of an earth-bound object, such as a stone, bullet or jet of water should be parabolic. The paradox is resolved by remembering that the value of g is taken as constant in magnitude and direction for the motion of earth-bound objects. Analysis with constant g yields a parabolic trajectory which is an approximation to the elliptical path obtained by taking variable g directed towards the centre of the earth.

Although it is admissible to have the eccentricity e between 0 and 1, it is usual to keep it very low, i.e., the orbit close to a circular orbit. The orbital eccentricities of the planets are also very low. For example,

Orbital eccentricity of the earth	*0.017*
Orbital eccentricity of the moon	*0.055*

Some salient features of an elliptic conic section are shown in Fig. D2.6.

Eccentricity $e = \frac{r}{d} \quad (<1)$

$$= \frac{r_{min}}{p - r_{min}}$$

$$= \frac{r_{max}}{p + r_{min}}$$

Semimajor axis $a = \frac{r_{max} + r_{min}}{2}$

Semiminor axis $b = \sqrt{r_{min} \cdot r_{max}} = a\sqrt{(1 - e^2)}$

Distance between the foci $= 2\sqrt{(a^2 - b^2)}$

Area of the ellipse $= \pi ab$

Equation of the orbit is given by

$$r = a(1 - e \cos \theta) \tag{D2.19}$$

whence $$r_{max} = a(1 + e)$$

and $$r_{min} = a(1 - e)$$

$$ep = a(1 - e^2)$$

Hyperbolic Trajectory

$$e > 1, \ \frac{DC^2}{GM} > 1$$

$$V_{\theta_0} > \sqrt{\frac{2GM}{r_0}}$$

From the equation of the trajectory,

$$\frac{1}{r} = \frac{GM}{C^2} + D\cos\theta$$

it may be observed that when $r \to \infty$

$$0 = \frac{GM}{C^2} + D\cos\theta$$

$$\cos\theta = -\frac{GM}{DC^2} = -\frac{1}{e}$$

and $$\theta = \cos^{-1}\left(-\frac{1}{e}\right)$$

describes the asymptote for the final direction of the particle.

It is observed that if the launching velocity equals or exceeds the escape value, i.e.,

$$V_{\theta_0} \geq \sqrt{\frac{2GM}{r_0}}$$

the object escapes the gravitational field of the earth. This is indeed the case of a mission to reach the Moon or Mars or any other planet in the solar system.

The time taken for a particle in a central-force motion to travel from a position θ_0 to a position θ can be computed by starting from

$$\frac{r^2 d\theta}{dt} = C$$

or $$dt = \frac{r^2 d\theta}{C}$$

The time taken T is given by

$$T = \int dt = \frac{1}{C}\int_{\theta_0}^{\theta} r^2 d\theta$$

$$= \frac{1}{C} \int_{\theta_0}^{\theta} \frac{d\theta}{\left(\frac{GM}{C^2} + D\cos\theta\right)^2} \tag{D2.20}$$

For an elliptical orbit, the area traversed in one revolution is πab and the areal velocity is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{C}{2}$$

The time period for one revolution is, therefore,

$$T = 2\pi\, ab/C \tag{D2.21}$$

For a circular orbit,

$$a = b = r_0$$

and the time period is

$$T = 2\pi\, r_0^2 / C = 2\pi\, r_0 / V_{\theta_0} \tag{D2.22}$$

It can again be observed that a synchronous or geo-stationary satellite would have a time period of 24 hours.

$$T = 2\pi\, r_0 / V_{\theta_0} = 24 \times 60 \times 60 \text{ s}$$

or
$$V_\theta = 0.000\,073 \text{ r m/s}$$

which is the same as determined earlier in Eq. (i) under 'Circular Orbit'.

D2.4 ENERGY EXPENDED FOR DIFFERENT TRAJECTORIES

Energy methods may be used to advantage for studying some aspects of the central-force motion. For the gravitational field,

$$F_r = -\frac{GM\,m}{r^2}, \quad F_\theta = 0$$

The potential energy PE is given by

$$\frac{d(PE)}{dr} = -F_r = \frac{GM\,m}{r^2}$$

or
$$PE = -\frac{GM\,m}{r} \tag{D2.23}$$

which implies that the gravitational force field is a conservative force field. The law of conservation of mechanical energy is, therefore, applicable in a central-force field:

$$KE + PE = \text{Constant}$$

or $$\frac{1}{2}mV^2 - \frac{GMm}{r} = \text{Constant} \tag{D2.24}$$

For a body at the surface of the earth of radius R

$$PE = -\frac{GMm}{R}$$

If a satellite is launched from a radial position at a distance r_0 from the centre of the earth,

$$KE + PE = \frac{1}{2}mV_0^2 - \frac{GMm}{r_0}$$

Conservation of mechanical energy

$$\frac{1}{2}mV^2 - \frac{GMm}{r} = \frac{1}{2}mV_0^2 - \frac{GMm}{r_0} \tag{D2.25}$$

provides a relation to relate the velocity with the radial position of a satellite in terms of the launching conditions.

For a circular orbit, the radial distance r is constant and equal to r_0 at all times; so is the velocity V equal to V_0 at all times.

The escape velocity V_e of a body may be determined from the fact that the body should continue indefinitely and reach

$$r \to \infty \quad \text{when} \quad V = 0$$

where $$KE + PE = \frac{1}{2}mV^2 - \frac{GMm}{r}$$

$$= 0 - 0 = 0$$

From the conservation of mechanical energy, the conditions at the launching position should be such that,

$$(KE + PE)_{\text{launching}} = 0$$

or $$\frac{1}{2}mV_e^2 - \frac{GMm}{r_0} = 0$$

whence $$V_e = \sqrt{\frac{2GM}{r_0}}$$

Since by definition,

$$\frac{GMm}{r_0^2} = mg_0$$

$$V_e = \sqrt{\frac{2GM}{r_0}} = \sqrt{2g_0 r_0} \tag{D2.26}$$

If a satellite is launched from a position r_0, the energy required to be expended to set the satellite into a trajectory may be determined if the intended velocity of the satellite at any position is known

$$E = \left(\frac{1}{2}mV^2 - \frac{GMm}{r}\right) - \left(-\frac{GMm}{r_0}\right)$$

$$= \frac{1}{2}mV^2 + GMm\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

$$= \frac{1}{2}mV_0^2 \tag{D2.27}$$

Energy requirements for different trajectories are given in Table D2.2.

Table D2.2 Energy Expended for Different Trajectories

Trajectory	*Velocity*	*Energy Expended*
Circular orbit	$V = \sqrt{\frac{GM}{r_0}}$	$E = \frac{GMm}{2r_0}$
Elliptic orbits	(i) $V < \sqrt{\frac{GM}{r_0}}$	$E < \frac{GMm}{r_0}$
	(ii) $\sqrt{\frac{GM}{r_0}} < V < \sqrt{\frac{2GM}{r_0}}$	$\frac{GMm}{2r_0} < E < \frac{GMm}{r_0}$
Escape trajectory (Parabolic-minimum values)	$V = \sqrt{\frac{2GM}{r_0}}$	$E = \frac{GMm}{r_0}$
Hyperbolic escape	$V > \sqrt{\frac{2GM}{r_0}}$	$E > \frac{GMm}{r_0}$

D2.5 LAUNCHING OF SATELLITES AT AN ANGLE

Consider a general case of launching at $r = r_0$ with an angle of launch α, as shown in Fig. D2.8. Let the launching velocity be $\mathbf{V}_0$ such that,

$$V_{\theta_0} = V_0 \cos\alpha$$

$$V_{r_0} = V_0 \sin\alpha$$

Let θ_0 be the angle between the axis of symmetry of the conic and the radius vector $\mathbf{r}_0$.

$$\frac{1}{r_0} = \frac{GM}{C^2} + D\cos\theta_0$$

for the position of launching.

In general,

$$\frac{1}{r} = \frac{GM}{C^2} + D\cos\theta$$

and

$$V_r = DC\sin\theta;\ V_{r_0} = DC\sin\theta_0$$

Since $$D\cos\theta_0 = \frac{1}{r_0} - \frac{GM}{C^2}$$

and $$D\sin\theta_0 = \frac{V_{r_0}}{C}$$

$$D = \left[\left(\frac{1}{r_0} - \frac{GM}{C^2}\right)^2 + \left(\frac{V_{r_0}}{C}\right)^2\right]^{1/2} \quad \text{(D2.28)}$$

$$e = \frac{DC^2}{GM}$$

Consider the condition for the escape of the satellite, $e = 1$. We can obtain that

$$r_0 V_0^2 \cos^2\alpha(r_0 V_0^2 - 2GM) = 0$$

whence, the escape velocity $V_0 = \sqrt{\dfrac{2GM}{r_0}}$

This is independent of the angle of launching; may it be 0°, 90° *or any other. A satellite projected vertically or radially outwards from the surface of the earth, as shown in Fig. D2.9, can also escape the earth's gravitational field at the same speed.* It may, however, be added that the V_0 for escape is measured with respect to the centre of earth considered as an inertial frame. The motion of the earth's surface adds to the final velocity of the satellite if the blast off takes place on the equator and along the rotation of the earth and subtracts from the final velocity if the blast off opposes the rotation. No such 'gain' or 'loss' is experienced in the velocity if the satellite is fired from the poles.

Another interesting fact is that if a space vehicle is launched in order to escape the gravitational field of the earth, it may so happen that the space vehicle escapes the entire solar system. This is due to the fact that the earth is rotating around the

It may be observed from here that if h must be made to approach infinity,

$$V \to \sqrt{V_0^2 - \frac{2GM}{R}} \tag{D2.30}$$

If, in addition, the velocity V at $h \to \infty$ becomes negligibly small or $V \to 0$, then the launching velocity V_0 equals $\sqrt{\frac{2GM}{R}}$ which is the escape velocity of the particle as is expected from the earlier analysis of launching at any arbitrary angle.

Alternatively, the equation

$$V^2 = V_0^2 - \frac{2GMh}{R(R+h)}$$

shows that the maximum height attained by a particle would correspond with the minimum residual velocity squared V^2, i.e., $V^2 = 0$

Hence, h_{max} is given by

$$V_0^2 - \frac{2GMh_{max}}{R(R+h_{max})} = 0$$

whence

$$h_{max} = \frac{V_0^2 R}{\left(2\frac{GM}{R} - V_0^2\right)} = \frac{V_0^2 R}{(2gR - V_0^2)} \tag{D2.31}$$

It is obvious from this expression that if

$$V_0^2 = 2\frac{GM}{R}$$

or

$$V_0 = \sqrt{2gR} = \sqrt{2\frac{GM}{R}}$$

h_{max} would become infinite, i.e., the particle would escape the gravitational field of the earth. On the contrary, if h_{max} is small, let $h_{max} = h$ and h/R is negligible,

$$(2gR - V_0^2)\frac{h}{R} = \frac{V_0^2 R}{R}$$

$$2gh - V_0^2 \frac{h}{R} = V_0^2$$

$$V_0 = \sqrt{2gh} \tag{D2.32}$$

which is a familiar expression for distance close to the surface of the earth.

D2.6 ASTRONOMICAL FACTS AND LAWS OF KEPLER

A system consisting of a star, planets and satellites is called a solar system. A star is a source of light, planets only reflect light and orbit around the star and the satellites revolve about the planets. Satellites may be natural or artificial. Man-made artificial satellites may be launched to revolve around the planets.

In our solar system, the sun is the star; the nine planets are Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto in order of their mean distance from the sun. Thirty satellite bodies orbit around these planets. The moon is the satellite of the earth. A number of artificial satellites are orbiting the earth with a view to weather forecasting, intercontinental television relaying and spying missions.

A study of the central-force motion would be incomplete without discussing the contribution of Kepler who laid the foundation of orbital mechanics as also of Newtonian mechanics.

The three laws of Kepler for planetary motion are:

1. *Every planet moves in an orbit which is an ellipse with the sun at one focus.*
2. *The radius vector drawn from the sun to any planet sweeps out equal areas in equal times.*
3. *The squares of the periods of the planets are proportional to the cubes of the semi-major axes of their orbits.*

These laws were enunciated by Kepler (1571-1630) from an analysis of the data recorded by Tycho Brahe. This was long before the enunciation of the laws of motion by Newton and the development of mathematical calculus. Newton (1642-1727) published his work in 1687 and set the stage for the development of classical mechanics.

Kepler's laws related to planetary motion can be derived from the laws of Newton and Newton's law can also be derived from Kepler's laws.

The first law refers to a conclusion derivable from the study of central-force motion; the motion in a central-force field must be plane and for the eccentricity $e < 1$, the conic section traced is an ellipse for an inverse square force field due to the Newton's law of gravitation.

The second law refers to the conclusion that the areal velocity is constant for a particle in a central-force field. It means that the areas swept by the particle are equal in equal time intervals. When the particle passes through the perigee, it must move faster than when it passes through the apogee. *Planets move faster when they are nearer the sun!*

In order to prove the third law of Kepler, we proceed as follows:

The time period for an elliptical orbit is

$$T = 2\pi ab/C$$

From the geometry of the ellipse,

$$b = a\sqrt{1 - e^2}$$

and by comparing with the standard form of the equation for an ellipse with that for the planetary motion,

$$ep = a(1 - e^2) = \frac{C^2}{GM}$$

Table D2.3 Astronomical Data for Our Solar System

Body	*Equatorial radius* R/R_e	*Mass* M/M_e	*Density* ρ/ρ_e	*Distance in astronomical units from the sun*	*Gravity* g/g_e	*Eccentricity* e	*Number of its satellites*	*Orbital time (Solar days)*	*Spin time (Solar days)*	*Escape velocity km/h*
Sun	109.1	3.33 × 105	0.255		28	—	—	—	25.38 days	2.22×10^6
Moon	0.273	0.0123	0.607	1.000	0.1645	0.055	—	27.32 days	27.32 days	8550
Mercury	0.38	0.056	0.96	0.387	0.36	0.206	0	87.97 days	59.0 days	15,460
Venus	0.96	0.81	0.90	0.723	0.87	0.007	0	224.7 days	243.0 days	37,000
Earth	*1.000*	*1.000*	*1.000*	*1.000*	*1.000*	*0.017*	*1*	*365.256 days*	*1 day*	*40,270*
Mars	0.53	0.107	0.715	1.523	0.38	0.093	2	687.0 days	24.6 hours	18,095
Jupiter	11.20	318.00	0.241	5.203	2.65	0.048	12	11.86 years	9.9 hours	214,580
Saturn	9.47	95.1	0.13	9.539	1.14	0.056	10	29.46 years	10.2 hours	127,600
Uranus	3.75	14.5	0.25	19.19	1.1	0.047	5	84.02 years	10.7 hours	250,410
Neptune	3.50	17.5	0.41	30.07	1.00	0.009	2	164.8 years	15.8 hours	90,050
Pluto	≈1	≈1	≈1	39.52	≈1	0.249	0	248.0 years	6.3 days	40,270

Rate of energy production in the sun = 3.90×10^{16} W

*Subscript e, for the earth (Universal $G = 6.673 \times 10^{-11}$ Nm²/kg²)

For the Earth

$M = 5.97 \times 10^{24}$ kg

$GM = 3.986 \times 10^{14}$ m³/s²

$g = 9.8065$ m/s²

$R = 6.371 \times 10^6$ m = 6371 km

From these relations,

$$T^2 = 4\pi^2 a^3/GM \qquad \text{or} \qquad T^2 \propto a^3 \tag{D2.33}$$

which shows that the squares of time periods of planets are proportional to the cubes of the semi-major axes of their elliptical orbits.

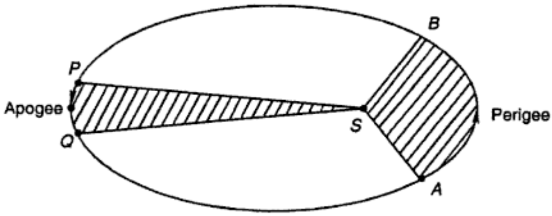


Fig. D2.10

***Example* D2.2** The Aryabhatta was launched by a Soviet launch pad and set into an orbit by being imparted a velocity of 7600 m/s at a distance of 600 km above the surface of the earth and parallel to it at that point.

Comment on its orbit and its salient features. It is given that GM for earth = 3.9860×10^{14} m^3/s^2 and the radius of the earth is 6371 km.

Solution The satellite Aryabhatta had the following initial conditions:

$$r_0 = 600 + 6371 = 6{,}971 \text{ km} = 6{,}971{,}000 \text{ m}$$

$$V_{\theta_0} = 7600 \text{ m/s} \qquad \text{and} \qquad \theta = 0$$

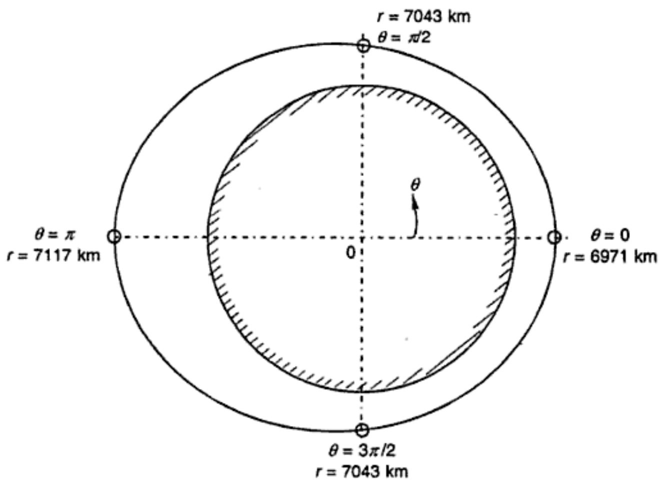


Fig. Ex. D2.2 ***(Solution a, b)***

$$C = r_0 V_{\theta_0} = 5.30 \times 10^{10} \text{ m}^2/\text{s}$$

The orbit is described by the equation

$$\frac{1}{r} = \frac{GM}{C^2} + D\cos\theta$$

and

$$D = \frac{1}{r_0} - \frac{GM}{C^2}$$

$$= \frac{1}{6{,}971{,}000} - \frac{3.9860 \times 10^4}{(5.30 \times 10^{10})^2}$$

$$= 1.50 \times 10^{-9} \text{ m}^{-1}$$

The eccentricity is given by

$$e = \frac{DC^2}{GM} = \frac{1.50 \times 10^{-9} \times (5.30 \times 10^{10})^2}{3.9860 \times 10^{14}} = 0.01$$

The Aryabhatta hs, threfore, gone into an elliptic orbit described by

$$\frac{1}{r} = \frac{3.986 \times 10^{14}}{(5.30 \times 10^{10})^2} + 1.50 \times 10^{-9} \cos\theta$$

or

$$\frac{1}{r} = (142 + 1.5\cos\theta) \times 10^{-9}$$

The coordinates of the satellite at salient locations are:

$\theta = 0$	$r = 6971$ km
$\theta = \pi/2$	$r = 7043$ km
$\theta = \pi$	$r = 7117$ km
$\theta = 3\pi/2$	$r = 7043$ km

The maximum distance of the satellite from the surface of the earth is

$$7117 - 6371 = 746 \text{ km}$$

It can be seen that the velocity that would have been required to set the satellite into circular orbit is

$$V_c = \sqrt{\frac{GM}{r_0}} = \sqrt{\frac{3.986 \times 10^{14}}{6{,}971{,}000}}$$

$$= 7562 \text{ m/s}$$

The velocity actually imparted to the Aryabhatta is 7600 m/s which is slightly more than this value. From this or from the fact that the eccentricity is 0.01, it can be concluded that the orbit is nearly circular as shown in Fig. Ex. D2.2(b)

(a) At the surface of the earth.

$$r_0 = 6371 \text{ km}$$

$$V_e = 28.2 \times 10^6 \sqrt{6371 \times 10^3}$$

$$= 11{,}170 \text{ m/s} = 11.17 \text{ km/s}$$

(b) At any altitude h km above the earth.

$$r_0 = 6371 + h$$

$$V_e = 28.2 \times 10^6 / \sqrt{6371 + h}$$

h(km)	V_e(m/s)
1000	10,390
3000	9210
5000	8360
10,000	6970

It may be seen from the curve plotted between V_e and h that the velocity of escape is less if the satellite is launched from higher altitudes as shown in Fig. Ex. D2.4 (Solution). On the other hand, it is a problem to reach a high altitude of the order of thousands of km. If we were to escape, we would attach the satellite to a multistage rocket motor which would take it up into the thin air and when all the stages drop off, the satellite would be on its own. We would also like to take advantage of the spinning motion of the earth and launch the satellite so as to gain a component of the velocity from the spin of the earth before leaving it.

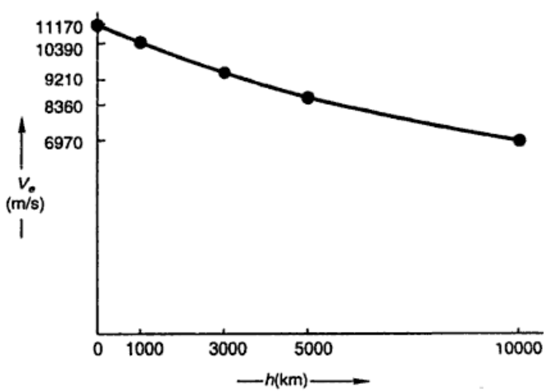


Fig. Ex. D2.4 ***(Solution)***

***Example* D2.5** A body is released at a distance far away from the surface of the earth. Calculate its velocity when it is near the surface of the earth, ignoring air resistance and all other forces except gravity.

Solution For the body, gain in kinetic energy equals loss in potential energy

$$\frac{1}{2}mV^2 = -GM\left(\frac{1}{L}-\frac{1}{R}\right)m = \frac{GMm}{R}$$

taking $1/L$ as zero for L to be very large.

Then,
$$V^2 = \frac{2GM}{R} = 2gR$$
$$= 2 \times 9.81 \times 6.37 \times 10^6$$

and
$$V = 11.17 \times 10^3 \text{ m/s}$$

***Example* D2.6** A satellite of mass 200 kg, initially at rest on the earth, is to be launched in a circular orbit at a height equal to the radius of the earth, i.e., 6.37×10^6 m. Calculate the minimum energy required.

Solution Radius of the circular orbit is $2R$.

Initial mechanical energy of the satellite at the surface of the earth is

$$E_1 = -GMm/R$$

Mechanical energy of the satellite in the circular orbit of radius $2R$ should be

$$\frac{1}{2}mV_0^2 + \left(-\frac{GMm}{2R}\right)$$

Since
$$V_0 = \sqrt{\frac{GM}{2R}}$$

the mechanical energy in rotation becomes

$$E_2 = \frac{1}{2}m\frac{GM}{2R} - \frac{GMm}{2R} = -\frac{1}{4}\frac{GMm}{R}$$

The minimum energy required to launch the satellite should be

$$E_2 - E_1 = -\frac{1}{4}\frac{GMm}{R} - \left(-\frac{GMm}{R}\right)$$
$$= \frac{3}{4}\frac{GMm}{R} = \frac{3}{4}gR$$
$$= \frac{3}{4} \times 9.81 \times 6.37 \times 10^6 = 9.365 \times 10^9 \text{ J}$$

***Example* D2.7** The Mars and the earth have their masses in proportion to 0.107 and their radii in the ratio of 0.53 (Mass being smaller), compare their (i) densities; (ii) gravitational intensities; (iii) escape velocities; and (iv) periods of their satellites.

Solution

$$\frac{M_m}{M_e} = 0.107, \qquad \frac{R_m}{R_e} = 0.53$$

(i) Density is mass divided by volume, i.e., $M \Big/ \frac{4}{3}\pi R^3$

$$\frac{d_m}{d_e} = \frac{M_m}{M_e} \cdot \left(\frac{R_e}{R_m}\right)^3 = 0.107 \times \frac{1}{0.53^3} = 0.718$$

(ii) Gravitational intensity $g = \frac{GM}{R^2}$

$$\frac{g_m}{g_e} = \frac{M_m}{M_e}\left(\frac{R_e}{R_m}\right)^2 = 0.107 \times \frac{1}{0.53^2} = 0.381$$

(iii) Escape velocity equals $\sqrt{\frac{2GM}{R}}$

$$\frac{V_{em}}{V_{ee}} = \sqrt{\frac{M_m}{M_e} \cdot \frac{R_e}{R_m}} = \sqrt{0.107 \times \frac{1}{0.53}} = 0.45$$

(iv) Period of a satellite $= \sqrt{\left(\frac{4\pi^2}{GM} R^3\right)}$

$$\frac{T_m}{T_e} = \sqrt{\frac{M_e}{M_m} \cdot \left(\frac{R_m}{R_e}\right)^3} = \sqrt{\frac{1}{0.107} \times 0.53^3} = 1.18$$

***Example* 2.8** A sky laboratory of mass 2000 kg has to be lifted from a circular orbit of radius $2R$ to another circular orbit of radius $3R$. Calculate the minimum energy required to do so.

Solution Mechanical energy of the satellite in a circular orbit of radius r is

$$\frac{1}{2} m V^2 - \frac{GMm}{r}$$

where

$$V = \sqrt{\frac{GM}{r}}$$

10 VIRTUAL WORK AND POTENTIAL ENERGY PRINCIPLES

10.1 INTRODUCTION

It is appropriate at this stage of presentation of the subject to look at the formulation of problems by methods other than Newton's laws of motion. We are here referring to the energy-based principles in mechanics. The advantages offered by the energy-based principles include freedom from drawing the free-body diagram, dealing with scalar quantities, namely energy and work instead of the vector quantities, such as force, velocity, acceleration, momentum, etc., ease of application to multibody systems and relative simplicity of analysis in the long run.

The principle of virtual work is introduced first for static equilibrium and then, together with the D'Alembert principle, it is extended to the analysis of dynamical systems. While we are on the subject of equilibrium, an alternative method known as the principle of potential energy, is also introduced. These principles are helpful in understanding the variational principles in mechanics.

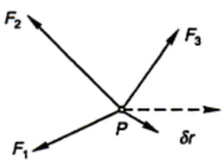
10.2 PRINCIPLE OF VIRTUAL WORK

Consider a body subjected to a system of forces. If the body is idealised as a particle, the system of forces

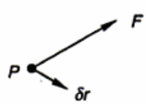
$$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$$

must be concurrent and replaceable by a single equivalent force **F** as shown in Fig. 10.1. If the body is idealised as a rigid body and the forces are non-concurrent then the system of forces and couple moments

$$\mathbf{F}_1, \mathbf{F}_2, \ldots; \qquad \mathbf{M}_1, \mathbf{M}_2, \ldots$$



(a) Given System of Forces



(a) Equivalent Force and Virtual Displacement of a Particle

Fig. 10.1 ***Equivalent Force on a Particle***

acting in a specified manner are replaceable, for an equivalent dynamic effect, by a single force **F** acting at a desired point and moment **M**. These facts have already been demonstrated under the study of equivalent system of forces.

Whether a body is actually at rest or in motion, a virtual or possible displacement can be visualised in view of the constraints. *Such a displacement is hypothetical or imaginary; it may or may not be along the actual displacement but it must be one of the physically possible displacements only.* Consider, for example, a body sliding up an incline under the action of some forces. The fact that the body can slide either up or down without violating the constraints, i.e., without being lifted from the incline or piercing into it or going sideways, the virtual displacement δr can be given either up or down the plane. Thus either of the modes of virtual displacements shown in Figs. 10.2(a) and (b) is acceptable. The fact that the body is actually moving up is of no consequence as far as the choice of virtual displacement is concerned. The actual mode of displacement is, however one of the ways a virtual displacement may be given. Virtual displacement δr is assumed to be very small or infinitesimal so that it is accompanied by an infinitesimally small amount of virtual work δW. *The term virtual work, therefore, implies the hypothetical work which would have been done to result in a virtual displacement under the application of the given system of forces and couple-moments.*

Fig. 10.2 ***A Body on an Incline***

The virtual work δW_1 due to a force **F** resulting in virtual displacement $\delta\mathbf{r}$ at the point of application of the force is

$$\delta W_1 = \mathbf{F} \cdot \delta\mathbf{r} \tag{10.1}$$

and the virtual work δW_2 due to a moment **M** resulting in an angular virtual displacement $\delta\theta$ of a rigid body is

$$\delta W_2 = \mathbf{M} \cdot \delta\theta \tag{10.2}$$

It follows that the virtual work for an infinitesimal virtual displacement of a particle subjected to a system of forces is

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} \tag{10.3}$$

where **F** is the resultant or equivalent force on the particle.

For a rigid body, the virtual work for an infinitesimal virtual displacement is

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} + \mathbf{M} \cdot \delta\theta \tag{10.4}$$

where $\delta\mathbf{r}$ is the linear virtual displacement of the point where the equivalent force **F** acts and $\delta\boldsymbol{\theta}$ is the angular virtual displacement of the body under the application of the equivalent moment **M**.

Alternatively, if a system of forces and couple-moments

$$\mathbf{F}_1, \mathbf{F}_2,...\mathbf{F}_n; \qquad \mathbf{M}_1, \mathbf{M}_2...\mathbf{M}_n$$

act on a rigid body in a specified manner and the linear possible virtual displacements at the points of application of the forces are

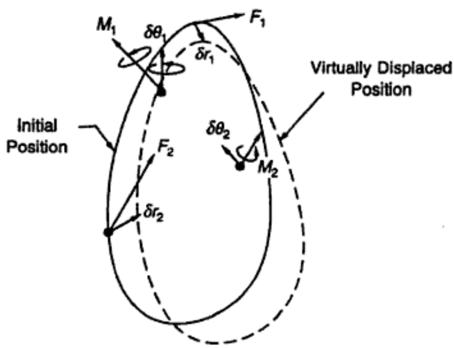
$$\delta\mathbf{r}_1, \delta\mathbf{r}_2,... \text{ respectively}$$

and the angular virtual displacements due to the moments are

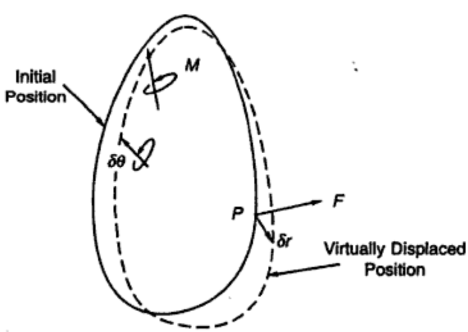
$$\delta\boldsymbol{\theta}_1, \delta\boldsymbol{\theta}_2, ...\text{respectively}$$

as shown in Fig. 10.3, then the total virtual work done on the body must be

$$\boxed{\begin{aligned}\delta W &= \mathbf{F}_1\cdot\delta\mathbf{r} + \mathbf{F}_2\cdot\delta\mathbf{r}_2 + ... + \mathbf{F}_n\cdot\delta\mathbf{r}_n \\ &\quad + \mathbf{M}_1\cdot\delta\boldsymbol{\theta}_1 + \mathbf{M}_2\,\delta\boldsymbol{\theta}_2 + ... + \mathbf{M}_1\cdot\delta\boldsymbol{\theta}_n\end{aligned}} \tag{10.5}$$



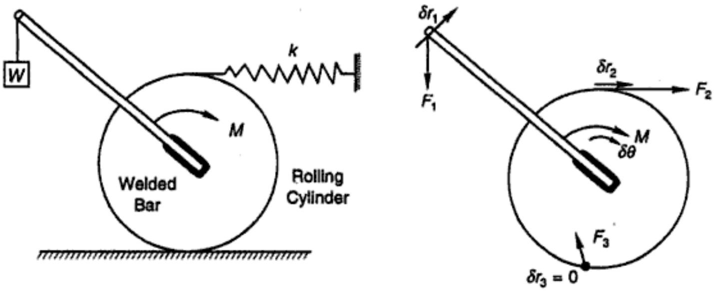
(a) Given System of Forces and Virtual Displacements



(b) Equivalent System of Forces and Virtual Displacements

Fig. 10.3 ***System of Forces Acting on a Rigid Body***

For example, the rolling cylinder shown in Fig. 10.4 is subjected to a system of forces as shown in the free-body diagram. From the constraints on the motion, the virtual displacements at the points of action of the forces can be visualised readily is also shown and this procedure of finding the virtual work can be used with advantage.



(a) A Rigid Body Subjected to a System of Forces

(b) Free Body Diagram and the Virtual Displacements

Fig. 10.4 ***Example of a Rigid Body Subjected to a System of Forces***

The principle of virtual work which we set out to enunciate is valid for a body in equilibrium. *It states that the virtual work should be zero for a body to be in equilibrium*

$$\delta W = 0 \tag{10.6}$$

This statement is initially due to Bernoulli and is often called the *Bernoulli Virtual Work Principle.*

It can be seen readily that this statement is equivalent to the conditions of equilibrium already formulated, viz.

$$\begin{aligned} \mathbf{F} &= 0 \\ \mathbf{M} &= 0 \end{aligned} \tag{10.7}$$

Substitution of these conditions in Eq. (10.4) shows that the virtual work δW must be zero for a body in equilibrium. Further, it was shown earlier that the above conditions are necessary but not sufficient for equilibrium, the same is also true for the virtual work principle.

The question that arises next is whether the virtual work principle offers any advantage over the conditions of equilibrium, Eq. (10.7), which follow directly from Newton's law and Euler's equation. The answer is "yes". There are definite advantages, some of which are explained as follows:

1. It may be easier and more convenient to determine the virtual work than to evaluate the resultant force and resultant moment. It is easier because the constraining forces which may be due to the normal reactions by the surfaces and the internal action-reaction forces do not contribute to the virtual work. For example, the reaction by a smooth surface can do no virtual work on a body when it is virtually

displaced by sliding over the surface because the reaction force and displacement are at right angles to each other. The merit of the virtual work principle is, therefore, that the reaction forces and other constraining forces need not be determined at all. This step is unavoidable if we were to evaluate the resultant force for equating it to zero for equilibrium. Some other forces which do not contribute to the virtual work and need not be evaluated are:

(i) Reaction at a smooth pin for rotation about the pin
(ii) Reaction at a roller moving along a track
(iii) Weight of a body when its centre of gravity moves horizontally
(iv) Friction force acting on a wheel rolling without slip.

2. The principle of virtual work applies to a system of connected particles or of rigid bodies as good as to a single rigid body or to a single particle. Only the forces external to the system of connected bodies need be considered for virtual work; the internal forces cannot contribute to the virtual work. Care must be taken to select the virtual displacements in a manner which do not violate the constraints.

3. It may appear, in the first instance, that the virtual work principle applies only to a body in equilibrium and provides an alternative, single condition of equilibrium which is a little simpler. A little consideration of the D'Alembert's principle will show that if a body of mass m is accelerating at an acceleration $\mathbf{a}$ under the action of an external force $\mathbf{F}$, then,

$$\mathbf{F} + (-m\mathbf{a}) = 0 \tag{10.8}$$

The form of the equation suggests that if a hypothetical force called inertia force

$$\mathbf{F}_I = -m\mathbf{a} \tag{10.9}$$

is to act on the body in addition to the external force $\mathbf{F}$, then the body would hypothetically come to a state of equilibrium. *It shows that the D'Alembert's principle used simultaneously with the virtual work principle should be able to extent the virtual work principle to bodies not in equilibrium.* As a closing comment, it should be mentioned here that this statement is so true that a complete formulation of energy method in mechanics has already overshadowed the Newtonian mechanics. Energy equations due to Langrange and Hamilton can be shown equivalent to the Newtonian formulation.

***Example* 10.1** A frictionless double-incline with angle α_1 and α_2 as shown in Fig. Ex. 10.1 carries a set of sliding masses m_1 and m_2 connected with an inextensible string and passing over a frictionless pulley at O. Obtain the relationship between

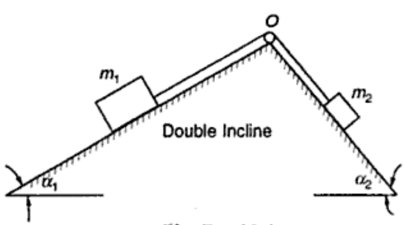


Fig. Ex. 10.1

α_1 and α_2 in terms of m_1 and m_2 for equilibrium and hence determine α_2 if

$$\alpha_1 = 30° \quad \text{and} \quad m_1 = 2m_2$$

How would the angle change if the surfaces had a coefficient of friction μ instead of being smooth?

Solution Let us draw and observe the free-body diagrams of the two masses in the absence of friction (Fig. Ex. 10.1 (Solution)).

Since the motion is possible only up and down the inclines, let us give a virtual displacement δr_1 up the α_1 incline to mass m_1. The virtual displacement of mass m_2 must then be δr_1 down the α_2 incline because the string is inextensible and hence is of constant length.

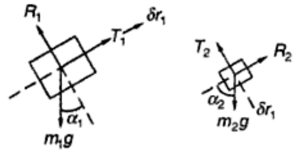


Fig. Ex. 10.1 (*Solution*)

The virtual work done by R_1 for the displacement δr_1 of mass m_1 must be zero because they are at right angles. Similarly, the virtual work by R_2 for the displacement δr_1 of mass m_2 must be zero. *R_1 and R_2 need not be determined.* Now, the virtual work done by T_1 for displacement δr_1 of mass m_1 along T_1 is $T_1\delta r_1$ and the virtual work done by T_2 for displacement δr_1 of mass m_2 along T_2 is $T_2\delta r_1$. The total work done by the tension forces in the inextensible string is

$$T_1\delta r_1 + T_2\delta r_1$$

which must be zero because T_1 is equal and opposite to T_2, there being no friction in the pulley.

*The only forces which indeed do virtual work are the external forces on the bodies. In this case m_1g and m_2g are the only external gravitational forces. The virtual work done by them adds up to

$$-\, m_1g \sin\alpha_1 \; \delta r_1 + m_2g \sin\alpha_2 \; \delta r_1$$

According to the virtual work principle, the virtual work must be zero for the system to be in equilibrium

$$-m_1g \sin\alpha_1 \; \delta r_1 + m_2g \sin\alpha_2 \; \delta r_1 = 0$$

Cancelling δr_1 and g from both terms,

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{m_2}{m_1}\,*$$

It may appear that the virtual work method is also lengthy and requires careful elimination of the non-contributory forces. However, this is untrue. *There was neither a need to draw the free-body diagram actually nor to eliminate the normal-reaction and internal equal tension forces.* Only a general statement is adequate and the required proof is between the two asterisks (*) above.

When $\quad m_1 = 2m_2 \quad$ and $\quad \alpha_1 = 30°$

$$\sin \alpha_2 = 2 \sin 30° = 1$$

whence $\alpha_2 = 90°$

Let us now investigate the effect of friction at the inclined surfaces. The virtual displacement still being the same as before, the frictional forces would enter into the virtual work.

On mass m_1, the frictional force would be

$$\mu\, m_1 g \cos\alpha_1$$

down the incline α_1 and the virtual work contributed by it

$$-\mu\, m_1 g \cos\alpha_1\, \delta r_1$$

Similarly, the virtual work done by the frictional force on mass m_2 would be

$$-\mu m_2 g \cos\alpha_2\, \alpha r_1$$

The virtual work principle would be applied to the total virtual work, i.e.,

$$-m_1 g \sin\alpha_1\, \delta r_1 + m_2 g \sin\alpha_2\, \delta r_1 - \mu\, m_1 g \cos\alpha_1\, \delta r_1$$
$$- m_2 g \cos\alpha_2\, \delta r_1 = 0$$

Cancelling δr_1 and g,

$$m_1(\sin\alpha_1 + \mu \cos\alpha_1) = m_2 (\sin\alpha_2 - \mu \cos\alpha_2)$$

or
$$\frac{\sin \alpha_1 + \mu \cos \alpha_1}{\sin \alpha_2 - \mu \cos \alpha_2} = \frac{m_2}{m_1}$$

A noteworthy comment at this stage is that this relationship applies to the equilibrium against sliding in accordance with the chosen mode of virtual displacement. If δr_1 was reversed in direction, frictional forces would also reverse and the virtual work principle would yield

$$\frac{\sin \alpha_1 - \mu \cos \alpha_1}{\sin \alpha_2 + \mu \cos \alpha_2} = \frac{m_2}{m_1}$$

for equilibrium. The application of the virtual work principle for systems with dissipative forces such as friction, aero-dynamic drag, etc., requires as much attention as the application of Newtonian laws.

***Example* 10.2** Determine the relationship between the moment M applied at the crank of radius R and the force F applied at the crosshead in the slider crank mechanism shown in Fig. Ex. 10.2.

Solution Let the system be virtually displaced as shown in Fig. Ex. 10.2 (Solution). The virtual work done by the moment M and the force F adds up to

$$-M\, \delta\theta - F\, \delta x \tag{i}$$

which must be equated to zero in the absence of any other external force or moment. Here, x is chosen positive to the right and θ positive anticlockwise as per our usual convention.

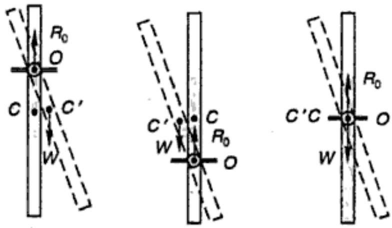


Fig. 10.7 ***State of Equilibrium***

In Case (a), the bar tends to return to its initial position. In Case (b), the ba. continues to deviate from the initial position and in Case (c) the bar stays wherever it is left. This is so because the moment created by the shift of the centre of gravity from C to C' tends to restore equilibrium in the first case, deviate it further in the second case and there is no such moment in the third case. The state of equilibrium in which a slight disturbance from the equilibrium position is accompanied by a restoring moment so as to bring the body back into its initial state of equilibrium is called *stable equilibrium*. On the other hand, if the slight disturbance is accompanied by an adverse moment to displace the body further, the state is said to be *unstable* and if the slight disturbance fails to generate a restoring or a worsening moment, the body is said to be in a state of *neutral* equilibrium.

Since the potential energy of a suspended body reckoned to its point of suspension O is minimum when the centre of gravity C is below O, maximum when the centre of gravity C is above O and remains unchanged when C and O coincide it is clear that *the states of stable, unstable and neutral equilibrium correspond to the minimum, maximum and 'stationary' potential energy respectively.* This fact can also be appreciated by the observation that a slight angular displacement of the bar is accompanied by a rise of C to C' in Case (a), lowering of C to C' in Case (b) and all-time coincidence of C and C' in Case (c) showing thereby that while in the vertical equilibrium position C must be the lowest possible in Case (a), highest in Case (b) and unchanged in Case (c).

From Differential Calculus, we can recollect that PE is a minima when

$$\frac{d^2\,(PE)}{ds^2} > 0 \tag{10.12}$$

and a maxima when

$$\frac{d^2\,(PE)}{ds^2} < 0 \tag{10.13}$$

and the point of inflexion and the state of 'stationary' PE are characterised by

$$\frac{d^2\,(PE)}{ds^2} = 0$$

It may be remarked that a point of inflexion of PE vs. s is not classified as stable, unstable or neutral equilibrium. This is because slight displacement one way may

Solution

$$PE = 10x^3 + 8x^2 - 9x$$

For equilibrium positions,

$$\frac{d(PE)}{dx} = 30x^2 + 16x - 9 = 0$$

whence $x_1 = -0.876$ m

$$x_2 = +0.3425 \text{ m}$$

Now, the second derivatives of the PE are

$$\frac{d^2(PE)}{dx^2} = 60x + 16$$

At $x_1 = -0.876$ m, $\frac{d^2(PE)}{dx^2} = -36.56; < 0$

At $x_2 = 0.3425$ m, $\frac{d^2(PE)}{dx^2} = +36.55; > 0$

The equilibrium position at $x = -0.876$ m corresponds to unstable equilibrium and that at $x = 0.3425$ m corresponds to stable equilibrium.

***Example* 10.5** A rectangular block and a semicylinder of the same length made of the same homogeneous material are secured together and placed on a flat rough surface in equilibrium as shown in Fig. Ex. 10.5. Determine the minimum radius R of the semicylinder in terms of the height H of the block if the width of the block is n times the radius of the semicylinder in order that the composite body be in stable equilibrium. Assume that the semicylinder may roll without slip on the flat surface.

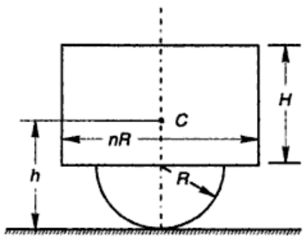


Fig. Ex. 10.5

Solution Let the centre of gravity of the composite body be h above the base in the equilibrium position.

When it is tilted by a small angle θ as shown, the centre of gravity drops to a new height

$$R + (h - R) \cos \theta$$

Referring to the initial or equilibrium position, the potential energy of the body is

$$PE = -mg\,\Delta h$$

$$= mg(R - h)(1 - \cos \theta)$$

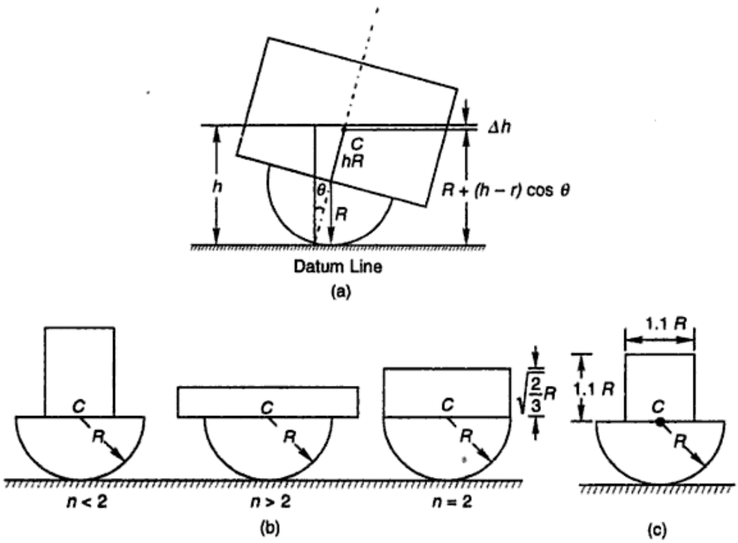


Fig. Ex. 10.5 ***(Solution)***

where m is the mass of the composite body.

Now, $$\frac{d(PE)}{d\theta} = mg(R - h)\sin\theta$$

This is zero when $\sin\theta = 0$ or $\theta = 0$ showing that the initial position is indeed the position of equilibrium,

$$\frac{d^2(PE)}{d\theta^2} = mg(R - h)\cos\theta$$

which, for $\theta = 0$, becomes

$$\frac{d^2(PE)}{d\theta^2} = mg(R - h)$$

This should be positive for stable equilibrium. The geometric condition for stability of the composite body is, therefore, that R should be greater than h. In other words, the centre of gravity should lie below the centre of the circle in the view shown in Fig. Ex. 10.5 (Solution). When $h = R$, the state is of neutral equilibrium: the body would stay where it is left. When h is greater than R, as initially assumed, the equilibrium is unstable.

In order that the centre of gravity be located at the centre of the circle for the limiting case,

$$\frac{H}{2} \times (nR \times H) = \frac{4R}{3\pi} \times \frac{\pi R^2}{2}$$

whence $$H = \frac{2R}{\sqrt{3n}}$$

It is interesting to observe that the wider the block, more is n and less is H. The thinner the block, less is n and more is H.

In the limits,

$$n \to \infty, \qquad H \to 0$$

and $$n \to 0, \qquad H \to \infty$$

In the special case when the width of the block equals the diameter of the semicylinder,

$$n = 2$$

and $$H = \sqrt{\frac{2}{3}}\,R = 0.8165R$$

Three such cases of $n < 2$, $n > 2$ and $n = 2$ are shown in the neutral equilibrium condition. The bodies exhibit stable equilibrium of the height of the block is reduced, but are unstable if the height is slightly increased beyond the heights shown for neutral equilibrium.

Another interesting case is of securing a block of square cross-section on the semicylindrical block. The maximum side of the square cross-section for stable equilibrium would be

$$a = H = nR$$

which together with

$$H = \frac{2R}{\sqrt{3n}}$$

provides $$n = (4/3)^{1/3} = 1.1$$

and $$a = 1.1R$$

as shown in Fig. Ex. 10.5(c) (Solution).

10.4 GENERALISED COORDINATES

The position of a particle on a given line is described by one coordinate; on a plane by two coordinates and in space by three coordinates. The coordinates may refer to any set of axes; these may involve distances and angles in any combination. To describe the position of two particles in space, we use six coordinates. For a larger number of particles moving in a certain fashion, particularly when the particles refer to one or more of the rigid bodies, we may have to use a larger number of coordinates. It is always possible to select the smallest number of independent variables to describe a system; *the smallest number of variables constitute the generalised coordinates.* The values of these generalised variables, whatever they may be, completely define a system and can be varied to define other states in keeping with the constraints on the system. There is no unique choice of the generalised coordinates.

10.2 Using the method of virtual work, determine the force in the top member of the simple truss consisting of equilateral triangles. **(Ans.** 0.577 P)

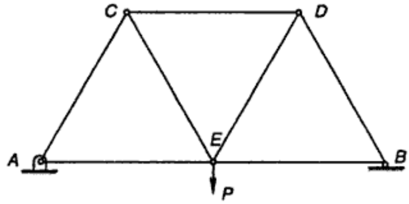


Fig. Prob. 10.2

10.3 A simple parallelogram linkage carries three forces as shown in Fig. Prob. 10.3 while the members AB and CD capable of oscillating about A and D respectively are at 60° with the base line AD. Determine the moment M required to maintain equilibrium of the linkage.

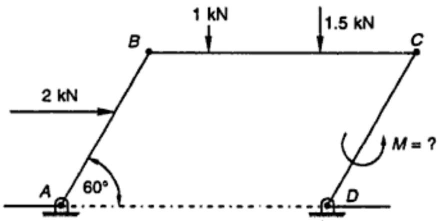


Fig. Prob. 10.3

10.4 A simple linkage consisting of two equal massless bars and a sliding block B is subjected to a set of two forces F_1 and F_2 as shown in Fig. Prob. 10.4. Express the angle θ as a function of F_1 and F_2 for equilibrium. **(Ans.** $\tan \theta = F_1/2F_2$)

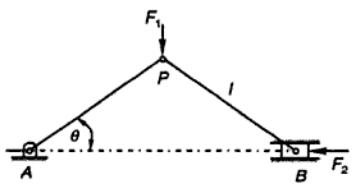


Fig. Prob. 10.4

10.5 Two inextensible strings of equal lengths and two equal concentrated masses comprise a double pendulum as shown in Fig. Prob. 10.5. A force F is applied horizontally at the lower end to keep it in equilibrium. Express the angles θ_1 and θ_2 with the vertical in terms of F and m.

(Ans. $\tan \theta_1 = F/2mg$, $\tan \theta_2 = F/mg$).

10.6 A uniform rectangular plank of height h, base width b and mass m rests on top of a convex circular surface of radius R. Establish the relationship between the height of the plank and the radius of the convex surface for the stable and unstable equilibrium of the plank in rolling without slipping over the convex surface.

(Ans. Equilibrium: OPQ vertical; Stable for $R > h/2$; Unstable for $R < h/2$)

(d) $\dfrac{d^2 PE}{ds^2} = 0$

5. The time taken by a small frictionless bead to slide on a thin wire in the gravitational field is the minimum if the shape of the wire is
   (a) a straight line (c) a parabola
   (b) a cycloid (d) an involute

Answers to Multiple-Choice Questions

1 (d), 2 (a), 3 (b), 4 (c), 5 (b)

11 VIBRATIONS OF SIMPLE MECHANICAL SYSTEMS

11.1 ELEMENTS OF MECHANICAL SYSTEMS

A system, by definition, is an assemblage of interacting elements constituted for a desired objective. The elements of a system may be certain simple mechanical, electrical, optical, thermo-mechanical, electrodynamical or some other devices which are characterised by their behaviour. Similarly, a system may belong to mechanical, electrical, optical, thermomechanical, electrodynamical or some other class. A system is termed *dynamical system* if the system response is a function of time. It is interesting, under the systems approach, to deal with different classes of systems in a unified fashion. This is made possible by the similarity of behaviour of the elements of different systems. With these comments, we will confine ourselves to the mechanical systems. The study can, however, be extended to other systems.

A mechanical system comprises of mechanical elements which provide the following properties or characteristics:

1. Inertia
2. Compliance
3. Damping.

These characteristics may either be translational or rotational. It may be noted that a suitable combination of the translational and the rotational characteristics can provide the desired characteristics in any motion, by simple superpositions, if the basic characteristics are linear. We shall restrict our discussion to the linear elements. Let us now focus our attention to each of the characteristics.

Inertia

	Translational	*Rotational*
Basic Concept	Mass m	Moment of inertia I
Acceleration	$a = \dfrac{d^2x}{dt^2}$	$\alpha = \dfrac{d^2\theta}{dt^2}$
Action	Force F	Moment M
Constitutive Equation	$F = ma = m\dfrac{d^2x}{dt^2}$	$M = I\alpha = I\dfrac{d^2\theta}{dt^2}$ (11.1)

Representation: Refer Fig. 11.1.

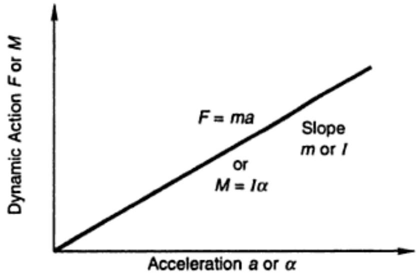


Fig. 11.1 Characteristic of an Inertial Element

Examples: Refer Fig. 11.2.

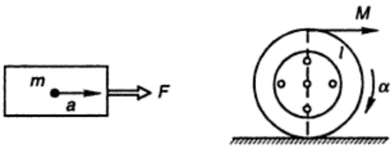


Fig. 11.2 Inertial Elements

Comments

Inertial elements, i.e., mass in translation and in rotation are pure translational and pure rotational elements respectively. By pure, it is implied that they have no other properties, such as deformability, etc. The motion of an inertial element is governed by the laws of Mechanics.

Compliance

	Translational	*Rotational*	
Concept	Elastic Elements		
	Translational Spring (spring constant) k	Rotational Spring (spring constant) K	
Displacement	x	θ	
Action	Force F	Moment M	
Constitutive Equation	$F = kx$	$M = K\theta$	(11.2)

Representation: Refer Fig. 11.3.

Examples: Refer Fig. 11.4.

Comments

A compliance element complies with a steady force to produce a corresponding steady deformation. Compliance elements such as springs are said to be pure if they are assumed to be devoid of mass and are capable of conserving energy under all modes of operation. A compliance element must return to its equilibrium position when the applied force is removed.

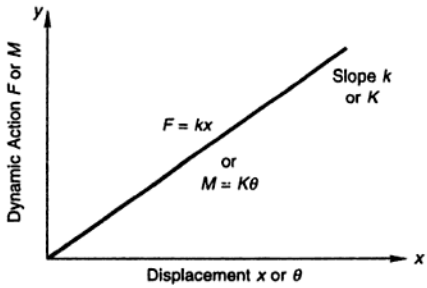


Fig. 11.3 ***Characteristic of a Compliance Element***

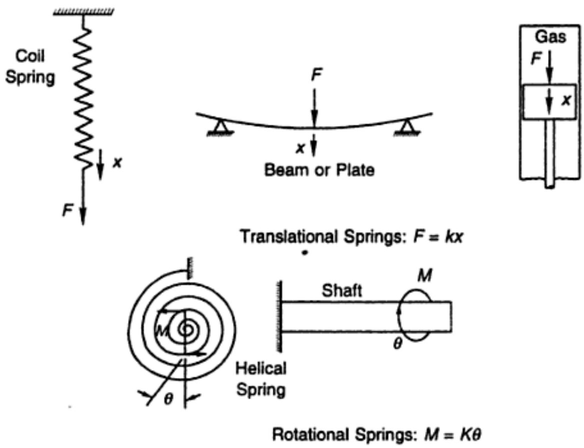


Fig. 11.4 ***Elements Producing Compliance***

Damping

	Translational	*Rotational*	
Concept	Dissipative Elements Translational Damper (damping constant) c	Rotational Damper (damping constant) C	
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$	
Action	Force F	Moment M	
Constitutive Equation	$F = cv = c\frac{dx}{dt}$	$M = C\omega = C\frac{d\theta}{dt}$	(11.3)

Representation: Refer Fig. 11.5.

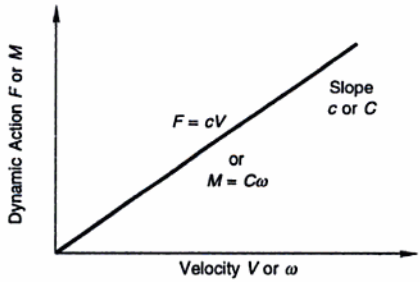
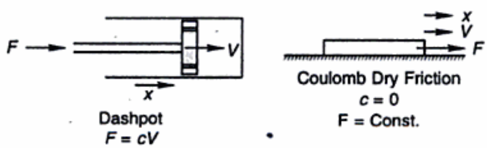


Fig. 11.5 ***Characteristic of a Damping Element***

Examples: Refer Fig. 11.6.



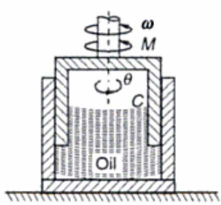


Fig. 11.6 ***Elements Producing Damping***

Comments

A damping element is associated with dissipative action. Dissipation of energy may be achieved through dry friction, viscous shearing or otherwise. Mass and elasticity of the components of the dampers is assumed to be zero for pure dampers. A damping element stays where it is, when the applied force is removed.

11.2 SIMPLE MECHANICAL SYSTEMS

Simple mechanical systems are those which can be conceived as simple combinations of the system elements.

A spring-mass combination with the mass subjected to a force along the line of the spring and passing through the centre of mass of the spring is called a linear spring mass system. If the mass is also subjected to a damping force, it is called a damped spring-mass system. The system may be arranged to operate in translation or in rotation and the action of forces and moments may be adjusted to comprise a variety of systems, examples of which are shown in Figs. 11.7 and 11.8.

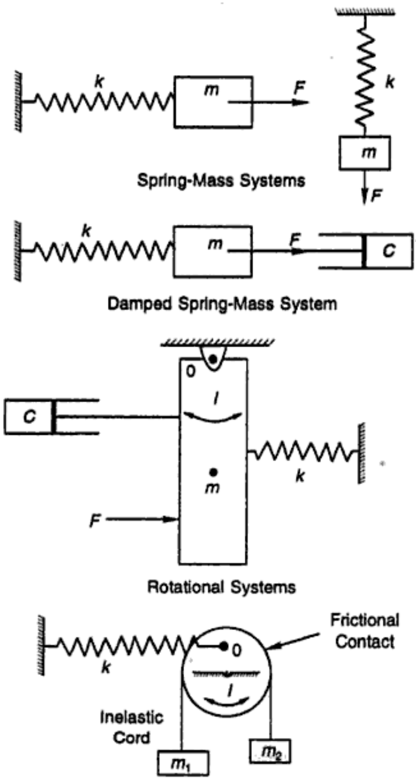


Fig. 11.7 ***Single-degree-of-freedom Systems***

We next define the number of degrees of freedom of a mechanical system. The number of independent coordinates required to specify the position of a body has been defined as the degrees of freedom of the body. For constrained motion of a body, the number of constraints must be subtracted from six to obtain the degrees of freedom. It is, therefore, the possible modes of motion of a body which amount to the degrees of freedom. Now, for a mechanical system of n interconnected bodies there can be a maximum number of 6 n degrees of freedom. The actual degrees of

7. The frequency of oscillation is the number of cycles per unit time described by the particle,

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

***Example* 11.1** A point moves with a simple harmonic motion such that it has a velocity of 9 m/s when it is at a distance of 2 m from the centre and a velocity of 4 m/s when it is at a distance of 3 m from the centre of the same side for the point moving in the same direction. Calculate (a) the amplitude of the motion, (b) the time period of the motion, (c) the time interval between the two positions, (d) the acceleration of the point at these positions, and (e) the greatest acceleration.

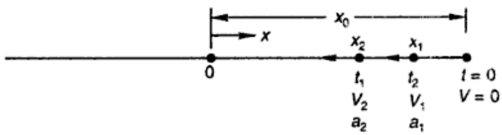


Fig. Ex. 11.1

Solution For the simple harmonic motion of a point,

$$x = x_0 \cos \omega t \tag{i}$$

$$V = -\omega\sqrt{x_0^2 - x^2} \tag{ii}$$

and

$$a = -x\omega^2 \tag{iii}$$

(a) For the data, from Eq. (ii),

$$9 = -\omega\sqrt{x_0^2 - 4} \text{ and } 4 = -\omega\sqrt{x_0^2 - 9}$$

whence,

$$x_0 = \pm\ 3.2 \text{ m and } \omega = 3.6 \text{ rad/s}$$

The amplitude of motion of the point is 3.2 m.

(b) The time period is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3.6} = 1.75 \text{ s}$$

(c) Assuming that $t = 0$ at $x = x_0$, times t_1 and t_2 for positions x_1 and x_2 are given by

$$3 = 3.2 \cos 3.6\, t_1;\ t_1 = 0.099 \text{ s}$$

$$2 = 3.2 \cos 3.6\, t_2;\ t_2 = 0.249 \text{ s}$$

and the time interval is

$$t_2 - t_1 = 0.249 - 0.099 = 0.15 \text{ s}$$

(d) From Eq. (iii), the accelerations are

$$a_1 = -3 \times 3.6^2 = -38.88 \text{ m/s}^2$$

$$a_2 = -2 \times 3.6^2 = -25.92 \text{ m/s}^2$$

(e) The maximum acceleration must be

$$a_{max} = -x_0\omega^2$$
$$= -3.2 \times 3.6^2 = -41.47 \text{ m/s}^2$$

at the right extremity and the same in magnitude but opposite in direction at the left extremity.

11.4 LINEAR FREE VIBRATIONS

A system of a single mass constrained to move along the axis of a horizontal spring attached to it, the other end of which is fixed, as shown in Fig. 11.11, is considered first. The equilibrium position of the mass corresponds to the unstretched length of the spring, which is also the reference position for the periodic motion.

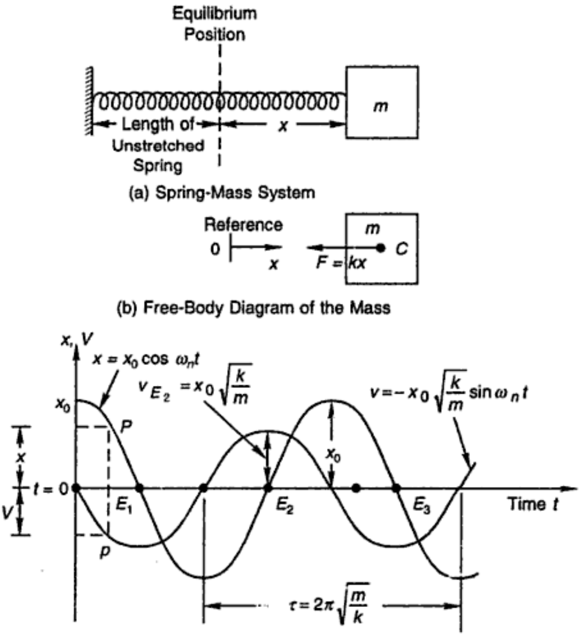


Fig. 11.11

At an instant when the mass is displaced by x, the restoring force acting on it is

$$F = -kx$$

The equation of motion in accordance with the Newton's law is

$$F = ma$$

or
$$-kx = m\frac{d^2x}{dt^2}$$

or
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The most general solution of this second-order linear differential equation is

$$x = A \sin \omega_n t + B \cos \omega_n t$$

whence, by differentiating with respect to time t,

$$\frac{dx}{dt} = \omega_n A \cos \omega_n t - \omega_n B \sin \omega_n t$$

$$\frac{d^2x}{dt^2} = -\omega_n^2 A \sin \omega_n t - \omega_n^2\ B \cos \omega_n t = -\omega_n^2 x$$

Substituting these values in the equation of motion,

$$-\omega_n^2 x + \frac{k}{m}x = 0$$

whence,

Angular frequency, $\omega_n = \frac{\sqrt{k}}{m}$,

Cyclic frequency, $f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and

Time period, $\tau = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$

It can be observed that the mass is set into vibrations by applying an initial force of $k\,x_0$, say towards right, to bring the mass at rest at a distance x_0 from the equilibrium position. If the time there is reckoned at $t = 0$, then from the general solution

$$x = A \sin \omega_n t + B \cos \omega_n t$$

the initial condition

$t = 0$ at $x = x_0$

gives $A = 0$ and $B = x_0$

rendering the solution as

$$x = x_0 \cos \omega_n t$$

The maximum displacement x_{max} corresponds to $\cos \omega_n t = 1$, i.e., at the extremities of the periodic motion

$$|x_{max}| = x_0$$

when the potential energy $PE_2 = 0$ for the spring, it being the zero-stretch position.

By the conservation principle,

$$PE_1 + KE_1 = PE_2 + KE_2$$

$$\frac{1}{2} k x_0^2 = \frac{1}{2} m v_E^2$$

whence
$$v_E = x_0 \sqrt{\frac{k}{m}}$$

From the knowledge of the maximum velocity and maximum displacement of the system, the frequency and time period can also be estimated

$$\omega_n = \frac{|v_{max}|}{|x_{max}|} = \frac{x_0 \sqrt{k/m}}{x_0} = \sqrt{\frac{k}{m}}$$

$$\tau = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{k}}$$

$$f = \frac{1}{\tau} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

It is interesting to observe the periodic motion graphically. This is shown in Fig. 11.11(c). The mass is released at $t = 0$ from a position $x = x_0$ from a state of rest. The mass passes through the equilibrium positions E_1, E_2, E_3, etc., when $x = 0$ and its velocity is maximum at these positions.

The case of free vibrations of a mass attached to a vertical spring fixed at the upper end is similar to the horizontal spring-mass system just considered. A reference to Fig. 11.12 shows that the equilibrium position of the spring-mass system in this case does not correspond to the unstretched state of the spring but to the spring stretched by the static deflection

$$\delta = \frac{W}{k}$$

At the equilibrium position, the weight W of the mass acting downward balances the tension T developed in the spring

$$W = mg = T = k\delta$$

For a displacement x from the equilibrium position, the restoring force acting on the mass equals

$$\begin{aligned} F &= -k(\delta + x) + W \\ &= -k(\delta + x) + k\delta \\ &= -kx \end{aligned}$$

The maximum velocity is $v_E = x_0 \sqrt{\dfrac{k}{m}} = x_0 \omega_n$

at the equilibrium position, directed alternatively upwards and downwards.

Acceleration of the mass $\quad a = -\omega_n^2 x \qquad (11.2)$

Maximum acceleration $\quad a_{max} = -\omega_n^2 x_0$

at the extremity and directed towards the equilibrium position.

***Example* 11.2** A small motor of mass 20 kg is symmetrically mounted on four equal springs, each with a spring constant of 25 N/cm. Estimate the frequency and period of vibration of the motor.

Solution The four springs arranged 'in parallel' may be considered equivalent to a single spring located in line with the centre of mass of the motor with

$$k = 4 \times 25 = 100 \text{ N/cm} = 10{,}000 \text{ N/m}$$

Cyclic frequency $\quad f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{10{,}000}{20}}$

$$= 3.56 \text{ Hz}$$

Time period $\quad \tau = \dfrac{1}{f} = 0.28 \text{ s}$

***Example* 11.3** A vertical U-tube manometer contains a liquid of mass density ρ as shown in Fig. Ex. 11.3. A sudden increase of pressure on one column forces the level of the liquid down. When the pressure is released, the liquid colums start vibrating. Neglecting the frictional damping, determine the period of vibration. Comment if the period is affected by changing the liquid, diameter of the tube or length l of the liquid column.

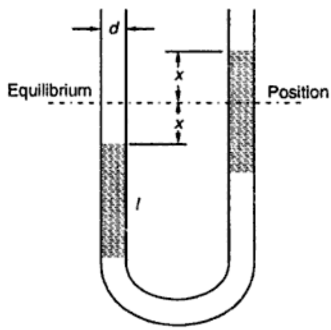


Fig. Ex. 11.3

Solution The equilibrium position of the liquid columns is when these are in level. At any instant during vibration, one column rises by a distance x while the other falls by the same distance. The restoring force, therefore, equals the weight of a liquid column $2x$ high.

$$F = -2x\pi (d^2/4)\, \rho g$$

This must equal the product of the mass of the total length of the liquid column in motion and the acceleration,

$$-2x \cdot \pi (d^2/4)\, \rho g = l\pi (d^2/4)\rho \frac{d^2 x}{dt^2}$$

which, on simplification, yields the equation of motion

$$\frac{d^2x}{dt^2} + \frac{2g}{l}x = 0$$

The general solution of this equation is

$$x = A \cos \omega_n t + B \sin \omega_n t$$

whence
$$\frac{d^2x}{dt^2} = -\omega_n^2 (A \cos \omega_n t + B \sin \omega_n t) = -\omega_n^2 x$$

and substituting it in the differential equation provides

$$\omega_n = \sqrt{\frac{2g}{l}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{2g}{l}}$$

$$\tau = 2\pi\sqrt{\frac{l}{2g}}$$

The period is unaffected by the density of the liquid and the diameter of the tube. The total length of the liquid column, however, does effect the period of vibration.

For a typical vertical manometer with a total column 0.5 m long,

$$\tau = 2\pi \times \sqrt{\frac{0.5}{2 \times 9.81}} = 1 \text{ s}$$

which implies that the manometer is capable of oscillating visibly at 1 cycle per second in the absence of damping affects which may arise from fluid friction between the liquid and the tube material.

Another interesting observation is that if the plane of the manometer is set at an angle θ to the horizontal, as is the case for low pressure-difference measurement, the restoring force becomes sin θ times the above value and the period of vibration becomes

$$\tau = 2\pi\sqrt{\frac{l}{2g \sin \theta}}$$

which shows that the time period increases or the frequency decreases; again neglecting any damping forces.

Experiment E13

Oscillation of a Simple Pendulum

Objective

To determine the time period of a simple pendulum and to estimate the value of g, the acceleration due to gravity.

If possible, the experiment may be repeated for a different length of string.

S. No.	l (m)	*Time for* 20 *oscillations*	τ (s)	g (m/s^2)

The average value of g for the observations is the best estimate of g.

RESULT AND POINTS FOR DISCUSION

1. Compare the value of g obtained by you with the standard value

$$g = 9.80665$$

and comment on the difference. Note that the difference is due to the latitude λ and height h above the sea level as well as the experimental errors. Count the sources of error.

2. Would an improved value of g be obtained if only
   (a) the length of the pendulum were increased?
   (b) the mass of the bob were increased?
   (c) the volume of the bob were decreased?
   (d) the surface finish of the bob were improved?
   (e) the simple pendulum were enclosed in a vacuum chamber?
   (f) the amplitude of oscillations is increased?
3. Is it necessary that the oscillations should be in one and the same vertical plane?
4. What would be the effect of the earth's spinning about its own axis on the motion of the simple pendulum over a prolonged period of time?
5. Compare the accuracy of this method of determining g with other methods known to you.
6. If a simple pendulum is taken to a location as far away from the surface of the earth as its radius, what would be the time period in relation to the time period at the surface of the earth?

Experiment E14

Multiple Elastic Impacts

OBJECTIVE

To understand the implications of the simultaneous conservation of momentum and energy in a multiple elastic impacts.

APPARATUS

A row of identical simple pendulums with steel bobs hanging in close contact with each other.

BACKGROUND INFORMATION

The momentum of a system of bodies is conserved if no external force acts upon them. The mechanical energy of a system of bodies is conserved if the motion or

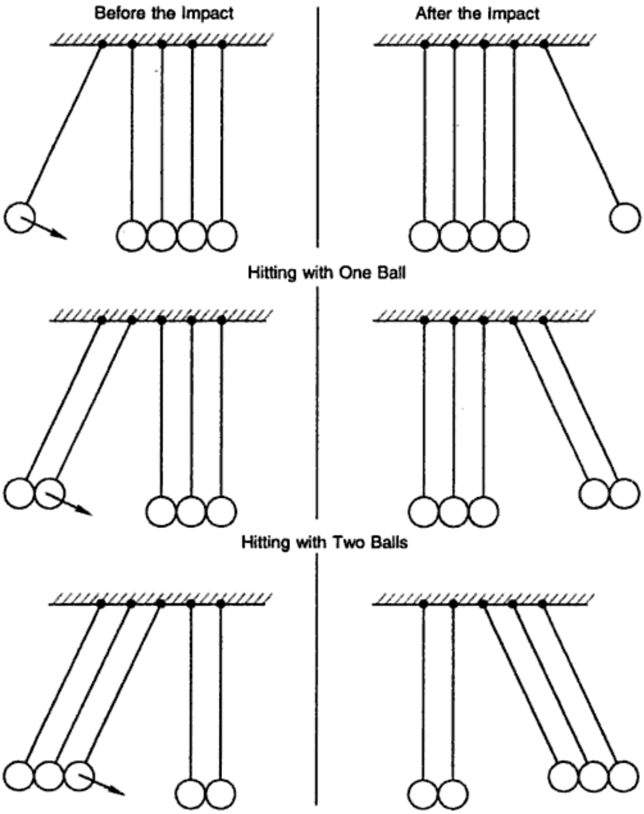


Fig. E14.1 ***Playing with Simple Pendulums***

interaction of the bodies takes place in a conservative force field. Simultaneous conservation of momentum and mecahnical energy may also take place. For example, if a number of balls, touching each other, and are struck by one or more balls directly and centrally, the motion is governed by the conservation of momentum and energy simultaneously. Assuming that the mass of each ball is m, let us analyse the effect of one or more balls striking the row of balls.

If the bob of a pendulum at one end is released from a small angle, its bob would move with a velocity v as it approaches the next bob. The momentum possessed by the first bob was mv and the mechanical energy was $1/2\ mv^2$. The number of balls that would be displaced, after multiple elastic impacts, would be such that the laws of conservation of momentum and energy are obeyed *simultaneously*.

If n balls moved out with velocities (v_1, v_2, v_3, ...v_n) respectively, then

$$v = v_1 + v_2 + v_3 + \dots + v_n$$

and

$$v^2 = v_1^2 + v_2^2 + v_3^2 + \dots v_n^2$$

These equations cannot be satisfied unless n equals 1, in which case $v = v_1$. The conclusion is that only the last one ball may move out with the same velocity with which the striking ball approached them. Only one pendulum will swing out. If two pendulums are released, causing two balls to approach with a velocity v each, the initial momentum of the system would be $2mv$ and the initial energy mv^2. It will be discovered that, in this case, two end balls would move out, each with velocity v after the series of impacts. Supposing only one ball were to move out at velocity v_1, then

$$2mv = mv_1; v_1 = 2v$$

and

$$mv^2 = \frac{1}{2} mv_1^2 ; v_1^2 = 2v^2$$

which is an inconsistent set of equations.

Similarly if three pendulums are released, then three pendulums would swing out!

Points for Discussion

1. When a ball rolls without slip on another surface, energy is conserved. Explain why and how?
2. Is the momentum conserved for every ball, taken one at a time, in the system? Recognize the balls for which momentum is conserved separately and for which it is not conserved separately. Is the energy conserved separately for each ball?
3. Would the pendulums keep on oscillating indefinitely?
4. Suggest some other experiment where you can gain visual experience on the conservation of momentum and energy, separately and simultaneously?

***Example* 11.4** A simple pendulum swings 5 oscillations in the same time as another 0.48 m longer swings 3 oscillations. Determine their lengths.

Solution Let the length of the first pendulum be l metres; the length of the second must be $(l + 0.48)$ metres.

In time T, the former swings 5 oscillations, its time period must be $T/5$.

Therefore, $$T/5 = 2\pi\sqrt{\frac{l}{g}}$$

Similarly, $$T/3 = 2\pi\sqrt{\frac{(l + 0.48)}{g}}$$

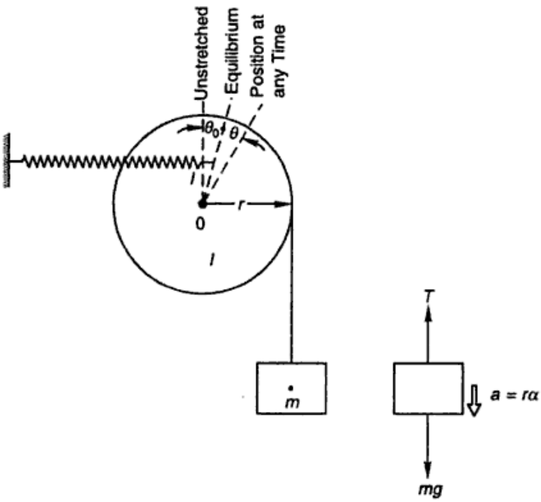


Fig. Ex. 11.10

In a slightly displaced position at any time t, the equations of motion, due to Newton and Euler, for the mass and the cylinder respectively are

$$mg - T = ma = mr\alpha \quad \text{(ii)}$$

$$Tr - kl\,(\theta_0 + \theta)l = I\alpha \quad \text{(iii)}$$

Substituting the value of T from Eq. (ii) into Eq. (iii),

$$m(g - r\alpha) \cdot r - kl^2\theta_0 - kl^2\theta = I\alpha$$

Employing Eq. (i) and rearranging,

$$(I + mr^2)\,\alpha + kl^2\theta = 0$$

or

$$(I + mr^2)\frac{d^2\theta}{dt^2} + kl^2\theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{kl^2}{I + mr^2}\theta = 0$$

from which the cyclic frequency f is obtained as

$$f = \frac{1}{2\pi}\sqrt{\frac{kl^2}{I + mr^2}}$$

***Example* 11.11** A uniform horizontal plank is resting symmetrically on two counter rotating drums, i.e., with equal and opposite angular velocities as shown in Fig. Ex. 11.11. Show that the plank performs simple harmonic motion, if displaced

The *simple pendulum* consists of a concentrated mass *m* at end of an inextensible string, of negligible mass, the upper end of which is tied to a rigid support. The initial disturbance is given by displacing the bob through a small angle θ and then releasing it to perform a periodic motion.

The *compound pendulum* consists of a rigid body that oscillates about a horizontal axis through the body at some point other than the centre of mass. The moment of inertia of the entire rigid body comes into play to establish the frequency of the periodic motion.

The *torsional pendulum* consists of a rigid body suspended by a vertical elastic shaft which when twisted develops a restoring moment. The rigid body is generally a horizontal circular disc or a spherical bob.

There are other pendulums, such as the conical pendulum, 'Foucault's pendulum', double and multiple-bob pendulums.

(a) Simple Pendulum

A simple pendulum, shown in Fig. 11.13(a), consisting of a massless and inextensible string of length *l* supporting a bob of mass *m* generates a free harmonic motion under the action of the following factors:

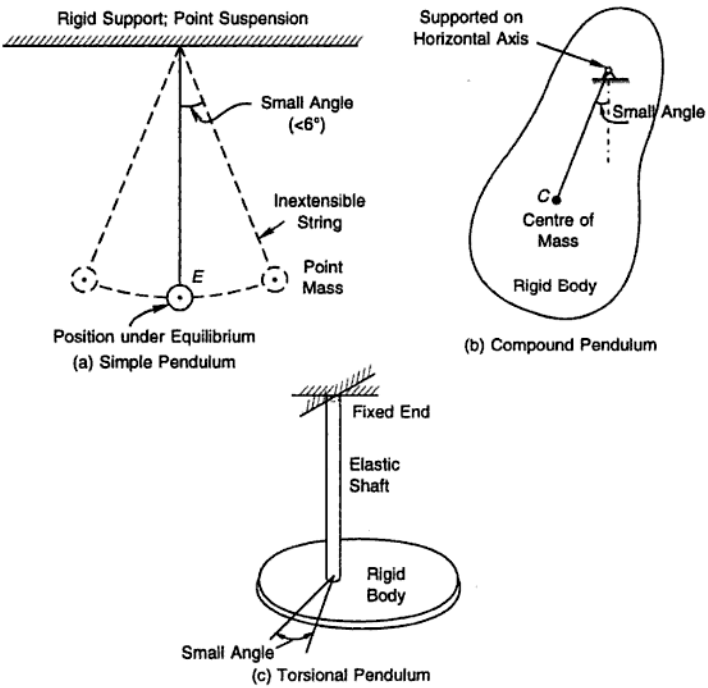


Fig. 11.13 ***Types of Pendulums***

4. The restoring force F acting on the bob equals $mg \sin \theta$ which, for small θ, is

$$F \approx mg\theta$$

implying that F varies linearly with θ. This fact can be appreciated with reference to Fig. 11.15 where the variation of the restoring force is shown as a function of θ.

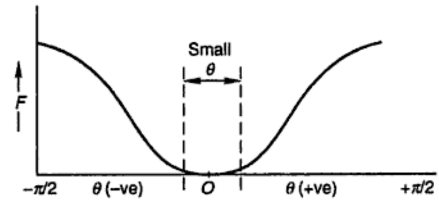


Fig. 11.15 ***Variation of the Restoring Force***

5. The frequency and time period of the motion are independent of the mass of the bob. The tension in the string is obviously proportional to the mass of the bob.
6. The frequency and time period are independent of the amplitude θ of motion. Their dependence upon the length of the string l and the acceleration due to gravity g is employed to determine the value of g.
7. If there were no resisting forces, the periodic motion would go on for ever but, in practice, there are resisting forces such as aerodynamic force on the bob which damp the motion to bring it to rest eventually.
8. The surface of the earth has been considered to provide an inertial frame. This is, however, untrue because the earth is spinning about its own axis. If the spinning of the earth is taken into account, the plane of oscillations of the pendulum precesses with time. Such precessional motion is ignored for a simple pendulum.

***Example* 11.12** Determine the following parameters for a simple pendulum at the mean surface of the earth:

(a) length of a 1-second pendulum

(b) period of a 1-metre pendulum.

If these pendulums are taken to the mean surface of the moon where $g = 1.67$ m/s^2, determine the corresponding periods.

Solution The value of g at the mean surface of the earth is 9.81 m/s^2. Employing the relation,

$$\tau = 2\pi \sqrt{\frac{l}{g}}$$

(a) For a 1-second pendulum,

$$l = \frac{1}{4\pi^2} \times 9.81 = 0.248 \text{ m}$$

(b) For a 1-metre pendulum,

$$\tau = 2\pi\sqrt{\frac{1}{9.81}} = 2.01 \text{ s}$$

At the surface of the moon where $g = 1.67$ m/s^2, the time periods of the two pendulums are

$$\tau_i = 2\pi\sqrt{\frac{0.248}{1.67}} = 2.42 \text{ s}$$

$$\tau_{ii} = 2\pi\sqrt{\frac{1}{1.67}} = 4.86 \text{ s}$$

The time period on the moon is indeed in the ratio of

$$\sqrt{\frac{g_{\text{earth}}}{g_{\text{moon}}}} = 2.42$$

(b) Compound Pendulum

A compound pendulum supported about a horizontal axis at a distance r above the centre of mass C, oscillating to produce a periodic motion, is shown in Fig. 11.16(a).

The restoring moment due to the weight of the pendulum about the pivot O is given by

$$M = (-mg \sin \theta)r$$

considering the moment to be positive along θ increasing.

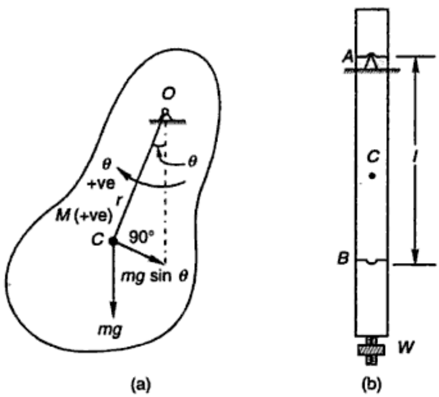


Fig. 11.16 ***(a) Analysis of Compound Pendulum***
(b) Kater Pendulum

By Euler's law, the equation of motion is

$$M = I\alpha = I\frac{d^2\theta}{dt^2}$$

where I is the moment of inertia about the pivotal axis.

It follows that

$$-mgr\sin\theta = I\frac{d^2\theta}{dt^2}$$

which, for small θ, becomes

$$\frac{d^2\theta}{dt^2} + \frac{mgr}{I}\theta = 0 \qquad (11.18)$$

The most general solution to the second order linear differential equation is given by

$$\theta = A\sin\omega_n t + B\cos\omega_n t$$

whence, $$\frac{d\theta}{dt} = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t$$

and $$\frac{d^2\theta}{dt^2} = -\omega_n^2\,(A\sin\omega_n t + B\cos\omega_n t) = -\omega_n^2\theta$$

Substituting the value of θ and $\frac{d^2\theta}{dt^2}$ in the differential equation

$$-\omega_n^2\theta + \frac{mgr}{I}\theta = 0$$

whence, $$\omega_n = \sqrt{\frac{mgr}{I}} \qquad (11.19)$$

The cyclic frequency is

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgr}{I}} \qquad (11.20)$$

and the time period is

$$\tau = \frac{1}{f} = 2\pi\sqrt{\frac{I}{mgr}} \qquad (11.21)$$

It is rather interesting to observe that the simple pendulum is a 'simple' case of the compound pendulum, i.e., when the mass is concentrated at the centre of mass and the distance r is large in comparison with the dimensions of the mass.

The moment of inertia about the pivotal axis is

$$I = m(k^2 + r^2)$$

where k is the radius of gyration.

For a simple pendulum $r >> k$

hence, $I = m r^2$

Substituting this value of I in the expression for the time period

$$\tau = 2\pi\sqrt{\frac{mr^2}{mgr}} = 2\pi\sqrt{\frac{r}{g}}$$

Recognising that r is essentially the length of the thread.

$$\tau = 2\pi\sqrt{\frac{l}{g}}$$

which is the expression desired for the simple pendulum.

An adaptation of the compound pendulum is the *Kater pendulum*. It consists of a bar with two knife-edges A and B, as shown in Fig. 11.15(b), such that they are not obviously at the same distance from the mass centre C. The distance l between them can be measured with great precision. It is the accuracy and certainty with which this distance can be measured that makes it superior to the compound pendulum where the mass centre cannot be ascertained easily and the simple pendulum whose string cannot be weightless and inelastic. There is a counterweight mg in the Kater pendulum which can be adjusted so that the period of oscillation τ is the same when a either knife-edge is used. The relation

$$\tau = 2\pi\sqrt{\frac{I}{mgr}} = 2\pi\sqrt{\frac{l}{g}}$$

applies to the Kater pendulum and the results obtained are more accurate than with either the simple or compound pendulum.

***Example* 11.13** A rod of length L and mass M is pivoted at one end to constitute a pendulum. Determine its period of oscillation and calculate its length if the period is desired to be 1 second.

If the rod was instead suspended from a point at one quarter of its length, what would be the expression for the period and what would be the length of the rod required for a period of 1 second?

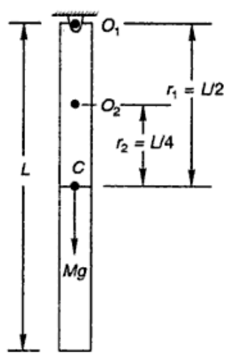


Fig. Ex. 11.13

Solution For the first case of suspension from the pivot O_1,

$$r = \frac{L}{2}$$

$$I = ML^2/12 + M(L/2)^2 = ML^2/3$$

and
$$\tau = 2\pi\sqrt{\frac{ML^2/3}{Mg\,L/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

For $\tau = 1$ s, $L = 3g/8\pi^2 = 0.373$ m

For the second case of suspension from the pivot O_2,

$$r = L/4$$

$$I = ML^2/12 + M(L/4)^2 = 7/48\ ML^2$$

and $$\tau = 2\pi\sqrt{\frac{7/48\ ML^2}{MgL/4}} = 2\pi\sqrt{\frac{7L}{12g}}$$

For $\tau = 1$ s, $$L = \frac{12g}{28\pi^2} = 0.426\text{ m}$$

An extension of this example leads us to an interesting question: What will τ be if the rod is suspended at the centre of mass C?

If the pivot is located at the centre of mass,

$$\tau = 2\pi\sqrt{\left(\frac{I}{mg \times 0}\right)} \rightarrow \infty$$

which shows that the periodic motion is not possible in this case; obviously because the restoring moment disappears as the weight of the body acts at the pivot itself.

***Example* 11.14** A disc of radius 10 cm is suspended from a point on its circumference. Determine its frequency of oscillation.

Solution (Referring to Fig. Ex. 11.4) The moment of inertia about the pivotal axis is

$$I = M\frac{r^2}{2} + Mr^2 = \frac{3}{2}Mr^2$$

$$= \frac{3}{2}M \times 0.1^2 = 0.015M\text{ kg m}^2$$

where M is the mass of the disc.

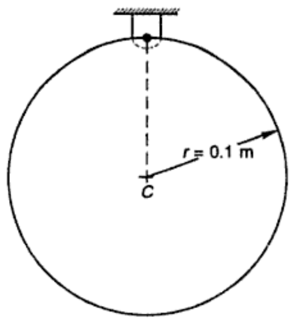


Fig. Ex. 11.14

The frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{M \times 9.81 \times 0.1}{0.015M}} = 1.287\text{ s}^{-1}$$

and the time period

$$\tau = \frac{1}{f} = \frac{1}{1.287} = 0.777\text{ s}$$

Experiment E15

Oscillation of a Compound Pendulum

OBJECTIVE

To determine the time period of oscillation of a compound pendulum and to estimate the value of g, the acceleration due to gravity.

APPARATUS

A bar-shaped compound pendulum with provision to suspend it from over a knife edge at different points along its length as shown in Fig. E15.1. A beam compass, a stop watch and a metre rod.

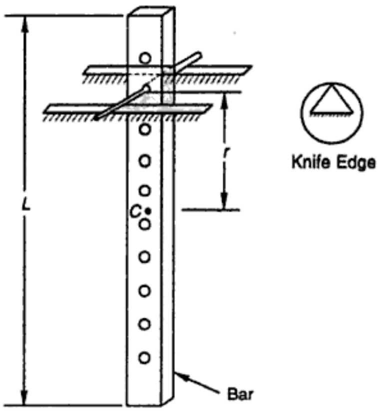


Fig. E15.1 ***Compound Pendulum***

BACKGROUND INFORMATION

From the equation of motion of a compound pendulum,

$$\frac{d^2\theta}{dt^2} + \frac{mgr}{I}\theta = 0 \tag{E15.1}$$

The time period is given by

$$\tau = 2\pi\sqrt{\frac{I}{mgr}}$$

$$= 2\pi\sqrt{\frac{mk^2 + mr^2}{mgr}}$$

$$= 2\pi\sqrt{\frac{(k^2 + r^2)/r}{g}} \tag{E15.2}$$

Comparing of this expression with the expression for the time period of a simple pendulum, i.e.,

$$\tau = 2\pi\sqrt{\frac{l}{g}}$$

shows that the length of an equivalent simple pendulum

$$l = \frac{k^2 + r^2}{r}$$

which, for a straight uniform bar becomes

$$l = \frac{L^2/12 + r^2}{r}$$

where L is the length of the bar.

From Eq. (E15.2), which is a quadratic equation in r,

$$r^2 - \frac{2}{4\pi^2}gr + k^2 = 0$$

it is noticed that there must be two values of r for the same time period τ of the pendulum such that

$$\frac{k^2 + r_1^2}{r_1} = \frac{k^2 + r_2^2}{r_2}$$

whence

$$r_1 r_2 = k^2 \tag{E15.3}$$

Since k^2 must be a positive quantity, it follows that r_1 and r_2 must have the same sign, i.e., the two points of suspension at a distance r_1 and r_2 from the centre of mass must be on the same side of it. Similarly, there must be two points of suspension on the other side of the centre of mass which yield the same time period. In all, there are four points of suspension on a bar pendulum which provide the same time period.

The value of g, the acceleration due to gravity may be obtained from a single measurement of the time period for a known r and by applying Eq. (E15.2). Averaging on the basis of a number of isolated readings would be an improvement over the method. Better still is to draw a curve of the time period vs. the distance of the point of suspension from the centre of mass on one side or on both sides of it. Averaging for the value of g on the basis of pairs of values of the distances on either side of the centre of mass for the same time period has the added advantage of using the faired curves drawn on the basis of all the measurements.

$$r_1 = OA,\ r_1' = OA';\ \text{average } r_1 = (OA + OA')/2$$

$$r_2 = OB,\ r_2' = OB';\ \text{average } r_2 = (OB + OB')/2$$

and
$$k^2 = r_1 r_2$$

From Eq. (E15.2),

$$g = \frac{4\pi^2}{\tau^2} \cdot \frac{k^2 + r^2}{r}$$

two estimates of g are obtained for the average r_1 and r_2 and k^2 as above and τ recorded from the plot as CO. Similarly, estimates of g are obtained from the set of points P, Q, P' and Q', and so on and an average value of g is determined.

RESULTS AND POINTS FOR DISCUSSION

1. A value of g can as well be obtained from each observation of r and τ by employing Eq. (E15.2). What is the advantage of the method adopted by you over this procedure?
2. Theoretically, for a uniform bar,

   $$k^2 = L^2/12$$

   while it is actually,

   $$k^2 = r_1 r_2$$

   Which is a closer to the true radius of gyration squared and why?
3. Determine the length of an equivalent simple pendulum for the compound pendulum and state why it must be less than the actual length of the pendulum below the point of suspension.
4. Which gives a more accurate value of the acceleration due to gravity, a simple pendulum or a compound pendulum?
5. What is the mean value of g over the surface of the earth and what is the value of g at the location of the experiment? Do the values compare with the estimate of g made by you?
6. Note that the moment of inertia of a rigid body about an axis may be determined by suspending the body about that axis and by noting the time period of oscillations by using Eq. (E16.2), if the acceleration due to gravity is known. Would you recommend this method of finding the moment of inertia of a rigid body?
7. What is the time period of oscillation of a bar when suspended from its centre of mass?
8. What is the minimum time period of the bar? Why can it not be less than this value?

(c) Torsional Pendulum

A torsional pendulum consisting of a rigid body attached to an elastic shaft operates by virtue of the restoring moment due to the elastic deformation of the shaft as shown in Fig. 11.17.

The restoring moment exerted by the shaft on the rigid body on a displacement θ is given by

$$M = -\frac{\pi r^4 G}{2L}\theta$$

The moment of inertia about the axis of the shaft is

$$I = \frac{mr^2}{2} = \frac{16.67}{2} \times (0.15)^2 = 0.187 \text{ kg m}^2$$

The angular frequency is

$$\omega_n = \sqrt{\frac{\pi r^4 G}{2LI}} = \sqrt{\pi \times \left(\frac{0.01}{2}\right)^4 \times 35 \times 10^9 /(2 \times 1 \times 0.187)}$$

$$= 13.6 \text{ s}^{-1}$$

The cyclic frequency f is given by

$$f = \frac{\omega_n}{2\pi} = \frac{13.6}{2\pi} = 2.16 \text{ s}^{-1}$$

The time period of oscillations is

$$\tau = \frac{1}{f} = \frac{1}{2.16} = 0.46 \text{ s}$$

(d) Conical Pendulum

A conical pendulum consists of a mass m suspended by an inextensible string of length l from a point O fixed on a vertical rotating rod as shown in Fig. 11.18. The equation of motion of the mass m can be written as

$$\mathbf{F} = m\,\mathbf{a}$$

where $\mathbf{F}$, the net external force equals

$$-mg\,\mathbf{k} - T\sin\theta\,\mathbf{i} + T\cos\theta\,\mathbf{k}$$

$\mathbf{i}$ and $\mathbf{k}$ being unit vectors as shown in Fig. 11.18.

The acceleration of the mass in rotation at a constant angular velocity is purely centripetal

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \omega\,\mathbf{k} \times (\omega\,\mathbf{k} \times l\sin\theta\,\mathbf{i}) = -\omega^2 l \sin\theta\,\mathbf{i}$$

Hence

$$-mg\,\mathbf{k} - T\sin\theta\,\mathbf{i} + T\cos\theta\,\mathbf{k} = -m\omega^2 l\sin\theta\,\mathbf{i}$$

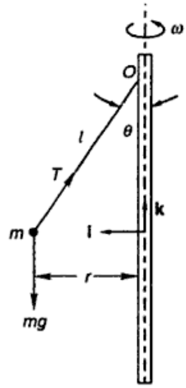


Fig. 11.18 ***Conical Pendulum***

whence, the two scalar equations appear as

$$-mg + T\cos\theta = 0$$

or $$T\cos\theta = mg$$

and $$T\sin\theta = m\omega^2 l\sin\theta$$

It follows that

$$T = m\omega^2 l \tag{11.26}$$

and $$\upsilon = \cos^{-1}(g/\omega^2 l) \tag{11.27}$$

The mass m describes a circle of radius r whereas the string and mass taken together describe the surface of a cone. The system has, therefore, acquired the name 'conical pendulum'.

(e) Foucault's Pendulum

The Foucault's pendulum is essentially a simple pendulum with a long string. A string of over 10 m length allows the bob to oscillate with an amplitude of 1 m with a small angular amplitude. The bob is capable of oscillation over a long period of time, say 24 hours or more. *A fact which stands out is that the plane of oscillation of a simple pendulum should precess about the vertical axis due to the spinning of the earth* as shown in Fig. 11.19. The period of precession would, of course, be different at different locations on the earth. The fact that the earth spins about its own axis is borne out by the experimental observations on the Foucault's pendulum. One such pendulum is installed at the entrance of the Science Museum, London. The pendulum is 'let go' in a plane motion in the mornings and, as the time passes, the plane of oscillations changes continuously with time, thus demonstrating the spinning of the earth about its own axis, day in and day out.

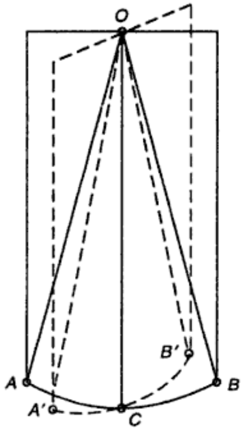


Fig. 11.19 ***Shift of Vertical Plane of Oscillation from OACB to OA'CB' with Time of Day due to Spinning of the Earth***

(f) Double Pendulum

A double pendulum consists of a mass m_1 suspended on a string of length l_1 from a fixed support and another mass m_2 suspended on a string of length l_2 from the first mass, as shown in Fig. 11.20. The mass m_2 can oscillate about the centre of mass m_1 whereas the mass m_1 can oscillate about the fixed point O. For a very special case when

$$m_1 = m_2 = m$$

$$l_1 = l_2 = l$$

and $\theta = \sin\theta$, $\cos\theta = 1$ for small angles θ_1 and θ_2.

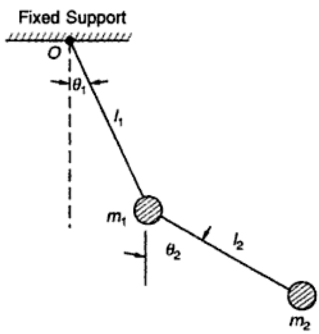


Fig. 11.20 ***Double Pendulum***

The equations of motion are

$$2l\frac{d^2\theta_1}{dt^2} + l\frac{d^2\theta_2}{dt^2} = -2g\theta_1 \qquad (11.28)$$

(iii) if and only if the system undergoes simple harmonic motion
(iv) if the initial state as well as the disturbed states are in equilibrium.

(c) The resisting force in an oscillatory system tends to
(i) reduce the time period
(ii) oppose the restoring force proportionately
(iii) reduce the amplitude
(iv) reduce the amplitude with time

6. Differentiate critically between
(a) simple pendulum and compound pendulum
(b) damping forces and driving forces
(c) angular frequency and cyclic frequency.
7. Make simple sketches to illustrate three oscillating systems
(a) capable of linear oscillations
(b) capable of angular oscillations.
8. Can every oscillating system be reduced to an equivalent spring-mass-damper system? Establish the equivalence of any one linearly and one angularly oscillating system.
9. Ill-effects of the resonance of a system can be avoided
(a) by keeping the resonating frequency as low as possible
(b) by making the frequency of the driving force as low as possible
(c) by passing the system through the state of resonance as quickly as possible.
10. Why is it that the equation of motion of a simple pendulum can be obtained either by Newton's law or by Euler's law or by the principle of conservation of mechanical energy, whereas the equation of motion for a compound pendulum cannot be obtained by the application of Newton's law alone?
11. Assuming that $g = \dfrac{GM}{(R+h)^2}$

where G is the gravitational constant, M is the mass of the earth, h is the vertical distance from the surface of the earth and R is the radius of the earth, establish a relationship between h and the length of a simple pendulum of period 1 second at all altitudes.

Tutorial Problems

11.1 A small ball of mass m is fixed at the mid-length of a taut wire of length l with tension P in it as shown in Fig. Prob. 11.1. Show that the ball executes a simple harmonic motion for small displacements. Also calculate the time period of the ball.

$$\left(\textbf{Ans. } \tau = 2\pi\sqrt{\frac{ml}{4P}}\right)$$

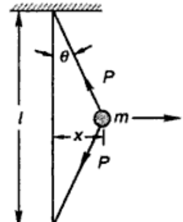


Fig. Prob. 11.1

11.2 A solid aluminium sphere with 50 cm diameter is attached to the lower end of 10 m long aluminium rod of 5 cm diameter, the upper end of which is fixed. Find the period of this pendulum. Take $G = 24 \times 10^9$ N/m^2 and $\rho = 27{,}000$ kg/m^3 for aluminium. (**Ans.** 0.12 s)

11.3 A block of steel of mass 50 kg is supported by an alternative spring arrangements as shown in Figs. Prob. 11.3(a) and (b) and (c). Determine the natural frequency of the block in vertical motion. Estimate also

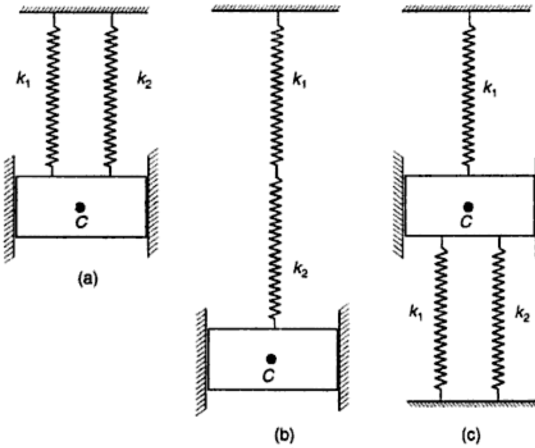


Fig Prob. 11.3

the maximum velocity and acceleration of the block if the amplitude is 5 cm. Take k_1 = 40 N/cm and k_2 = 50 N/cm.

(**Ans.** (a) 0.214 s^{-1}, 0.067 m/*s*, 0.09 m/s^2
(b) 0.106 s^{-1}, 0.0334 m/s, 0.0222 m/s^2
(c) 0.257 s^{-1}, 0.08 m/s, 0.13 m/s^2)

11.4 A vertical *U*-tube manometer containing mercury for a total length of 0.3 m is subjected to a sudden pressure differential which is removed and the columns are set to vibrate. Determine the frequency of vibration and comment on its value if

(a) the size of the manometer is changed to a tube of half its diameter and a length half as much and if
(b) the mercury is replaced by water
(c) the *U*-tube is inclined with its plane making an angle of 30° with the horizontal.

(**Ans.** f = 1.29 Hz; (a) no change with diameter but f = 1.82 Hz for half length (b) no change, (c) 0.912 Hz)

11.5 A semicircular cylinder of radius r and mass density ρ rests in the equilibrium position on its curved surface on a flat plate. Establish that the semi-cylinder is capable of oscillating and determine its period of oscillation.

$$\left(\textbf{Ans. } \tau = 2\pi\sqrt{\frac{(9\pi-16)r}{2g}}\right)$$

11.6 A mass m attached to a rigid rod which is held in position by two horizontal springs is made to vibrate in the plane of the springs (Fig. Prob. 11.6). What condition must be satisfied for equilibrium in the vertical position? Obtain the equation of motion and express the natural frequency of the system in terms of the given parameters.

When will the period of oscillations be infinite?

$$\left(\textbf{Ans. } l_1^2 = \frac{mgl_2}{2k}\,;\ \tau = 2\pi \Big/ \sqrt{\frac{2kl^2}{ml_2^2} - \frac{g}{l_2}}\right)$$

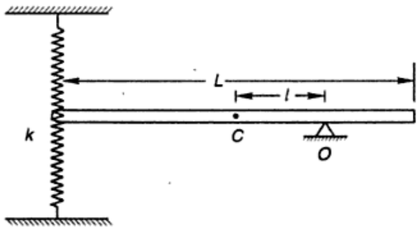


Fig. Prob. 11.8

11.9 A particle P of mass m oscillates along a vertical line in a system of springs shown in Fig. Prob. 11.9. Determine the frequency of oscillation assuming all other members as massless elements.

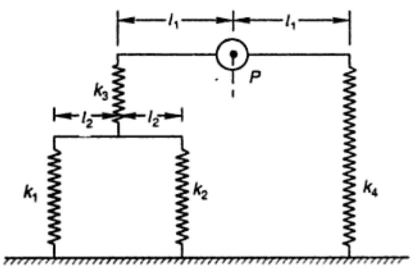


Fig. Prob. 11.9

$$\left(\textbf{Ans. } \omega^2 = \frac{4}{m\left(\frac{1}{4k_1} + \frac{1}{4k_2} + \frac{1}{k_3} + \frac{1}{k_4}\right)}\right)$$

11.10 A circular disc of diameter d and mass m is free to roll without sliding on a horizontal plane. Two identical springs of stiffness k are attached to as shown in Fig. Prob. 11.10. Determine the period of small oscillations of the disc.

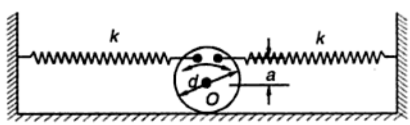


Fig. Prob. 11.10

$$\left(\textbf{Ans. } f = \frac{1}{\pi}\sqrt{\frac{6k\,(L/2 + l)^2}{m\,(L^2 + 12l^2)}}\right)$$

11.11 Find the period for small oscillations if a rod of length L and mass M is suspended from a point $L/4$ from one end.

$$\left(\textbf{Ans. } 2\sqrt{\frac{7L}{24g}}\right)$$

11.12 A pulley having a moment of inertia I about its axis of rotation supports a rope which carries a mass m at one end, while the other end is connected to a spring of spring constant k as shown in Fig. Prob. 11.12. Find the period of oscillation of the system. Assume that the rope does not slip on the pulley.

$$\left(\text{Ans. } 2\pi\left(\frac{I + mR^2}{kR^2}\right)\right)$$

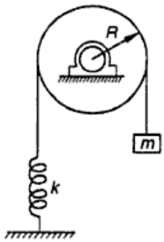


Fig. Prob. 11.12

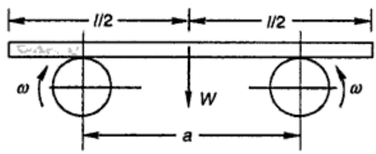


Fig. Prob. 11.13

11.13 Two parallel cylindrical rollers rotate in opposite directions as shown in Fig. Prob. 11.13. The distance between the centres of the rollers is a. A straight, uniform horizontal rod of length l and weight W rests on top of the rollers. The coefficient of kinetic friction between the rod and the roller is μ. Taking x as the distance from the centre of the rod to the midpoint between the rolls, write the equation of motion of the rod, assuming that it has been initially displaced from the central position. Find the frequency of the resulting vibratory motion.

(**Ans.** $\sqrt{2\mu g/a}$)

Look up Hints to Tutorial Problems!

Multiple-Choice Questions

Select the correct or the most appropriate response from among the available alternatives in the following multiple-choice questions:

1. A linear response implies that
   (a) the elements are in one line
   (b) the response is along the given line
   (c) the sensitivity is constant
   (d) the response is not exponential
2. A vibrating system
   (a) does not pass through an equilibrium position
   (b) passes through the equilibrium position once every cycle
   (c) passes through the equilibrium position twice every cycle
   (d) starts from an equilibrium position and does not return to it

3. Resonance occurs when
   (a) a freely vibrating system is made to vibrate at increasingly higher frequencies
   (b) the forcing frequency equals the natural frequency of the system
   (c) the system vibrates at its natural frequency
   (d) the amplitude of vibrations exceeds twice the amplitude of free vibrations

Answers to Multiple-Choice Questions

1 (c), 2 (a), 3 (b)

A
APPENDICES

This section consists of appendices with information useful for the study of mechanics:

APPENDIX 1

Rules for Differentiation and Integration

Some rules of differentiation and integration commonly referred to by the science and engineering students are listed as follows:

DIFFERENTATION

$y = f(x)$	$y' = \frac{d}{dx} f(x)$	$y = f(x)$	$y' = \frac{d}{dx} f(x)$
a	0	$\sin^{-1} x$	$(1 - x^2)^{-1/2}$
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$	$\cos^{-1} x$	$-(1 - x^2)^{-1/2}$
		$\tan^{-1} x$	$(1 + x^2)^{-1}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\cot^{-1} x$	$-(1 + x^2)^{-1}$
		$\sec^{-1} x$	$x^{-1}(x^2 - 1)^{-1/2}$
u^n	$nu^{n-1}\frac{du}{dx}$	$\text{cosec}^{-1} x$	$-x^{-1}(x^2 - 1)^{-1/2}$
$f(u)$	$\frac{d}{du} f(u) \frac{du}{dx}$	$\sinh^{-1} x$	$(x^2 + 1)^{-1/2}$
		$\cosh^{-1} x$	$(x^2 - 1)^{-1/2}$
e^x	e^x	$\tanh^{-1} x$	$(1 - x^2)^{-1}$
u^r	$vu^{r-1}\frac{du}{dx} + u^r \frac{dv}{dx} \log_e u$	$\coth^{-1} x$	$-(x^2 - 1)^{-1}$
		$\text{sech}^{-1} x$	$x(1 - x^2)^{-1/2}$
$\log_e x$	$\frac{1}{x}$	cosech^{-1}	$-x^{-1}(x^2 + 1)^{-1/2}$
$\log_{10} x$	$\frac{1}{x}\log_{10} e$		

$f(\theta)$	$f' = \frac{d}{d\theta} f(\theta)$	$f(\theta)$	$f' = \frac{d}{d\theta} f(\theta)$
$\sin\theta$	$\cos\theta$	$\sinh\theta$	$\cosh\theta$
$\cos\theta$	$-\sin\theta$	$\cosh\theta$	$\sinh\theta$
$\tan\theta$	$\sec^2\theta$	$\tanh\theta$	$\text{sech}^2\theta$
$\cot\theta$	$-\text{cosec}^2\theta$	$\coth\theta$	$-\text{cosech}^2\theta$
$\sec\theta$	$\tan\theta\sec\theta$	$\text{sech}\,\theta$	$-\text{sech}\,\theta\tanh\theta$
$\text{cosec}\,\theta$	$-\cot\theta\,\text{cosec}\,\theta$	$\text{cosech}\,\theta$	$-\text{cosech}\,\theta\coth\theta$

For a curve $y = f(x)$,

$y' = 0$ corresponds to a stationary or extremum point and

$$y'' \begin{cases} < 0 & \text{for maxima} \\ = 0 & \text{for inflexion} \\ > 0 & \text{for minima} \end{cases}$$

$$\text{Radius of curvature} = \frac{1}{\text{Curvature}} = \left|\frac{ds}{d\theta}\right| = \left|\frac{(1 + y'^2)^{3/2}}{y''}\right|$$

$$= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Partial differentiation, $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{\delta x}{\delta t} + \dfrac{\partial f}{\partial y}\dfrac{\delta y}{\delta t}$ for $f = f(x, y)$

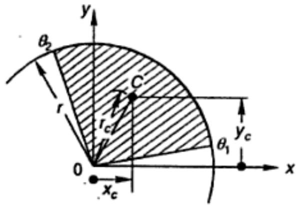
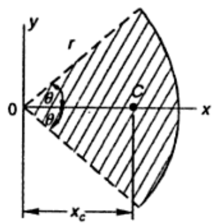
The nth derivative, $D^n(uv) = D^n u\, v + {}^nC_1\, D^{n-1}u\, Dv + \ldots + {}^nCr\, D^{n-r}u.\, D^r v + \ldots + uD^n v$

INTERGRATION

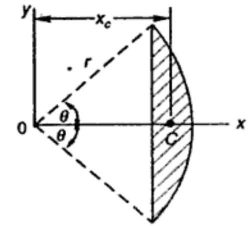
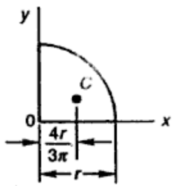
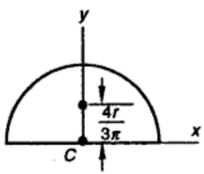
$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
a	ax	$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\dfrac{x}{a}$
ax^n	$\dfrac{ax^{n+1}}{n+1}$	$a^2 + x^2$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
$\dfrac{1}{x}$	$\log_e \lvert x \rvert$	$\dfrac{1}{\sqrt{x^2 \pm a^2}}$	$\log_e\left(\dfrac{x}{a} + \sqrt{\dfrac{x^2}{a^2} \pm 1}\right)$
e^{ax}	$\dfrac{e^{ax}}{a}$	$\sec^2\theta$	$\tan\theta$
		$\sec\theta\tan\theta$	$\sec\theta$
a^x	$\dfrac{a^x}{\log_e a}$	$\sec\theta$	$\log^e(\sec\theta + \tan\theta)$
$\sin a\theta$	$-\dfrac{\cos a\theta}{a}$	$\text{cosec}\,\theta$	$\log^e(\text{cosec}\,\theta - \cot\theta)$
		$\cot\theta$	$\log^e \lvert\sin\theta\rvert$
		$\text{cosec}\,\theta\cot\theta$	$-\text{cosec}\,\theta$
$\cos a\theta$	$\dfrac{\sin a\theta}{a}$		

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\tan a\theta$	$\dfrac{\log_e \sec a\theta}{a}$	uv	$u\int v\,dx - \int\left(\dfrac{du}{dx}\int v\,dx\right)dx$
$\log_e x$	$x \log_e x - x$	$\sin^2\theta$	$-(\sin\theta\cos\theta - \theta)/2$
$\sqrt{x^2 + a^2}$	$\dfrac{x}{2}\sqrt{x^2 + a^2} - \dfrac{a^2}{2}$	$\cos^2\theta$	$(\sin\theta\cos\theta + \theta)/2$
	$\times \left\lvert \log_e \dfrac{x + \sqrt{x^2 + a^2}}{a} \right\rvert$	$f(ax + b)$	$\dfrac{\int f(x)dx}{a}$
$\sqrt{x^2 - a^2}$	$\dfrac{x\sqrt{x^2 - a^2}}{2} - \dfrac{a^2}{2}$	$\dfrac{1}{\sqrt{x^2 - a^2}}$	$\text{Cosh}^{-1}\left(\dfrac{x}{a}\right)$
	$\times \left(\log \dfrac{x + \sqrt{x^2 - a^2}}{a}\right)$		
$\sqrt{a^2 - x^2}$	$\dfrac{x\sqrt{a^2 - x^2}}{2} + \dfrac{a^2}{2}$	$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\text{Sin}^{-1}\left(\dfrac{x}{a}\right)$
	$\times \sin^{-1}\left(\dfrac{x}{a}\right)$		
$\dfrac{1}{a^3 - x^2}$	$\dfrac{1}{2a}\log\left\lvert\dfrac{a + x}{a - x}\right\rvert$	$\dfrac{1}{\sqrt{x^2 + a^2}}$	$\text{Sinh}^{-1}\left(\dfrac{x}{a}\right)$

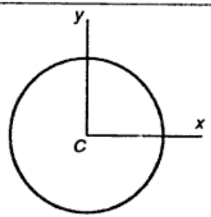
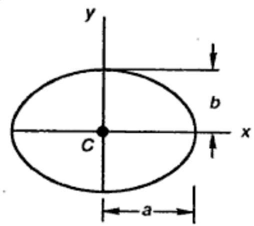
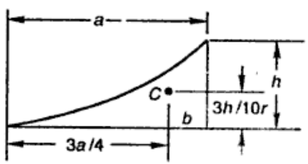
Appendix 2 (*Contd.*) Properties of Plane Figures

Figure	*Description*	*Area*	*Centroid*	*Area Moment of Inertia*
	Circular sector	$A = r^2(\theta_2 - \theta_1)/2$	$x_c = \dfrac{2r(\sin\theta_2 - \sin\theta_1)}{3(\theta_2 - \theta_1)}$ $y_c = \dfrac{-2r(\cos\theta_2 - \cos\theta_1)}{3(\theta_2 - \theta_1)}$	$I_{xx} = (r^4/8)[(\theta_2 - \theta_1) - \sin(\theta_2 - \theta_1)\cos(\theta_2 + \theta_1)]$ $I_{yy} = (r^4/8)[(\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1)\cos(\theta_2 + \theta_1)]$ $I_{zz} = (r^4/4)(\theta_2 - \theta_1)$ $I_{xy} = (r^4/8)(\sin^2\theta_2 - \sin^2\theta_1)$
	Circular sector	$A = r^2\theta$	1. General circular sector $x_c = \dfrac{2r\sin\theta}{3\theta}$ $y_c = 0$ 2. Quarter circle, $\theta = \dfrac{\pi}{4}$ $x_c = \dfrac{4r\sqrt{2}}{3\pi}$ $y_c = 0$ 3. Semicircle, $\theta = \dfrac{\pi}{2}$ $x_c = \dfrac{4r}{3\pi}$ $y_c = 0$	 $I_{xx} = I_{yy} = \pi r^4/16$ $I_{zz} = \pi r^4/8$ $I_{xx} = I_{yy} = \pi r^4/8$ $I_{zz} = \pi r^4/4$

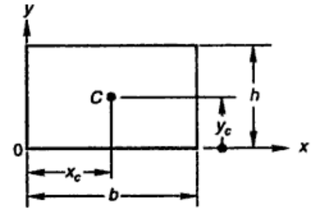
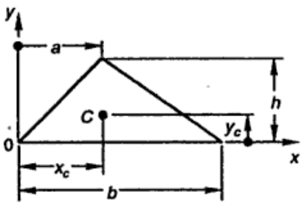
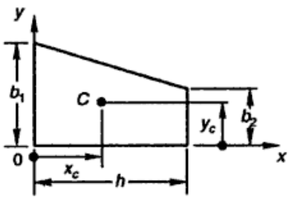
Appendix 2 (*Contd.*) Properties of Plane Figures

Figure	*Description*	*Area*	*Centroid*	*Area Moment of Inertia*
	Circular segment	$A = \frac{r^2}{2}(2\theta - \sin 2\theta)$	$x_c = \frac{4a \sin^3 \theta}{3(2\theta - \sin 2\theta)}$ $y_c = 0$	—
	Quarter circle	$A = \frac{\pi r^2}{4}$	$I_{xx} = I_{yy} = \frac{\pi r^2}{16}$ $I_{zz} = \frac{\pi r^4}{8}$	—
	Semicircle	$A = \frac{\pi r^2}{2}$	$I_{xx} = I_{yy} = \frac{\pi r^4}{8}$ $I_{zz} = \frac{\pi r^4}{4}$	—

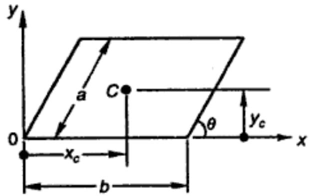
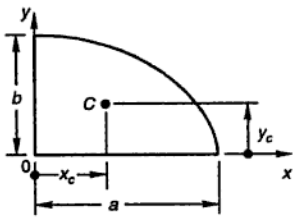
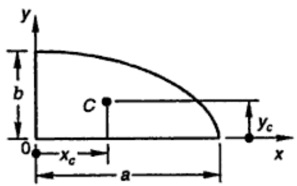
Appendix 2 (*Contd.*) Properties of Plane Figures

Figure	*Description*	*Area*	*Centroid*	*Area Moment of Inertia*
	Circle	$A = \pi r^2$	$I_{xx} = I_{yy} = \frac{\pi r^4}{4}$ $I_{zz} = \frac{\pi r^4}{2}$	—
	Ellipse	$A = \pi ab$	$I_{xx} = \pi ab^3/4$ $I_{yy} = \pi ba^3/4$ $I_{zz} = \frac{\pi ab}{4}(a^2 + b^2)$	—
	Parabolic spandrel	$A = ah/3$	—	—

Appendix 2 (*Contd.*) Properties of Plane Figures

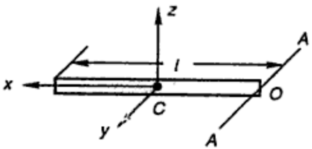
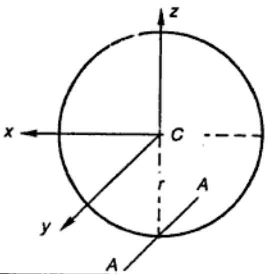
Figure	Description	Area	Centroid	Area Moment of inertia
	Rectangle	$A = bh$	$x_c = b/2$ $y_c = h/2$	$I_{xx} = bh^3/3$ $I_{yy} = hb^3/3$ $I_{xy} = b^2h^2/4$ $I_{xx_c} = bh^3/12$ $I_{yy_c} = hb^3/12$
	Triangle	$A = bh/2$	$x_c = (a + b)/3$ $y_c = h/3$	$I_{xx} = bh^3/12$ $I_{yy} = bh\,(a^2 + ab + b^2)\,/12$ $I_{xy} = bh^2\,(2a + b)\,/24$ $I_{xx_c} = bh^3/36$
	Trapezoid	$A = h(b_1 + b_2)/2$	$x_c = \dfrac{h(b_1 + 2b_2)}{3(b_1 + b_2)}$ $y_c = \dfrac{(b_1^2 + b_1 b_2 + b_2^2)}{3(b_1 + b_2)}$	$I_{xx} = h(b_1^4 - b_2^4)/12(b_1 - b_2)$. $I_{yy} = h^2(b_1 + 3b_2)/12$ $I_{xy} = h^2\,(b_1^2 + 2b_1 b_2 + 3b_2^2)/24$

Appendix 2 (*Contd.*) Properties of Plane Figures

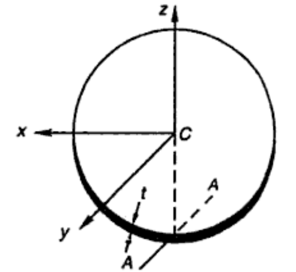
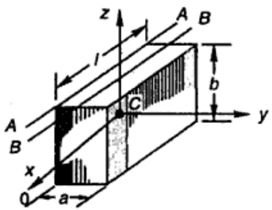
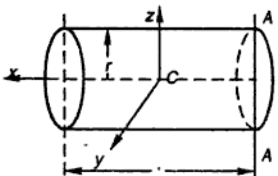
Figure	*Description*	*Area*	*Centroid*	*Area Moment of inertia*
	Parallelogram	$A = ab \sin\theta$	$x_c = (a\cos\theta + b)/2$ $y_c = a\sin\theta/2$	$I_{xx} = a^2 b \sin^3\theta/3$ $I_{yy} = ab(2a^2\cos^2\theta + 3ab\cos\theta + 2b^2)/6 \sin\theta$ $I_{xy} = a^2 b \sin^2\theta\,(4a\cos\theta + 3b)/12$
	Parabola	$A = 2ab/3$	$x_c = 2a/5$ $y_c = 3b/8$	$I_{xx} = 2ab^3/15$ $I_{yy} = 16a^3b/15$ $I_{xy} = 7a^2b^2/60$
	Quarter-ellipse	$A = \pi ab/4$	$x_c = 4a/3\pi$ $y_c = 4b/3\pi$	$I_{xx} = \pi ab^3/16$ $I_{yy} = \pi a^3 b/16$ $I_{xy} = a^2b^2/8$

APPENDIX 3

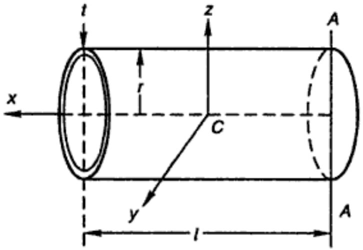
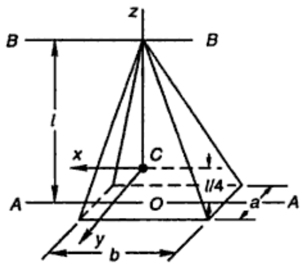
Properties of Solid Bodies

Figure	Description	Mass	Moment of Inertia
	Slender rod	$m = \rho l$	$I_{AA} = m l^2/3$ $I_{yy} = I_{zz} = m l^2/12$ $I_{xx} = 0$
	Thin hoop	$m = 2 \rho \pi r$	$I_{xx} = I_{zz} = m r^2/2$ $I_{yy} = m r^2$ $I_{AA} = 2m r^2$

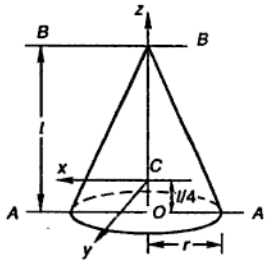
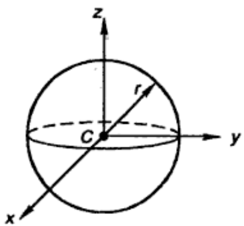
Appendix 3 (*Contd.*) Properties of Solid Bodies

Figure	*Description*	*Mass*	*Moment of Inertia*
	Thin circular disc	$m = \rho \cdot \pi r^2 \cdot t$	$I_{xx} = I_{zz} = m r^2/4$ $I_{yy} = m r^2/2$ $I_{AA} = 3m r^2/2$
	Rectangular parallelopiped (and thin plate)	$m = \rho ab\, l$	$I_{xx} = m(a^2 + b^2)/12$ $I_{yy} = m(b^2 + l^2)/12$ $I_{zz} = m(a^2 + l^2)/12$ $I_{xy} = I_{yz} = I_{xz} = 0$ $I_{AA} = m(a^2 + b^2)/3$ $I_{BB} = m(a^2 + 4b^2)/12$
	Right circular cylinder (and thin circular disc)	$m = \rho\pi r^2 l$	$I_{xx} = m r^2/2$ $I_{yy} = I_{zz} = m(3r^2 + l^2)/12$ $I_{AA} = m(3r^2 + 4l^2)/12$

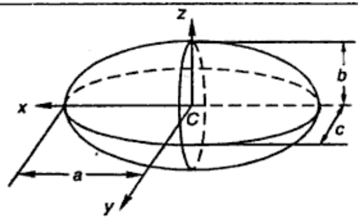
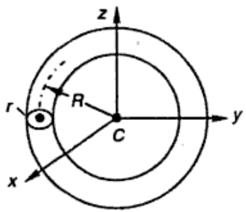
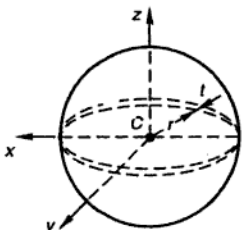
Appendix 3 (*Contd.*) Properties of Solid Bodies

Figure	*Description*	*Mass*	*Moment of Inertia*
	Thin cylindrical shell	$m = 2\pi r t l$	$I_{xx} = m r^2$ $I_{yy} = I_{zz} = m(6r^2 + l^2)/12$ $I_{AA} = m(3r^2 + 2l^2)/6$
	Right rectangular pyramid	$m = \rho abl/3$ $OC = l/4$	$I_{xx} = m(4b^2 + 3l^2)/80$ $I_{yy} = m(4a^2 + 3l^2)/80$ $I_{zz} = m(a^2 + b^2)/20$ $I_{AA} = m(b^2 + 2l^2)/20$ $I_{BB} = m(b^2 + 12l^2)/20$

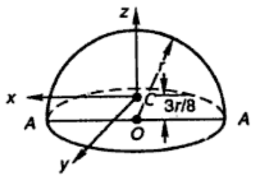
Appendix 3 (*Contd.*) Properties of Solid Bodies

Figure	*Description*	*Mass*	*Moment of inertia*
	Right circular cone (and thin bevelled disc)	$m = \rho\pi r^2 l/3$ $OC = l/4$	$I_{yy} = I_{xx} = 3m(4r^2 + l^2)/80$ $I_{zz} = 3mr^2/10$ $I_{AA} = m(3r^2 + 2l^2)/20$ $I_{BB} = 3m(r^2 + 4l^2)/20$
	Sphere	$m = 4\rho\pi r^3/3$	$I_{xx} = I_{yy} = I_{zz} = 2/5\ m r^2$

Appendix 3 (*Contd.*) Properties of Solid Bodies

Figure	*Description*	*Mass*	*Moment of Inertia*
	Ellipsoid	$m = 4\rho\pi\, abc/3$	$I_{xx} = m(b^2 + c^2)/5$ $I_{yy} = m(a^2 + b^2)/5$ $I_{zz} = m(a^2 + c^2)/5$
	Torus	$m = 2\rho\pi^2 r^2 R$	$I_{xx} = m(4R^2 + 3r^2)/4$ $I_{yy} = I_{zz} = m(4R^2 + 5r^2)/8$
	Spherical shell	$m = \rho \cdot 4\pi r^2 \cdot t$	$I_{xx} = I_{yy} = I_{zz} = 2/3\, m\, r^2$

Appendix 3 (*Contd.*) Properties of Solid Bodies

Figure	*Description*	*Mass*	*Moment of Inertia*
	Hemisphere	$m = \rho \cdot 2/3\pi r^3$ $OC = 3r/8$	$I_{xx} = I_{yy} = 83m r^2/320 \approx 1/4\, mr^2$ $I_{zz} = 2/5\, m r^2$ $I_{AA} = 2/5\, m r^2$

Hints to Tutorial Problems

CHAPTER R2

R2.1 Vector **OP** = [(10 – 3) **i** + (5 + 1) **j** + (8 – 2) **k**]/$(7^2 + 6^2 + 6^2)$.

R2.2 The feasible path is from $P(0, 0)$ to x, y passing through (3, 4) and then to the left horizontally to reach (0, 10). Unit vector along the incline is $\overline{\mathbf{u}} = (3\,\mathbf{i} + 4\,\mathbf{j})/5$. One must go until $y = 10$; hence $x = 7.5$.

Distance along the incline is $\sqrt{10^2 + 7.5^2}$ = 12.5 units.

Total distance travelled = 12.5 + 7.5 = 20 units.

R2.3 $\overline{\mathbf{r}} = (-5 - 0)\,\mathbf{i} + (2 - 0)\,\mathbf{j} + (14 - 0)\,\mathbf{k}$

Magnitude of $\overline{r}$ is $\sqrt{5^2 + 2^2 + 14^2}$ = 15 units.

$\overline{\mathbf{r}} = 15\,(l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k})$. By comparison with above,
$l = -5/15$, $m = 2/15$, $n = 14/15$, etc.

R2.4 For an arbitrary triangle *ABC*, drop a perpendicular from *B* to side *AC* and use Pythagorus theorem to prove the two laws.

R2.5 The direction of projection is along the cross product of *A* and *B* vectors which is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & 1 \end{vmatrix}, \text{ i.e., } 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}.$$

Find the unit vector along it and multiply by 70 m/s.

R2.6 By definition, $\mathbf{A} \cdot \mathbf{B} = AB \cos 60$

Find A = 101 units, $B = \sqrt{18 + x^2}$ units and hence, by comparing, $x = -35.15$ units as the feasible answer.

R2.7 Use the rules as stated in the text.

R2.8 (a) You need to show that

$e_1 \cdot e_2 = 0$,
$e_2 \cdot e_3 = 0$ and
$e_3 \cdot e_1 = 0$

(b) Find $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{C}$.
Now show that the dot product of the two results is not zero; hence non-coplanar.

R2.9 For (a), differentiate term-by-term. For (b), differentiate the scalar after dot product, $16t^3$ and get the answer $48t^2$. Likewise for (c) and (d).

R2.10 Velocity $\mathbf{V} = d\mathbf{r}/dt$ and substitute $t = 2$ seconds.
Acceleration = $d\mathbf{V}/dt$ and substitute $t = 2$ seconds.

R2.11 Unit vector along $PA = [(4-0)\,\mathbf{i} + 0\,\mathbf{j} + (0-8)\,\mathbf{k}]\Big/\sqrt{4^2+8^2}$

Force along $PA = 20\,(4\,\mathbf{i} - 8\,\mathbf{k})\Big/\sqrt{80}$

Forces in other cables are $T_1\,(-\mathbf{i} + 4\,\mathbf{j} - 8\,\mathbf{k})\Big/\sqrt{81}$

and $T_2\,(-2\,\mathbf{i} - 3\,\mathbf{j} - 8\,\mathbf{k})\Big/\sqrt{77}$.

Equating the sum of all three **i** and **j** components and zero, $T_1 = 22$ kN, $T_2 = 28.6$ kN.
Forces exerted on the pole = sum of **k** components which is 63 kN.

R2.12 Get $\mathbf{r}_{PQ} = -\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$ with $\mathbf{u} = (-\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k})/3$. The force is $100\,\mathbf{u}$. The distance PO is $6\,\mathbf{j} - 4\,\mathbf{k}$ and the moment $\mathbf{r} \times \mathbf{F}$ can be found.

R2.13 Use the definitions and apply the rules.

R2.14 (a) that **r** is a constant vector or zero.
(b) that the two are collinear or parallel or one is zero.
(c) irrotationality!

R2.15 $\mathbf{r} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 2x & -2y & xy \end{vmatrix}$, etc.

R2.16 $\mathbf{V} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 100 & 25 & 0 \\ 0.01 & -0.001 & 0 \end{vmatrix}$, etc.

R2.17 $\omega = 2\pi \times 50/60\ \mathbf{i} = 5.236\ \mathbf{i}$ rad/s
For C_1, $\mathbf{r} = 0.5\,\mathbf{k}$, $\mathbf{V} = \omega \times \mathbf{r}$, etc
For C_2. $\mathbf{r} = 0.2\,\mathbf{j} + 0.5\,\mathbf{k}$, $\mathbf{V} = \omega \times \mathbf{r}$, etc.

R2.18 Construct the force triangle with the desired conditions and show that the components are 200 and 282.6 N. For equally inclined components,

$$F_1 \cos\theta + F_2 \cos\theta = R$$
$$F_1 \sin\theta = F_2 \sin\theta;\ F_1 = F_2 = R/(2\cos\theta)$$

As θ increases, $\cos\theta$ decreases until $\theta = 90°$ where $\cos\theta = 0$ and F_1 and F_2 tend to infinity.

R2.19 For equilibrium,
$-T_1 \cos 30 - T_2 \cos 25 - T_3 \cos 30 = -1000$
and $T_1 \sin 30 \cos 30 - T_3 \sin 30 \cos 30 = 0$
and $T_1 \sin 30 \sin 30 + T_2 \sin 25 - T_3 \sin 30 \sin 30 = 0$
which provide, $T_1 = T_3 = 356.6$ N, $T_2 = 421.9$ N.

R2.20 The resultant force $100(0.6\ \mathbf{i}\ + 0.8\ \mathbf{j})$ must be the vectorial sum of $F_1(0.5\,\mathbf{i} + 0.5\,\mathbf{j} + 0.707\,\mathbf{k})$, etc. Find the values of the magnitudes F_1, F_2 and F_3.

CHAPTER 2

2.1 While at (2, 1), it exerts a moment (2 **i** + 1 **j**) × (– 50 **k**), i.e., 100 **j** – 50 **i** about 0. Now, while at a (1, 2), it exerts 50 **j** – 100 **i**. The difference is required to maintain equivalence.

2.2 By the priniciple of parallel transfer of force, it is necessary to accompany with a moment 100 × 0.25 for *O* and 100 × 0.5 for *B*.

2.3 Resolve them into components and add, e.g., 150 cos 30 **i** + 150 sin 30 **j** + 180 cos 45 **i** – 180 sin 45 **j** . . .

2.4 Resolve them into *x*, *y* and *z* components and add.

2.5 They result in a couple of moment 0.3 × 10 Nm clockwise normal to the plane. If one force was 11 N, the resultant force would be 1 N in that direction and additional moment of 0.15 × 1 Nm.

2.6 Resultant force is zero but the resultant moment about any arbitrary point is –30 Nm.

2.7 Let the fourth force be *F* located at a distance *x* from *A*. Write the equations for resultant action and solve for *F* and *x*.

2.8 Proceed by considering the resultant as the summation of forces and summation of moments about any arbitrary chosen points.

2.9 Summation of forces and moments about any chosen point provides the answer. Addition of a force along the bar is made vectorially.

2.10 The resultant force is (3 **i** – 0.5 **j**) kN. Find the magnitude and the angle.

2.11 The pivot provides force-reactions to balance the forces. Sum of moments is zero. If the 10 N force moves to *P*, there is a net moment!

2.12 Proceed by summing the forces and moments and locating the single equivalent force.

2.13 The equivalent action at *O* should consist of *F* and *M* which are summations of forces and their moments about *O*.

2.14 Resolve the loads into *x* and *y* components and add. Find the magnitude of the load and the unit vector only along which it acts.

2.15 Consider the sum of forces and moments about the fulcrum.

2.16 Single resultant = –400 cos 60 **i** – (–300 –200 + 400 sin 60) **j** with the moment 400 sin 60 × 0.3 × 50 – 200 × 0.5 = $F_y \times x$.

2.17 **F** at *A* and *B* = 3/0.4 = 7.5 N. The force **R** referred to *A* is 100 **i** acting at *A* together with a moment 100 × 0.2 in the z-direction. Equivalent couple force are 50 N, 0.4 m apart. Hence the replacement.

2.18 **F** = – 50 **i** + 80 **j** – 50 **k** passing through 0 and $\mathbf{M}_0$ = 1250 **k** – 500 **i** + 750 **j**. Find the unit vector along **F** and the component of the moment along it is $\mathbf{M}_0$ = 351.1 units. This could be eliminated by transferring the force parallel to itself such that $\mathbf{r} \times \mathbf{F} = \mathbf{M}_n$.

2.19 Find the angle θ_1 of the 50 N force with the base diagonal as 20.4 degrees. The angle θ_2 between the *x*-axis and the base diagonal is 68.2 degrees. Write for the force and then the moments as **r** × **F**. Find the resultant force and moment.

2.20 Consider two parallel elementary discs of radii r_1 and r_2 of the spheres at a distance $(r - x - y)$ between them. Write the expression for the elementary force of attraction and integrate over the radii of the spheres.

2.21 Consider two rod elements equidistant from the point mass. The horizontal components of the forces cancel and the vertical components add up. Mass per unit length of the rod is $M/2a$. Integrate the force from 0 to a.

2.22 Limiting friction = $0.3 \times 20 \times 10^3$ = 6 kN. With relative motion the frictional force decreases.

2.23 It all depends upon the masses and coefficient of friction between different pairs of surfaces. Draw a figure with, say, five plates and apply a horizontal force at the mid plate 3 with plates 2 and 1 above it and 4 and 5 below it. If F is greater than $\mu_3 (m_1 g + m_2 g + m_3 g)$ and $\mu_2 (m_1 g + m_2 g)$ is greater than F_1 them the plate number 3 and those above it will slide. Like this, discuss different possibilities.

2.24 Resultant force = k **a** + k **b** + k **c** but **a** + **b** = −**c** by the parallelogram law of forces. Therefore, the resultant is zero. Moment about an arbitrary point equals 2 **k** times area of the triangle. A polygon can be considered to be composed of constituent triangles; hence the fact!

2.25 Resultant = 200 **i** – 400 **j** – 500 **k** units. If it is passed through the point O, the accompanying moment would be 3**i** × (−500 **k**) + 6 **i** × (−400 **j**).

2.26 $\mathbf{M}_A = 4 \times 4 \sin 40° - T \times 6 \cos (90 - (180 - (90 + 30 + 40)))$. Equate it to zero for equilibrium and find T.

2.27 Consider a particle of mass m at a distance d from the earth. Then, $GM_e \cdot m/d^2 = GM_m \cdot m/(3.8 \times 10^5 - d)^2$. Find d.

2.28 The upward force at B must equal the weight of the punch, 2 × 9.81 N. Force in AB = 2 × 9.81/cos 30 and the moment = 0.2 × force in AB.

2.29 Express the force in components and express the position vector **CE** or **CB** to find the moment about C, similarly, about H. Moment along the axis CF is given by the dot product of the moment about C with the unit vector along CF.

2.30 Proceed by taking summation of forces and take moments about the point D. Next, employ the 'parallel force transfer' principle to get the answer for the second part.

CHAPTER 3

3.1 Consider the free-body diagram of the boat including force **F** exerted by the flow and tension T in OC to find their magnitudes. Now draw the *fbd* without T and determine the tensions in OA and OB. Both being positive, the boat will remain in equilibrium.

3.2 Consider the forces at O for equilibrium and apply the Lami's theorem.

3.3 For a concurrent force system, Σ **M** = 0. Check if it is true in this case.

3.4 Find Σ **F** and Σ $\mathbf{M}_0$. Let (x **i** + y **j**) x – 8 **k** = – 12 **i** + 15 **j**.
For (d), yes by choosing three forces which add to 8 **k** and their $\mathbf{M}_0$ = 12 **i** – 15 **j**.

3.5 First, consider the equilibrium at O. Force in OC is 29 kN.
Then, P = 29 cos 15 = 28 kN.

3.6 First, consider the *fbd* of the cylinder, and then for the link taking into account the 570 N force exerted by the cylinder normal to the link.

3.7 For (a), apply the Lami's theorem. For (b), subsitute for T and W; $d = 10$ m.

3.8 Find the tensions from the *fbd* of the rod. Determine x from A so that $\mathbf{M}_A = 0$.

3.9 First, use $\Sigma M_0 = 0$ for the arm. Then, at a time t, let the load be at x and at $t + \Delta t$, it beat $x + \Delta x$. By subtraction and using $x = vt$, $dx = vdt$ get the answers! Substitute $\theta = 30°$ at the end.

3.10 From the *fbd* of cylinder 1, get the force in the bar as 1200 N. Then, from the *fbd* of the cylinder 2, get P.

3.11 For the tetrahedron joining the four centres, locate the centre of the top ball; $\sin\theta = 1/\sqrt{3}$. From the *fbd* of the top ball, $3N\cos\phi = mg$. For a lower ball, $N\sin\theta = 2T\cos 30$

3.12 For equilibrium at B apply the Lami's theorem.

3.13 Use $\Sigma\mathbf{F} = 0$ and take moments about the axis 2 – 3 and then about axis CD for $\Sigma\mathbf{M} = 0$

3.14 Taking the origin at O, the unit vector along OA is $(b\ \mathbf{i} - a\ \mathbf{j} - h\ \mathbf{k}/\sqrt{b^2 + a^2 + h^2}\ )$. Also, find for OB and consider the equilibrium at O.

3.15 Tension in BC is $T_1\ (-8\ \mathbf{k} + 4\ \mathbf{j} - 2\ \mathbf{i})/\sqrt{84}$ and in DE is $T_2\ (-5\ \mathbf{k} - 4\ \mathbf{j})/\sqrt{41}$. Consider $\Sigma\mathbf{M}_A = 0$ for equilibrium. $R_x = 0$, $R_y = 840$ and $R_A = 5100$ N.

3.16 Locate B, C, G and H. Find the unit vector along BH and BG and consider the *fbd* of the boom.

CHAPTER S1

S1.1 Reaction at D consists of x and y components, D_x and D_y. Reaction at roller E is only horizontal, E_x. Take moments about E and then at D and use $\Sigma F_y = 0$. Consider equilibrium at A; then at C and B.

S1.2 Consider equilibrium at A to get forces in AB and AE. Then consider equilibrium at B and then at E. Find reaction at D and force in DC by equilibrium at D.

S1.3 Determine the inclinations of AB and AC by geometry. Reactions are found by considering the equilibrium of whole truss. By equilibrium at A, find forces in AB and AC. Force in BC can be found by considering equilibrium at B or C. Same way, attempt the next problem.

S1.4 Reaction at R has both components A_x and A_y; reaction at D is only vertical, D_y. Find them by equilibrium of the total truss. Then consider equilibrium at D, C, F and A in turn.

S1.5 Determine F_x, F_y and E_y. Cut a section through AH, AC and BC and consider the equilibrium of the upper part. Either cut another section through HG, CG and CD or proceed by taking joints B, H, C and D, in turn.

S1.6 Determine F_x, F_y and D_y by equilibrium of the whole truss. Then use the method ot joints by taking D, C, H, etc., reaching F at the end only.

S1.7 Find the reactions A_x, A_y and C_y first. Then take up joints C, D and A or B, in turn. One of the last two can be used for verification.

S1.8 Find the reactions A_x, A_y and B_y by equilibrium of the whole frame. Consider the f.b.d. separately for BEC and ADC and DE. Each of them is in equilibrium. BE must be a 2-force member (only in compression) because reaction at B must be vertical only.

4.10 Draw the intersecting curves tentatively, intersecting in the first quadrant. Locate the points of intersection (0, 0) and (6.87, 7.86). Now, $x_c = \int x\, dA / \int dA = \int x\,(y_2 - y_1)\, dx / (y_2 - y_1)\, dx$ and simplify. Similarly proceed for y_c.

4.11 Use $y_c = \left(y_{c_1}\ m_1 - y_{c_2}\ m_2 + y_{c_3}\ m_3\right) / (m_1 - m_2 + m_3)$ where suffix 1 stands for the complete perspex cylinder, 2 for the hole and 3 for the lead filling.

4.12 $y_c = 0$ by symmetry about the x-axis. Side of the square is $R/\sqrt{2}$ and area is $R^2/2$, etc.

4.13 $x_0 = 0$ by symmetry about the y-axis. Volume $= \pi/3(r_1^2\ (h_1 + h_2) - r_2^2\ h_2)$. Use the fact that y_c for a complete cone is at $h/4$ from the base.

4.14 For the hemisphere, $y_c = (R - 3/8R) = 5/8R$ and $V = 2/3\pi R^3$. For the cylinder $y_c = R + h/2$, etc.

4.15 For equilibrium, the centre of gravity should be vertically below the point of suspension and hence at the lowest position.

4.16 For the circular arc, $l_s = \pi R$ and $Y_{cs} = 2R/\pi + a$. Find a such that

$L_s \cdot y_{cs} + 2a \cdot a/2 = (L_s + 2a) \cdot$ a. Get $a = \sqrt{2}\, R$ and

total length $= (\pi + 2\sqrt{2}\,)R$.

4.17 The centroid of the semicircular wire, which is at $2R/\pi$ from the diameter should be vertically below the point of suspension, for equilibrium. If W is suspended from B, then take moments about the hinge and equate the sum to zero.

4.18 Empoly the Pappus-Guldinus theurem. For a hemisphere, $4\pi\ r^2/2 = \pi\ r\ x$ distance covered by the generating arc $\pi\ y_c$. Hence $y_c = 2r/\pi$ and $x_c = 0$. Similarly y_c for a semicircular disc $= 2\ /3\pi\ r^3/(\pi\ r^2/2.\ \pi = 4r/3\pi)$.

4.19 For the arc, $y_c = 0.2 + 2 \times 0.15/\pi = 0.295$ m; $l = \pi \times 0.15$ and proceed.

CHAPTER 5

5.1 Integrate $a = \dfrac{dv}{dt}$ w.r.t. t in order to get $v = t^4/4 - t^3 + 5t + c_1$ and evaluate the constant c_1 by substituting $t = 1$ s, $v = 6.25$ m/s. Integrate again to get s.

5.2 From $a = \dfrac{dv}{dt}$, find v and then find s using $v = 0$ at $t = 0$ and $s = 0$ at $t = 0$. It will stop again when $v = 0$. Find t at that instant.

5.3 Final velocity is $(2\,\mathbf{i} - 3\,\mathbf{j}) \times 10$ more than the initial velocity! Determine the x, y and z components by employing $s = u\,t + 1/2a\,t^2$ along each direction.

5.4 At $t = 0$, $x = 0$, $y = 3$, $z = 0$; at $t = 25$, $x = 20$, $y = 3$, $z = -4$. Determine the constants A, B and C to come out as 5, 0 and –2 respectively. Then find the velocity at $t = 5$ s.

5.5 Let $x = x_0 \cos\ \omega\, t$; $v = -\omega x_0 \sin\ \omega\, t$ and $a = -\omega^2 x_0 \cos\ \omega t$ referred to an extremity. Use $x_0 = 1$m, $r = 2\pi/\omega = 2$s.

5.6 Let $x = x_0' \sin\ \omega\, t$, referred to the mid-point. Here $\omega = \pi/3$. Use the boundary conditions.

5.7 Here $\tau = 0.6 = 2\pi/\omega$. At the mean position, $v_{max} = \omega x_0 = 1.5$. At half way position, $0.5 = \sin \omega t$. So, $\omega t = 0.523$ and $v = 1.5 \cos 0.523 = 1.3$ m/s.

5.8 For the projectile path, $x_{max} = v \cos \theta . t$ and $z_{max} = v \sin \theta . t/2 - \frac{1}{2} g (t/2)^2$.

Equate x_{max} to z_{max} and use the fact that $z = 0$ at x_{max}; $t = 2v \sin \theta/g$.

5.9 The horizontal distance between them, 30 cos 30° is the sum of the distances traversed by the bullets, i.e. $350 \cos 30°.t + 300 \cos(-30°).\ t$ whence $t = 0.0462$ s, etc.

5.10 For simplicity consider the target at x_m horizontally $= v \cos \theta . t$ and $t = 2v \sin \theta/g$. $x_m = v^2 \sin 2\theta/g$ Since $\sin \psi = \sin(\pi - \psi)$, $\theta_1 = 1/2 \sin^{-1}(gx/v^2)$; $\theta_2 = \pi/2 - \theta_1$.

5.11 $z = x \tan \alpha - gx^2/(2v^2 \cos^2 \alpha)$. For $x = 5z \geq 2$ and $x = 0$, $z = 0$. From the equation, $\sin 2\alpha = 0.2725$, $2\alpha = 15.81°$ and 164.2°. Hence find the appropriate α.

5.12 $z = -120 = x \tan \alpha - gx^2/(2r^2 \cos^2 \theta)$. Find x for $\theta = 30$. Use $x = v \cos \theta . t$.

5.13 At any time t, separation between them is given by $x = h - (2v \cos \alpha t)$ and $z = z_2 - z_1$. Distance d between them is $\sqrt{(x^2 + z^2)}$. For it to be minimum, differentiate it w.r.t. time and equate to zero!

5.14 For the first half distance $s/2 = \frac{1}{2} g (t - 1)^2$ and for the next half distance, $s/2 = g(t - 1) + 1/2g.1^2$. Equate the two expressions and proceed.

5.15 The path of the ball is given by $z = -gx^2/(2 \times 1.5^2 \times 1)$, i.e., $z = -2.18x^2$. Find the point of its intersection with $z = -x$, the slope of the tips of the stairs; $x = 0.45$ m which means the third step.

5.16 For the path, $z = \tan \theta - gx^2/(2V^2 \cos^2 \theta)$,, substitute $x = 12$ m, $z = -2$ m, $\theta = 30°$, and with $g = 9.81$ m/s^2, determine the speed V as 37 km/hour.

5.17 Assuming that the helicopter is descending with a vertical u, initial height h, use $h = u t + 1/2gt^2$. For the second packet, the height is $(h - 5u)$, etc.

5.18 Obtain $dx/dt = 8t$ and also dy/dt by using $x y = 16$. Substitute $t = 1$ second to get the velocity components. For $x = 2$ m, get $t = 0.707$ second and proceed the same way.

5.19 Given that, for $\alpha = 90°$, $h = z_{max}$

Now, $x_{max} = V_0^2 \sin 2a/g = 2h$ at $\alpha = 45°$.

5.20 Proceed by taking $s/2 = 1/2a\, t_1^2$, $s/2 = 1/2r\, t_2^2$, $t = t_1 + t_2$, $a\, t_1 = r\, t_2$.

5.21 Given that $\alpha = kt = d\omega/dt$. Intergrate w.r.t. time and substitute the data.

5.22 Draw a line joining G to E and write vector expressions for position, velocity and acceleration.

5.23 First differentiate the given relations w.r.t. time for velocity and acceleration and then substitute the data each time.

5.24 The time taken for the bullet to hit it is expressed as $h/(V \cos \alpha - v) \tan \alpha$ from $\tan \alpha = h/x'$ and $x = x' + vt$ where x' is the horizontal distance at the instant of firing and x is at the instant of striking the shell. Then, find h by using the expression $V \sin \alpha t - 1/2gt^2$.

5.25 Use $z = 1/2gt^2$ to find the time and $x = V_x . t$. $x/h = \tan \theta$; $\theta = 48.9°$.

5.26 Differentiate r and θ w.r.t. time and use $\mathbf{V} = \dot{r}\,\mathbf{e}_r + r\omega\,\mathbf{e}_\theta$ and substitute t = 3 seconds. Differentiate again $\mathbf{V}$ w.r.t. time to get the acceleration.

5.27 Notice that $\omega = 10$ rad/s, $\alpha = 2$ rad/s, $\dot{r} = 3$ m/s, $\ddot{r} = 2$ m/s^2, $r = 0.5$ m. Substitute in the terms for velocity and acceleration in cylindrical co-ordinates.

5.28 From the given expression for r and θ, determine, r, θ, at $t = 20$ seconds and differentiate twice to get velocity and acceleration respectively in cylindrical co-ordinates.

5.29 From $y = 2x^3$, $\dot{y} = 6x^2 \cdot \dot{x}$. Now, $v = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}$ and $|v| = \sqrt{\dot{x}^2 + \dot{y}^2} = 3$ whence $\dot{x} = 0.493$ and $\dot{y} = 2.96$ m/s. Similarly, from $y = 2x^3$, get $\ddot{y}$ and proceed.

5.30 By energy conservation until A and B, $V_A = V_B = V = \sqrt{2gh}$; $h = 0.5 + l\cos 30°$. The acceleration at A is $g\cos 30°$ along the slope and hence $-\ g\cos 60°$ tangential to the arc; the radial component being V^2/r towards the centre of the arc. Using $v = 5.18$ m/s from the above, obtain the total acceleration.

5.31 Use $\mathbf{a}_{Pf} = \mathbf{a}_{Pm} + \mathbf{a}_0 + \alpha \times \mathbf{r} + 2\omega \times \mathbf{V}_{Pm} + \omega \times (\omega \times \mathbf{r})$. Place the moving frame at B on the link AB.
Then $\alpha = 4$, $\omega = 3$, $\mathbf{V}_{Pm} = 2 \times 5 = 10\ \mathbf{i}$, $\mathbf{a}_{Pm} = 4\ \mathbf{i} + 50\ \mathbf{j}$, $\mathbf{a}_0 = 0$. Substitute these values above to get the result.

5.32 $\mathbf{V}_p = 5$ m/s radial plus 5×10, i.e. 50 m/s tangential velocity. It can never reach B.

5.33 From $x = l\tan\theta$, get $\dot{x}$ by using $\omega = 2\pi N/60$ rad/s. Again, differentiate and substitute the data and get the acceleration.

5.34 Consider the chain at an instant t when a length x has dropped and $(l - c - x)$ is flat on the table. Consider the tension T in the chain at the edge; then, $T + (c + x)\,w/g\ \ddot{x} = (c + x)\,w$ or $T = (c + x)\,w\,(1 - \ddot{x}/g)$ for the vertical part. Also, $T = (l - c - x)\,w/g\ \ddot{x}$ for the horizontal part. The two to get the differential equation describing the motion. Solve the equation as well.

5.35 $V_r = V\sin\alpha\ (= \dot{r})$, $l_r = l\sin\alpha\ (= r)$. Now, $\mathbf{a} = -r\omega^2\,\mathbf{e}_r + 2\dot{r}\,\omega\,\mathbf{e}_\theta$. The magnitude of acceleration is given by $\sqrt{(l\sin\alpha\,\omega)^2 + (2V\sin\alpha\,\omega)^2}$ which simplifies to yield the answer.

CHAPTER 6

6.1 The splash of water is heard after the stone drops a distance and the sound travels up the same distance. Hence $2.5 = \sqrt{2d/g} + d/330$ which gives $d = 28.6$ m.

6.2 Here $9.\sqrt{x} = \dfrac{dv}{dt} = \dfrac{dv}{dx}\cdot\dfrac{dx}{dt} = v\cdot\dfrac{dv}{dx}$ whence $v\,dv = 9x^{1/2}\,dx$, $v = \sqrt{12}\,x^{3/4}$

Now $v = \dfrac{dx}{dt}$. Substitute and integrate to get $x = 67.5$ m, $v = 81.6$ m/s and $a = 73.9$ m/s^2.

6.3 For the free body diagram of the block, write the equations of motion along the horizontal and vertical axes. Also, write for the free body diagram of the wedge. Solve the equations.

6.4 Find the acceleration by using $v = u + at$ and write the equation of motion along the incline for the verticle. Then
$F = 2000 \times 0.33 + 200 + 2000 \times 9.81 \times 1/20 = 1848$ N.

6.5 For the total system of masses accelerating at a, the net external force along the motion is $m_3g - \mu_1 m_1 g - \mu_2 m_2 g$; equate it to $(m_1 + m_2 + m_3)$ times acceleration. Then, consider the free body diagram of the blocks m_3 and m_1 and solve the equations of motion.

6.6 Consider the free body diagram of block B. $ma = \mu\, m\, g$; $a = \mu\, g$. The force F on block A causes resistance $\mu\, m\, g$ and inertia force $m\, a$ on B.

6.7 Net force on the object is $mg - kv^2 = ma = m\dfrac{dv}{dt}$. Separating the variables, $dt = dv/(g - k/m \cdot v^2)$. Intergrate it logarithmically and obtain the expression.

6.8 For the upward motion of the particle, $-mg - kv - ma = 0$. Expressing $a = \dfrac{dv}{dt}$, separating the variables and integrating both sides logarithmically, get the answers. For the second case $-mg - kv^2 - ma = 0$ and proceed the same way.

6.9 From the free body diagram of the block A, $F = \mu_A N_A - T\cos\theta - Ma = 0$. and $N_A - Mg - T\sin\theta = 0$. Eliminate N_A. Now consider block B and get $T = m\,(a + \mu_B g)/(\cos\theta + \mu_B \sin\theta)$. Then use their given values and solve for T and a.

6.10 For the position at an angle θ, $mv^2/r - mg\cos\theta - N = 0$ and $N = 0$ at the instant of leaving the sphere. Referred to the initial position, mechanical energy is conserved. Solve for θ.

6.11 For the conical pendulum, at the instant shown, $T\cos\theta - mg = 0$ and $T\sin\theta - m\, v^2/r = 0$. Use $v = r\,\omega$ and $\sin\theta = r/l$ to get the answers.

6.12 For the free body diagram of the rotating block,
$T\cos 30° = m\,\omega^2 r + N\cos 60°$ and
$T\sin 30° = mg - N\sin 60°$.
Solve the equations. At the instant it loses contact, $N = 0$. Find T and ω.

6.13 Kinetic energy of the wagon just before it touches the spring is $1/2\, m\, v^2$ and the potential energy stored in the spring after compression is $1/2\,(k_1 x_1^2 + k_2 x_2^2)$. Equate the two, using $x_2 = 2/3\, x_1$. Next, for the free body diagram of the wagon, $m\, a + k\, x = 0$. Using $a = \dfrac{dv}{dt}\cdot\dfrac{dx}{dt}$, separating the variables and integrating $-v^2/2 = k/m \cdot x^2/2 - 50$, etc.

6.14 For (a), $t = 22/\, V\cos 45°$, $V\sin 45°\, t - 1/2 g\, t^2 = 0$.
For (b), $t = 22/V\cos 45°$, $V\sin 45°\, t - 1/2 g\, t^2 = -2$.

6.15 Use conservation of mechanical energy principle,
$v = \sqrt{2 \times 9.81 \times 3\,(\cos 45° - \cos 30°)} = 3.06$ m/s

7.10 Let a moving frame of reference be placed on pin P fixed with CD and let Q be on the link AB. Use cosine law and sine law. $\mathbf{V}_Q = 12.5\ \mathbf{i} - 6.4\ \mathbf{j}$ and $\mathbf{V}_P = \mathbf{V}_Q + \mathbf{V}_{PQ}$. Finally, ω of $CD = V_P/12 = -10.7/12 = -0.89$ rad/s.

7.11 From geometry, $AO = 11.33$ cm, $OP = 10$ cm, etc. Take A' on CD, coincident with A. Then $\mathbf{V}_{A'} = \mathbf{V}_A + \mathbf{V}_{A'A}$. Also, from geometry, $\theta = 27.9°$. Finally, ω of $OP = 99.7/10 = 9.97$ rad/s.

7.12 In this case, angular momentum must be conserved. $\mathbf{r} \times m\ \mathbf{V}$ = constant, $r^2\ m\ \omega^2$ = constant = $R\ m\ \omega_0^2$. From geometry of moment, $R = r + 2\pi a.\ \theta/2\pi = r + a\ \theta$. Express r and $\dot{r}$ and integrate w.r.t t to get the desired expressions.

7.13 Let the inner cylinder rotate at ω_i. The velocity at the bottom is $r_3\omega_i$. For 'no-slip' condition, this is also the velocity of the inside of the hollow cylinder. The targential velocity of the hollow cylinder is $r_3\omega_i/r_2 = v/r_i$, whence $\omega_i = r_2/(r_1 r_3).\ v$. Similarly, proceed for the angular acceleration.

7.14 The velocity of C must be the average of the two. It can also be determined by equating the values of the rotational velocity ω of the cylinder as seen from C, both above and below; $(\dot{x}_1 - v)/R = (v - \dot{x}_2)/R$.

7.15 Write the expressions for $\mathbf{V}_A$ and $\mathbf{V}_B$ and use $\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{BA}$. Solve the equation.

7.16 From the geometry, $\angle OCQ = 25.7°$, $\angle OCQ = 94.3°$, etc. Find V_C and V_{CP} and hence compute V_B and V_A, taking into account the fact that A can move only horizontally.

7.17 From the geometry, find the angle ACB and proceed either graphically or vectorially.

CHAPTER 8

8.1 Take a strip of width dy and length $2x$ at a distance y from the x axis. Use the equation $x^2/a^2 + y^2/b^2 = 1$ and substitute $x = a \cos \theta$ and $y = b \sin \theta$. Integrate, $\theta = -\pi/2$ to $+\pi/2$.

8.2 Equate the I_{xx} to I_{yy}, i.e. $bh^3/12 = hb^3/48$. Hence $b = 2h$.

8.3 Consider an elemental strip of length a, cross section $dx\ x\ dy$ and find I_{xx}. Integrate between the limits $-a/2$ to $+a/2$.

8.4 Consider an elemental disc of radius r_0 at a distance z from the apex. Then, $r_0/z = R/h$ and $m = \rho.1/3\pi R^2 h$. Find I_{zz} by double integration.

8.5 The proof is given in Ex. 8.14 for a solid sphere, set R_1 to zero and $R_2 = R$.

8.6 Consider a circular disc of radius y at a distance x from the y axis. First, find the volume and the mass of the ellipsoid. Then, use $I_{xx} = my^2/2$ for the disc and integrate.

8.7 $I_{xx} = I_{yy} = m\ R^2/4 = 100\ R^2$. Further, $I_{AA} = 100\ R^2 + 400\ (0.5)^2 = 109$ etc.

8.8 Consider an element dx at a distance x from the centre. Find I_{yy} about the desired points and substitute $l = 30$ cm.

8.9 Polar moment of inertia $I_0 = \pi\ D^4/32 = \pi\ R^4/2$ for the circular section and $\pi\ r^2/4$ for the semicircle removed therefrom.

8.10 First find the height of the centroid above the base, 4.48 cm and then find I_{xx} by using components.